2D p-wave superconductor

A. Lattice model

B. Edge state

C. Vortex and its bound states

D. Topological qubit

    1. Braiding 2 Majorana fermions
    2. Braiding 4 Majorana fermions

A. Lattice model (spinless electrons)

$$
\begin{aligned}
H \;=\; \sum_{mn} \Big[& -t(c^\dagger_{m+1,n} c_{mn} + c^\dagger_{m,n+1} c_{mn}) + h.c. \\
- \;& (\mu - 4t) c^\dagger_{mn} c_{mn} \\
+ \;& \frac{\Delta_0}{2} c^\dagger_{m+1,n} c^\dagger_{mn} + i\frac{\Delta_0}{2} c^\dagger_{m,n+1} c^\dagger_{mn} + h.c. \\
- \;& \frac{\Delta_0}{2} c^\dagger_{m-1,n} c^\dagger_{mn} - i\frac{\Delta_0}{2} c^\dagger_{m,n-1} c^\dagger_{mn} + h.c. \Big]
\end{aligned}
$$

$$c^\dagger_{mn} = \frac{1}{\sqrt{N}} \sum_k e^{i(k_x m + k_y n)} c^\dagger_{k_x k_y}$$

$$\Rightarrow \quad H = \frac{1}{2} \sum_{\mathbf{k}} (c^\dagger_k \;\; c_{-k}) \mathsf{H}(\mathbf{k}) \begin{pmatrix} c_k \\ c^\dagger_{-k} \end{pmatrix}$$

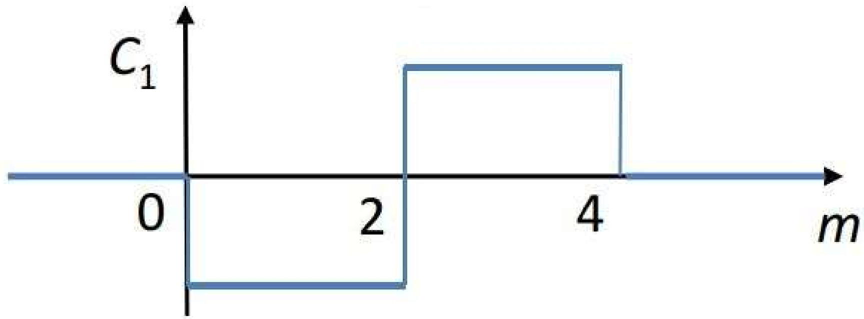
p-wave gap function

$$\mathsf{H} \;=\; \begin{pmatrix} \varepsilon(\mathbf{k}) & 2i\Delta_0(\sin k_x + i \sin k_y) \\ -2i\Delta_0(\sin k_x - i \sin k_y) & -\varepsilon(\mathbf{k}) \end{pmatrix}$$

$$\varepsilon(\mathbf{k}) \;=\; -2t(\cos k_x + \cos k_y) - (\mu - 4t)$$

Recall QWZ model:

$$
\begin{aligned}
&\mathsf{H}(\mathbf{k}) = \mathsf{H}_0 + \mathsf{H}_m + \mathsf{H}_{so}, && (1.1)\\
&\mathsf{H}_0 = \varepsilon_0(\mathbf{k}) -\\
t &\begin{pmatrix} 2-\cos k_x a - \cos k_y a & 0 \\ 0 & -(2-\cos k_x a - \cos k_y a) \end{pmatrix}, \\
&\mathsf{H}_m = m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
&\mathsf{H}_{so} = \lambda \begin{pmatrix} 0 & \sin k_x a - i \sin k_y a \\ \sin k_x a + i \sin k_y a & 0 \end{pmatrix}.
\end{aligned}
$$



Compare QWZ model with p-wave model:

$$2t = -t_{QWZ}, \ \mu = -m, \ \text{and} \ 2i\Delta_0 = \lambda.$$

Choose t=1/2

$$\begin{aligned}\mathrm{H}(\mathbf{k}) \ &= \ \overbrace{(2-\mu-\cos k_x - \cos k_y)}^{\equiv M(\mathbf{k})}\tau_z \\ &- \ 2\Delta_0(\sin k_x \tau_y + \sin k_y \tau_x).\end{aligned}$$

$$E_{\pm}(\mathbf{k}) = \pm\sqrt{M(\mathbf{k})^2 + 4\Delta_0^2(\sin^2 k_x + \sin^2 k_y)}$$

$$\mathrm{H}(\mathbf{k}) = \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}$$

$$C_1 = \frac{1}{4\pi}\int_{BZ} d^2k \frac{1}{h^3}\mathbf{h} \cdot \frac{\partial \mathbf{h}}{\partial k_x} \times \frac{\partial \mathbf{h}}{\partial k_y}$$

B. Edge state

$$\mathsf{H}(\mathbf{k}) = \begin{pmatrix} tk^2-\mu & 2i\Delta_0(k_x+ik_y) \\ -2i\Delta_0(k_x-ik_y) & -tk^2+\mu \end{pmatrix}$$

$$\mu(x) \simeq \tanh x$$

$$\begin{pmatrix} -\mu & 2i\Delta_0\left(\frac{1}{i}\frac{d}{dx}+ik_y\right) \\ -2i\Delta_0\left(\frac{1}{i}\frac{d}{dx}-ik_y\right) & \mu \end{pmatrix}\psi(x) = \varepsilon_{k_y}\psi(x)$$

$\Rightarrow$ $$\psi(x) = e^{-\frac{1}{2\Delta_0}\int_0^x dx'\mu(x')}\psi_0 \qquad \psi_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

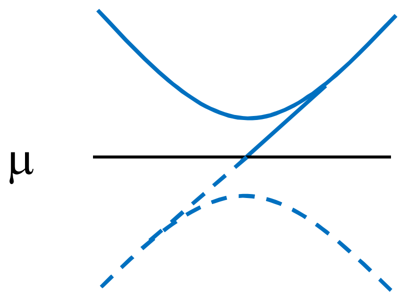
$$\begin{aligned} \gamma_{k_y} &= \int d^2r\left[u^*(\mathbf{r})\psi(\mathbf{r}) + v^*(\mathbf{r})\psi^\dagger(\mathbf{r})\right] \\ &= \int d^2r e^{-ik_y y} e^{-\frac{1}{2\Delta_0}\int_0^x dx'\mu(x')}\left[e^{-i\pi/2}\psi + e^{i\pi/2}\psi^\dagger\right] \end{aligned}$$

$\Rightarrow$ $\gamma^\dagger_{-k_y} = \gamma_{k_y}$

$\Rightarrow$ $\gamma_0^+ = \gamma_0$

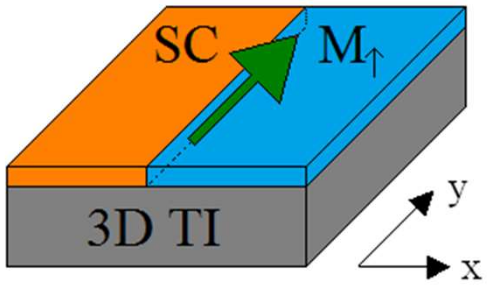
Majorana zero mode (MZM)

Not isolated

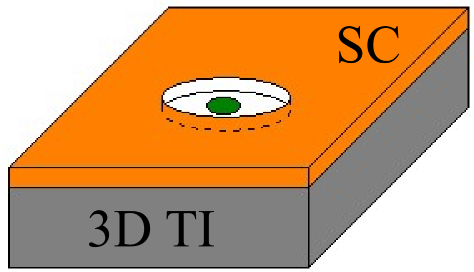


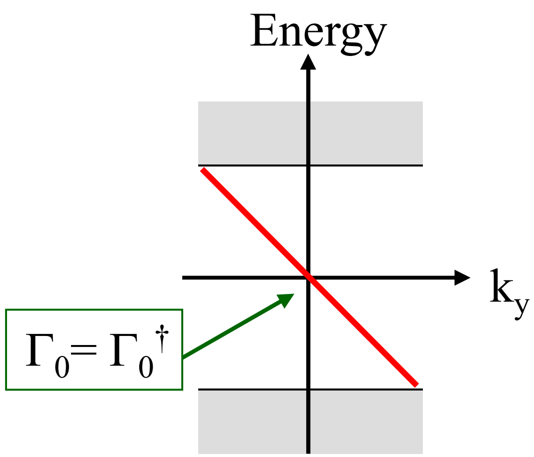
Chiral
superconductor

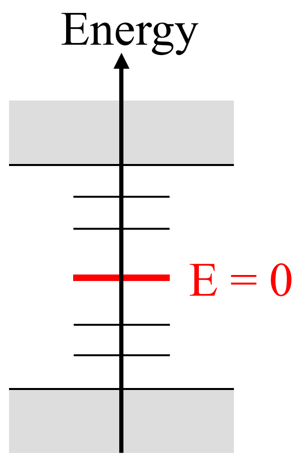
Edge



vortex







Chiral MF

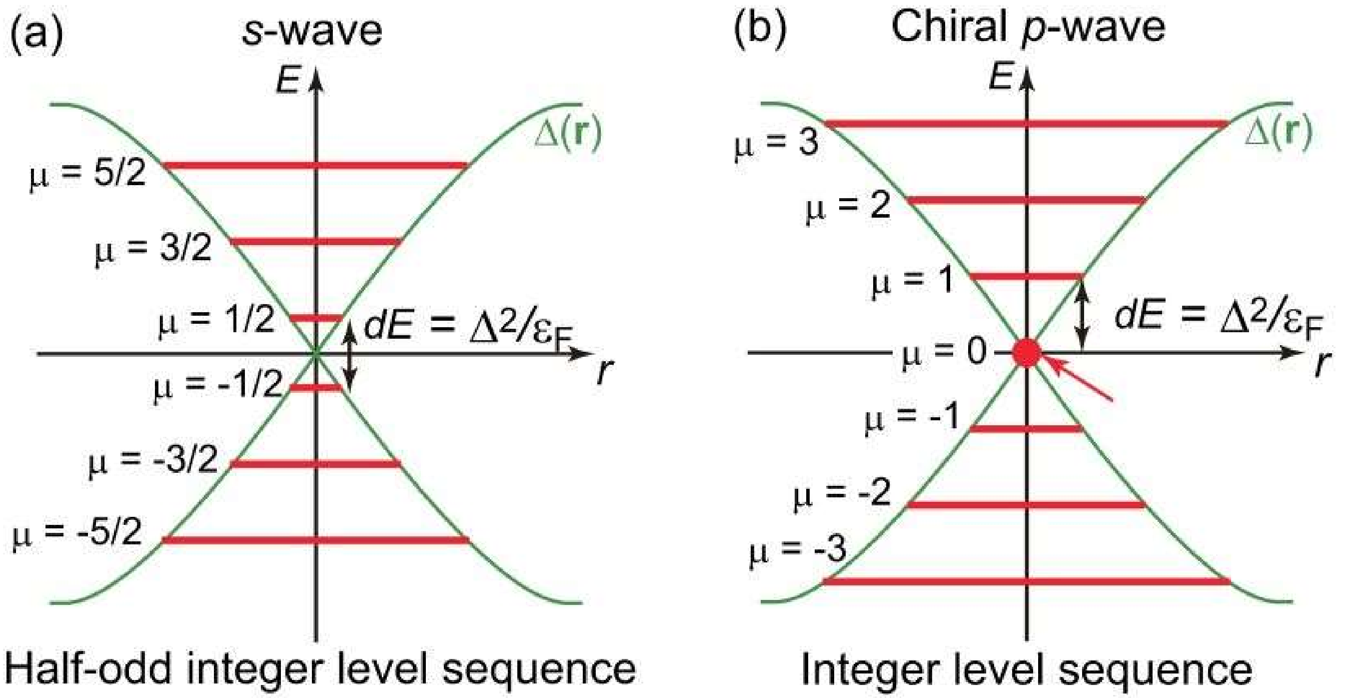
$\Gamma_E = \Gamma_{-E}^\dagger$

Localized MF

$\Gamma_0 = \Gamma_0^\dagger$

From Fleury's ppt

Bound states in the vortex of *s*-wave and *p*-wave SC



- dE ~ 1 μeV in conventional SC

  dE ~ 0.1 meV in iron-based SC

  (Δ~ 2 meV, E_F~10-20 meV)

Figs from Machida and Hanaguri,
Prog. Theor. Exp. Phys 2024

Ginzburg-Landau theory (1950)

A phenomenological theory of the order parameter Ψ(r) (App. I of Kittel)

(Later this can be derived from the BCS theory, see Fetter and Walecka)

- ψ(r) is complex-valued. It can be seen as a macroscopic wave function.
  ψ(r)=0 for normal state.
- It satisfies the Ginzburg-Landau eq

$$\left[-\frac{\hbar^2}{2m^*}\left(\boldsymbol{\nabla}-i\frac{q^*}{\hbar}\mathbf{A}(\mathbf{r})\right)^2+a(T)+b\,|\psi(\mathbf{r})|^2\right]\psi(\mathbf{r})=0$$

$$\boxed{\begin{aligned} q^* &= -2e \\ m^* &= 2m \end{aligned}}$$

- Connection with the gap function

$$\Psi(\mathbf{r}) \simeq \Delta(\mathbf{r})$$

Fetter and Walecka

$$\Psi(\mathbf{x}) \equiv \left[\frac{7\zeta(3)\,n}{8(\pi k_B T_c)^2}\right]^{\frac{1}{2}} \Delta(\mathbf{x})$$

- Connection with SC electron density and SC current density

SC electron density $\rho_S(\vec{r}) = |\psi(\vec{r})|^2$

Current density

$$\vec{j}=\frac{q^*}{2m^*}\left[\psi^*\left(\frac{\hbar}{i}\nabla-q^*\vec{A}\right)\psi+\psi\left(\frac{\hbar}{i}\nabla-q^*\vec{A}\right)^*\psi^*\right]$$

C. Vortex and its bound states

$$\begin{aligned}\mathbf{j} &= \frac{q^*}{2m^*}\left[\Psi^*\left(\frac{\hbar}{i}\nabla - q^*\mathbf{A}\right)\Psi + c.c.\right] \\ &= -\frac{e\hbar}{2mi}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{2e^2}{m}|\Psi|^2\mathbf{A}\end{aligned}$$

$$\Psi(\mathbf{r}) \simeq \Delta(\mathbf{r}) \quad \text{if } \Delta(\mathbf{r}) = |\Delta(\mathbf{r})|e^{-i\xi(\mathbf{r})}$$

$$\Rightarrow \quad \mathbf{j} \propto \frac{\hbar}{2e}\nabla\xi - \mathbf{A}$$

The phase of Δ is adjustable under the gauge transformation

$$\begin{aligned} \Delta &\rightarrow \Delta' = \Delta e^{i\chi}, \\ \text{then } \xi &\rightarrow \xi' = \xi - \chi, \\ \text{choose } \mathbf{A} &\rightarrow \mathbf{A}' = \mathbf{A} - \frac{\hbar}{2e}\nabla\chi. \end{aligned}$$

so that ***j*** is not changed

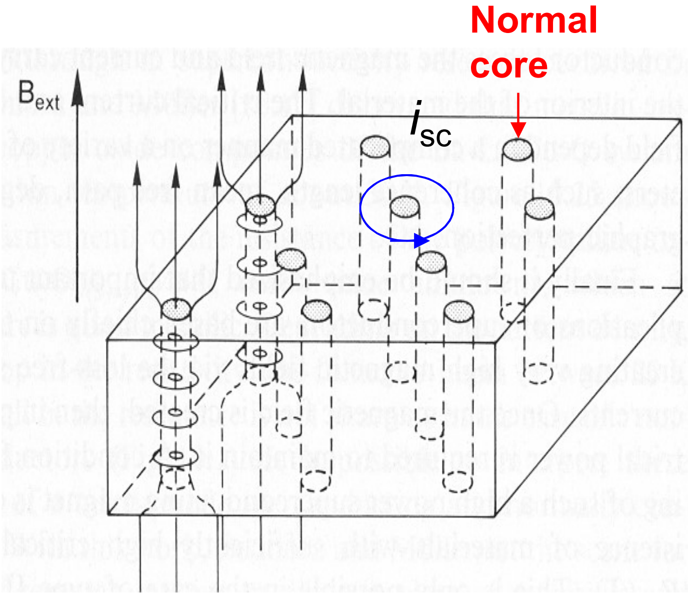
BdG equation:

$$\begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0^* \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = E_n\begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

It is invariant under the GT if

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} e^{i\chi/2}u \\ e^{-i\chi/2}v \end{pmatrix}$$

SC current around a vortex



Far way from a vortex, ***j***=0

$$\oint_C d\mathbf{r} \cdot \mathbf{j} = 0$$

$$\oint_C d\mathbf{r} \cdot \mathbf{A} = -\frac{\hbar}{2e}[\xi(2\pi) - \xi(0)]$$
$$= \frac{h}{2e}n, \quad n \in Z,$$

The magnetic flux through a vortex is quantized!

Gauge transformation

choose $\chi = \xi \; (= n\theta)$ to remove the SC phase, $\Delta' = |\Delta|$

Consider *n*=1,
after a 2π rotation of θ,
$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} e^{i\xi/2}u \\ e^{-i\xi/2}v \end{pmatrix}$$
$$= (-1)^n \begin{pmatrix} u \\ v \end{pmatrix}$$

We now study the bound states inside a vortex

Requantizing

$$\mathsf{H}(\mathbf{k}) = \begin{pmatrix} tk^2 - \mu & 2i\Delta_0(k_x + ik_y) \\ -2i\Delta_0(k_x - ik_y) & -tk^2 + \mu \end{pmatrix}$$

Polar coordinate

$$\boxed{\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \end{aligned}}$$

$$\begin{aligned} i(k_x + ik_y) &\to \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ &= e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) \end{aligned}$$

$$\Rightarrow \begin{pmatrix} -\mu & 2\Delta_0 e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) \\ 2\Delta_0 e^{-i\theta}\left(-\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) & \mu \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

$$= E_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

Zero-mode solution:

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \underbrace{\frac{i}{\sqrt{r}} e^{-\frac{1}{2}\int_0^r dr' \frac{\mu}{\Delta_0(r')}}}_{\equiv ig(r)} \begin{pmatrix} -e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix}$$

Bogoliubov QP for the zero mode

$$\begin{aligned}\gamma_0 &= \int d^2r \left[u_0^*(\mathbf{r})\psi(\mathbf{r}) + v_0^*(\mathbf{r})\psi^\dagger(\mathbf{r})\right] \\ &= \int d^2r \; ig(r) \left[e^{-i\theta/2}\psi(\mathbf{r}) - e^{i\theta/2}\psi^\dagger(\mathbf{r})\right]\end{aligned}$$

➡ $\gamma_0^+ = \gamma_0$ Majorana zero mode

D. Topological qubit

$$f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2),$$
$$f_1^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2),$$
$$\rightarrow \quad f_1^\dagger f_1 = \frac{1 + i\gamma_1\gamma_2}{2} \sim 0, 1.$$

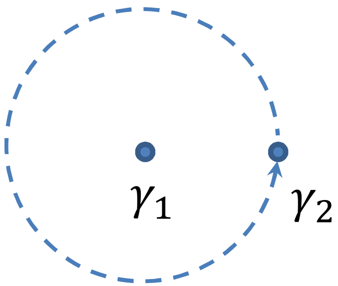
$-i\gamma_1\gamma_2$ is the fermion parity operator

For a MZM located at $\boldsymbol{R}_j$

$$\gamma_j = \int d^2r \left[h_j(\mathbf{r})e^{-i\theta_j/2+i\Gamma_j/2}\psi_j + h_j^*(\mathbf{r})e^{i\theta_j/2-i\Gamma_j/2}\psi_j^\dagger\right]$$

$$h_j(\mathbf{r}) = ig(\mathbf{r} - \mathbf{R}_j),$$

$$\theta_j = \arg(\mathbf{r} - \mathbf{R}_j),$$
$$\Gamma_j = \sum_{\ell \neq j} \arg(\mathbf{R}_j - \mathbf{R}_\ell).$$



The phase Γ_j arises because of a sign change under a 360-degree rotation.

For example, consider only 2 MZMs. If we move γ_2 around γ_1 once, then

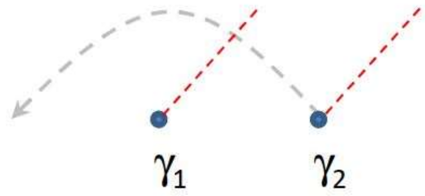
$\Gamma_2 = \arg(\mathbf{R}_2 - \mathbf{R}_1)$ changes by 2π, and γ_2 changes sign.

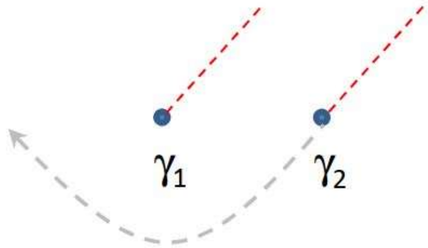
1. Braiding 2 Majorana fermions

Two ways to exchange γ_1, γ_2

(a) $\begin{cases} \gamma_1 & \rightarrow & \gamma_2, \\ \gamma_2 & \rightarrow & -\gamma_1. \end{cases}$

(b) $\begin{cases} \gamma_1 & \rightarrow & -\gamma_2, \\ \gamma_2 & \rightarrow & \gamma_1. \end{cases}$





braiding operator

$$\gamma_j \rightarrow B_{12} \gamma_j B_{12}^{\dagger}$$

For clockwise rotation,

$$B_{12} = \frac{1}{\sqrt{2}}(1 + \gamma_1 \gamma_2)$$

$$B_{12}^2 = \gamma_1 \gamma_2$$

$$\gamma_j \rightarrow B_{12}^2 \gamma_j (B_{12}^{\dagger})^2 = -\gamma_j$$

Write the states with fermion numbers 0, 1 as $|0\rangle$, $|1\rangle$, then,

$$|1\rangle = f_1^\dagger |0\rangle,$$

$$\begin{cases} f_1^\dagger f_1 |0\rangle = 0, \\ f_1^\dagger f_1 |1\rangle = |1\rangle. \end{cases}$$

Also,
$$\begin{cases} -i\gamma_1\gamma_2 |0\rangle = |0\rangle, \\ -i\gamma_1\gamma_2 |1\rangle = -|1\rangle. \end{cases}$$

It follows that,

$$\begin{aligned} B_{12}|0\rangle &= \frac{1}{\sqrt{2}}(1+i)|0\rangle = e^{i\pi/4}|0\rangle, \\ B_{12}|1\rangle &= \frac{1}{\sqrt{2}}(1-i)|1\rangle = e^{-i\pi/4}|1\rangle. \end{aligned}$$

Under the $(|0\rangle, |1\rangle))^T$ basis,

$$\begin{aligned} \gamma_1 &= \sigma_x, \\ \gamma_2 &= \sigma_y, \\ -i\gamma_1\gamma_2 &= \sigma_z, \text{ and } B_{12}|1\rangle = e^{-i\frac{\pi}{4}\sigma_z}. \end{aligned}$$

2. Braiding 4 Majorana fermions

$$
\begin{aligned}
f_1 &= \frac{1}{2}(\gamma_1 + i\gamma_2), \\
f_2 &= \frac{1}{2}(\gamma_3 + i\gamma_4).
\end{aligned}
$$

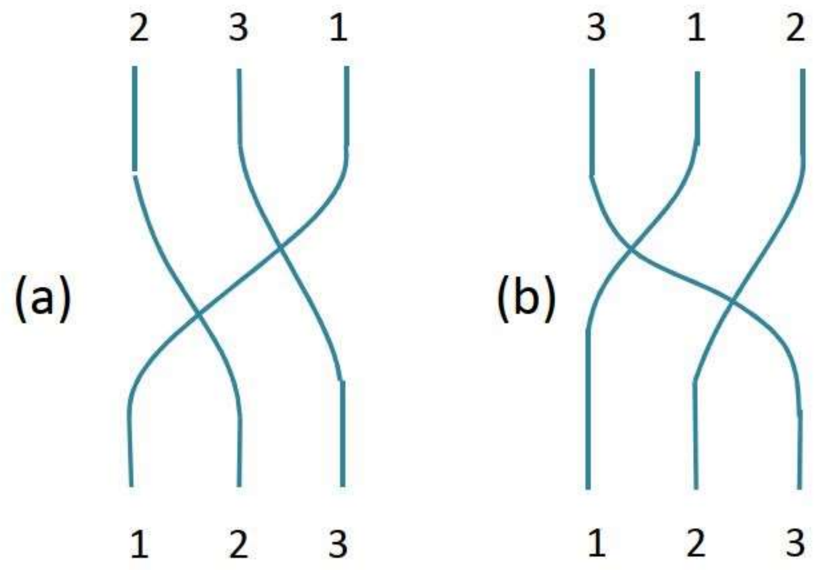
The basis of the Hilbert space are $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

For *intra*-fermion braiding,

$$
\begin{aligned}
B_{12}|00\rangle &= \frac{1}{\sqrt{2}}(1+i)|00\rangle, \\
B_{34}|00\rangle &= \frac{1}{\sqrt{2}}(1+i)|00\rangle.
\end{aligned}
$$

For *inter*-fermion braiding,

$$
\begin{aligned}
B_{23}|00\rangle &= \frac{1}{\sqrt{2}}(1+\gamma_2\gamma_3)|00\rangle \\
&= \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle).
\end{aligned}
$$



$$B_{12} = e^{\frac{\pi}{4}\gamma_1\gamma_2} = e^{i\frac{\pi}{4}\sigma_z \otimes 1}$$

$$= \begin{pmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 & 0 \\ 0 & 0 & e^{-i\pi/4} & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \end{pmatrix},$$

$$B_{34} = e^{\frac{\pi}{4}\gamma_3\gamma_4} = e^{1\otimes i\frac{\pi}{4}\sigma_z}$$

$$= \begin{pmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \end{pmatrix},$$

$$B_{23} = e^{\frac{\pi}{4}\gamma_2\gamma_3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}.$$

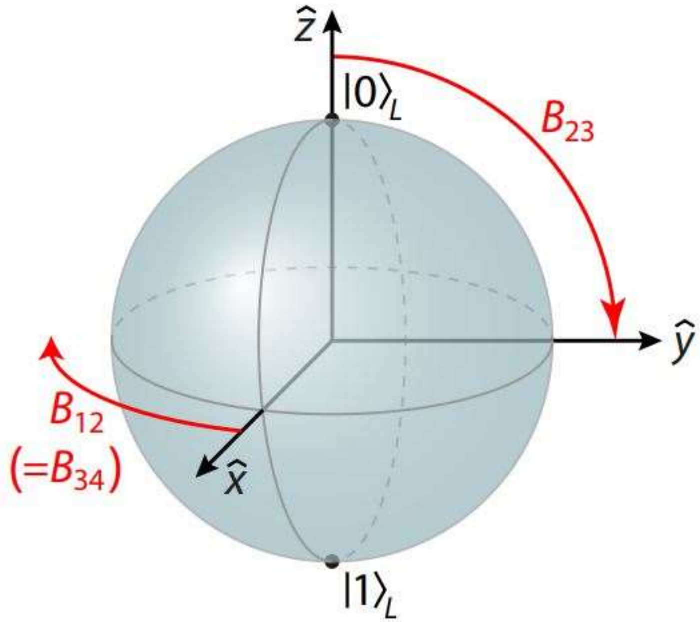
$\Rightarrow$

$$[B_{12}, B_{34}] = 0$$

$$[B_{j-1,j}, B_{j,j+1}] = \gamma_{j-1}\gamma_{j+1}$$

$$|\bar{0}\rangle \equiv |00\rangle, \ |\bar{1}\rangle \equiv |11\rangle$$

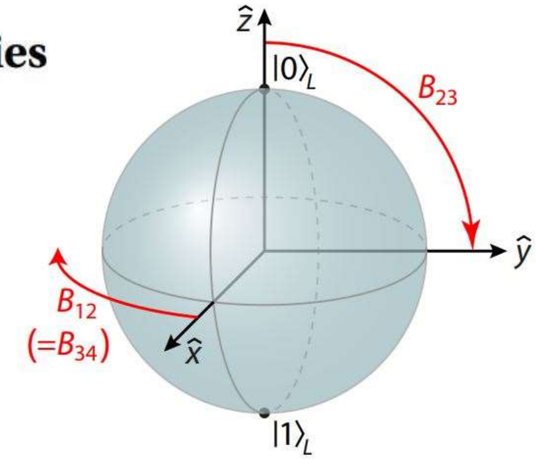
$$B_{12} = B_{34} = e^{i\frac{\pi}{4}\tau_z},$$
$$B_{23} = e^{i\frac{\pi}{4}\tau_x}.$$



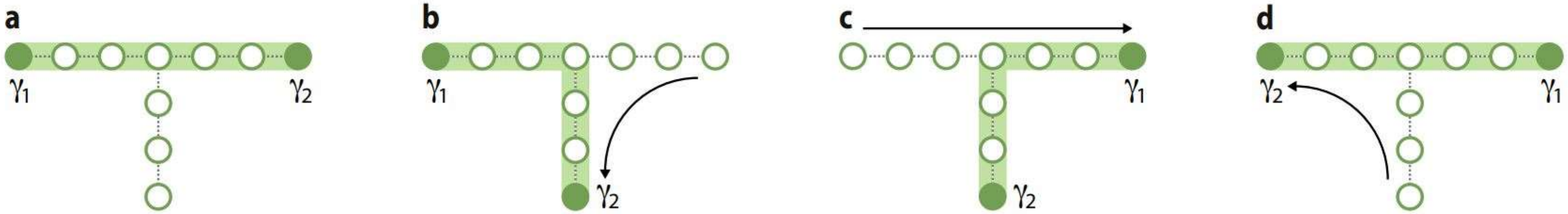
Engineering the Kitaev chain, Bordin

Back to Kitaev chain,

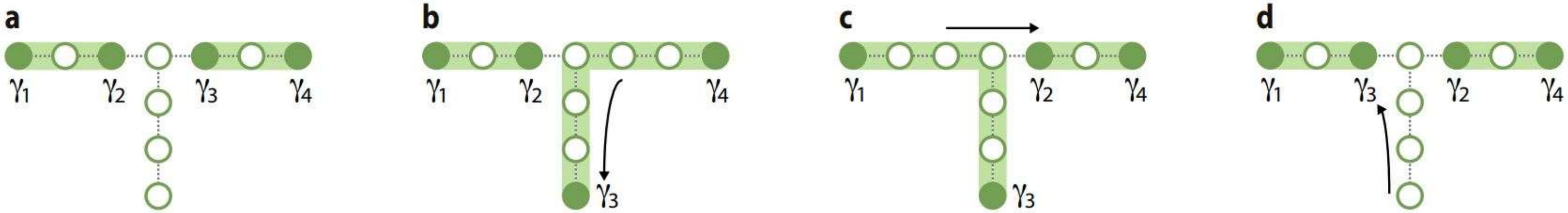
Shuttling Majorana modes by acting on the onsite energies



B_{12} braiding sequence using a Kitaev chain T-junction

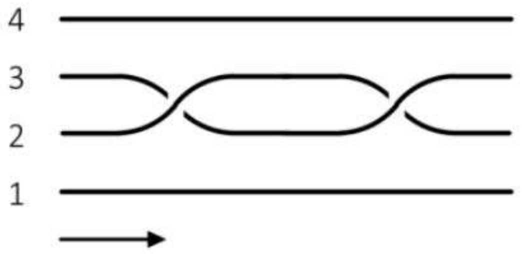


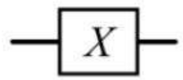
B_{23} braiding sequence using a Kitaev chain T-junction



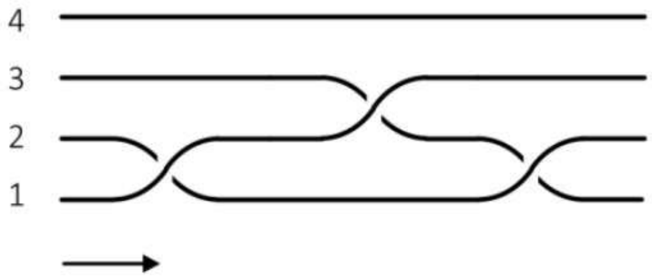
Engineering the Kitaev chain, Bordin

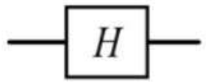
Building quantum gates with the braiding of MZMs



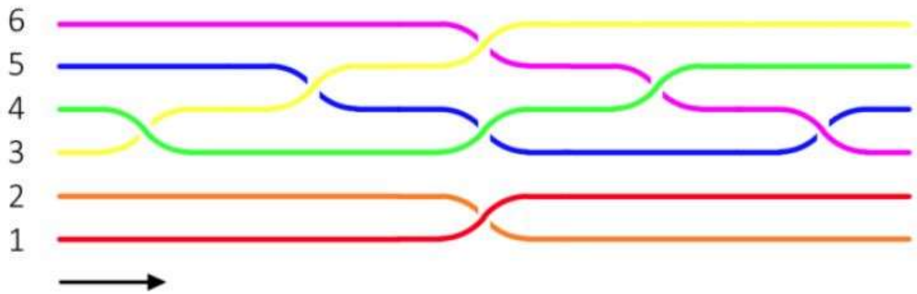


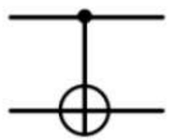
Pauli X quantum gate





Hadamard quantum gate





CNOT quantum gate

(need to change fermion parity, so 4 MFs are not enough)

https://www.quantum-bits.org/?p=2226

https://arxiv.org/pdf/1602.05035