Majorana fermion (1937)

- Real-valued solution of Dirac eq.

➡ A fermion that is its own anti-particle

- Possible candidate: neutrino

Majorana fermion in solid state?

- Superconductor has intrinsic *p-h* symmetry

  Look for non-degenerate zero-energy state, usually need *p*-wave SC

- To avoid doublet, need ***spinless*** p-wave SC
  - 1D: end state (this chap)
  - 2D: edge state, bound state in vortex (next chap)

1D p-wave superconductor

A. Continuum model

   1. Edge state

B. Kitaev model

   1. Topological number
   2. Kitaev chain with open ends
   3. Fermion parity of the ground state

1D p-WAVE SUPERCONDUCTOR

A. Continuum model

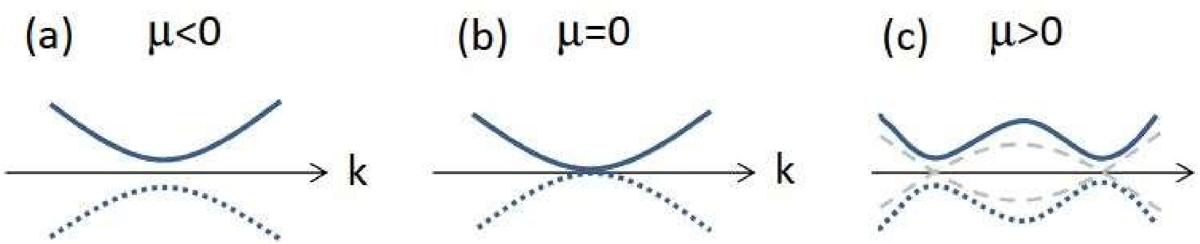
$$H = \frac{1}{2}\sum_k (c_k^\dagger c_{-k}) \begin{pmatrix} \varepsilon_k & \Delta_k \\ \Delta_k^* & -\varepsilon_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

$\varepsilon_k = \hbar^2 k^2/2m - \mu$, and $\Delta_k = \Delta_0 k$. (p-wave)

- Eigenenergies $E_\pm(k) = \pm\sqrt{\varepsilon_k^2 + |\Delta_k|^2}$
- Eigenstates for $+E$ (u_k, v_k)

$$u_k = \sqrt{\frac{1}{2}\left(1 + \frac{\varepsilon_k}{E_k}\right)}; \quad v_k = \sqrt{\frac{1}{2}\left(1 - \frac{\varepsilon_k}{E_k}\right)}\frac{\Delta_k^*}{|\Delta_k|}.$$

- Low-energy:



Strong-coupling phase

weak-coupling phase
(with nontrivial topology)

Edge state: assume $\mu(x > 0) > 0$, and $\mu(x < 0) < 0$

Ignore higher-*k* terms $\begin{pmatrix} -\mu(x) & \Delta_0 k \\ \Delta_0 k & \mu(x) \end{pmatrix} \psi = E\psi$

- Re-quantization

try $\psi(x) = \psi_0 e^{-\frac{1}{\Delta_0}\int_0^x dx' \mu(x')}$

$k \to \partial / i\partial x$

$$\Rightarrow \begin{pmatrix} -\mu(x) - E & i\mu(x) \\ i\mu(x) & \mu(x) - E \end{pmatrix} \psi_0 = 0.$$

$$\text{At } E = 0, \quad \psi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \gamma_0 &= \int dx \left[u^*(x)\psi(x) + v^*(x)\psi^\dagger(x) \right] \\ &= \frac{e^{i\pi/4}}{\sqrt{2}} \int dx e^{-\frac{1}{\Delta_0}\int_0^x dx' \mu(x')} \left[e^{-i\pi/4}\psi(x) + e^{i\pi/4}\psi^\dagger(x) \right] \end{aligned}$$

Removing the overall phase of $\pi/4$, we have

$$\gamma_0^\dagger = \gamma_0.$$

- A quasi-particle that is its own anti-particle
- Spinless, chargeless, massless

B. Kitaev model 2001

$$H = \sum_{j=1}^{N}\left[-\frac{t}{2}(c_{j+1}^{\dagger}c_j + c_j^{\dagger}c_{j+1}) - \mu c_j^{\dagger}c_j \right.$$

periodic BC, $c_{N+1} = c_1$.

$$\left. +\frac{\Delta_0}{4}\left(c_{j+1}^{\dagger}c_j^{\dagger} - c_j^{\dagger}c_{j+1}^{\dagger} + h.c\right)\right], t > 0, \Delta_0 \in R.$$

Fourier transformation,

$$c_j^{\dagger} = \frac{1}{\sqrt{N}}\sum_k e^{ijk}c_k^{\dagger}$$

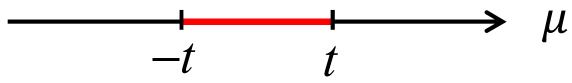
➡ $$H = \sum_k \left[-t\cos k c_k^{\dagger}c_k - \mu c_k^{\dagger}c_k \right.$$
$$\left. + \frac{\Delta_0}{4}\left(e^{ik}c_k^{\dagger}c_{-k}^{\dagger} - e^{-ik}c_k^{\dagger}c_{-k}^{\dagger} + h.c\right)\right]$$

➡ $$H = \frac{1}{2}\sum_k (c_k^{\dagger} c_{-k}) \begin{pmatrix} -t\cos k - \mu & i\Delta_0 \sin k \\ -i\Delta_0 \sin k & t\cos k + \mu \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^{\dagger} \end{pmatrix}$$
$$+ \frac{1}{2}\sum_k t\cos k + \mu.$$

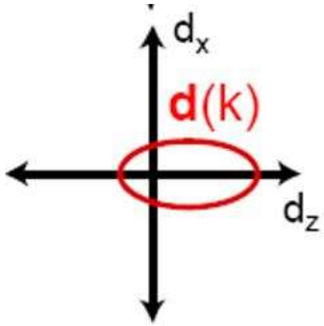
➡ $$E_{\pm}(k) = \pm\sqrt{(t\cos k + \mu)^2 + \Delta_0^2 \sin^2 k}$$

The energy gap closes when both

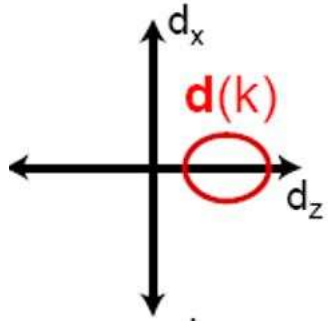
$$\begin{cases} \sin k = 0, \\ t\cos k + \mu = 0. \end{cases}$$



$$\mathsf{H}(k) = \mathbf{d}\cdot\boldsymbol{\sigma} \qquad \mathbf{d} = -(0, \Delta_0 \sin k, t\cos k + \mu)$$



$|\mu| < t$: Weak pairing phase
topological superconductor



$|\mu| > t$: Strong pairing phase
trivial superconductor

Topological number $\boxed{(-1)^\nu = \mathrm{sgn}[d_z(0)]\mathrm{sgn}[d_z(\pi)]}$

Majorana Fermions in Nanowires, by Frolov

Kitaev chain with open ends

$$\{c_j, c_{j'}^\dagger\} = \delta_{jj'}$$

Let a_j be Majorana fermion operators, with $a_j^\dagger = a_j$.

define $\{a_j, a_{j'}^\dagger\} = 2\delta_{jj'}$

$\Rightarrow$ $\{a_j, a_{j'}\} = 2\delta_{jj'}$, and $a_j^2 = 1$ (not zero!)

Decompose a fermion into 2 MFs
$$c_j = \frac{1}{2}(a_{2j-1} + i a_{2j}),$$
then
$$c_j^\dagger = \frac{1}{2}(a_{2j-1} - i a_{2j}).$$

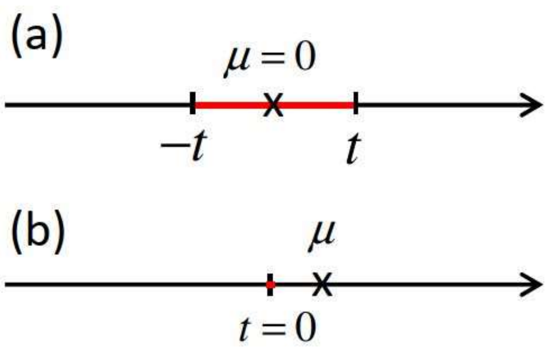
Consider a Kitaev chain with two open ends:

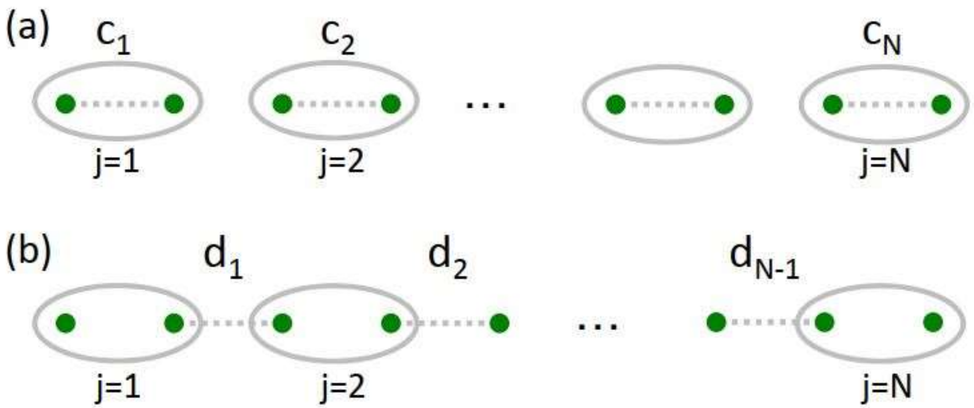
$$\begin{aligned}
H &= -\frac{t}{2}\sum_{j=1}^{N-1} c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} - \mu \sum_{j=1}^{N} c_j^\dagger c_j \\
&+ \frac{\Delta_0}{2}\sum_{j=1}^{N-1} c_{j+1}^\dagger c_j^\dagger + c_j c_{j+1} \\
&= \frac{i}{4}\sum_{j=1}^{N-1} (t+\Delta_0) a_{2j} a_{2j+1} + (-t+\Delta_0) a_{2j-1} a_{2j+2} \\
&- \frac{1}{2}\sum_{j=1}^{N} \mu\,(i a_{2j-1} a_{2j} + 1)\,.
\end{aligned}$$

For simplicity, consider the case with $\Delta_0 = t$,

$$H = \frac{i}{2}\sum_{j=1}^{N-1} t a_{2j} a_{2j+1} - \frac{i}{2}\sum_{j=1}^{N} \mu a_{2j-1} a_{2j}$$

- When |μ| < t, it is a topological SC with Majorana edge states. This fact is trivial when μ = 0: the 2nd term vanishes, and thus the 2 MFs on the ends (with zero energy) decouple from the rest of the MFs.
- For the trivial phase (|μ| > t), one can choose t = 0 to simplify H. Then every MF in the chain is coupled with its neighbor, and there is no lone MF at the ends.



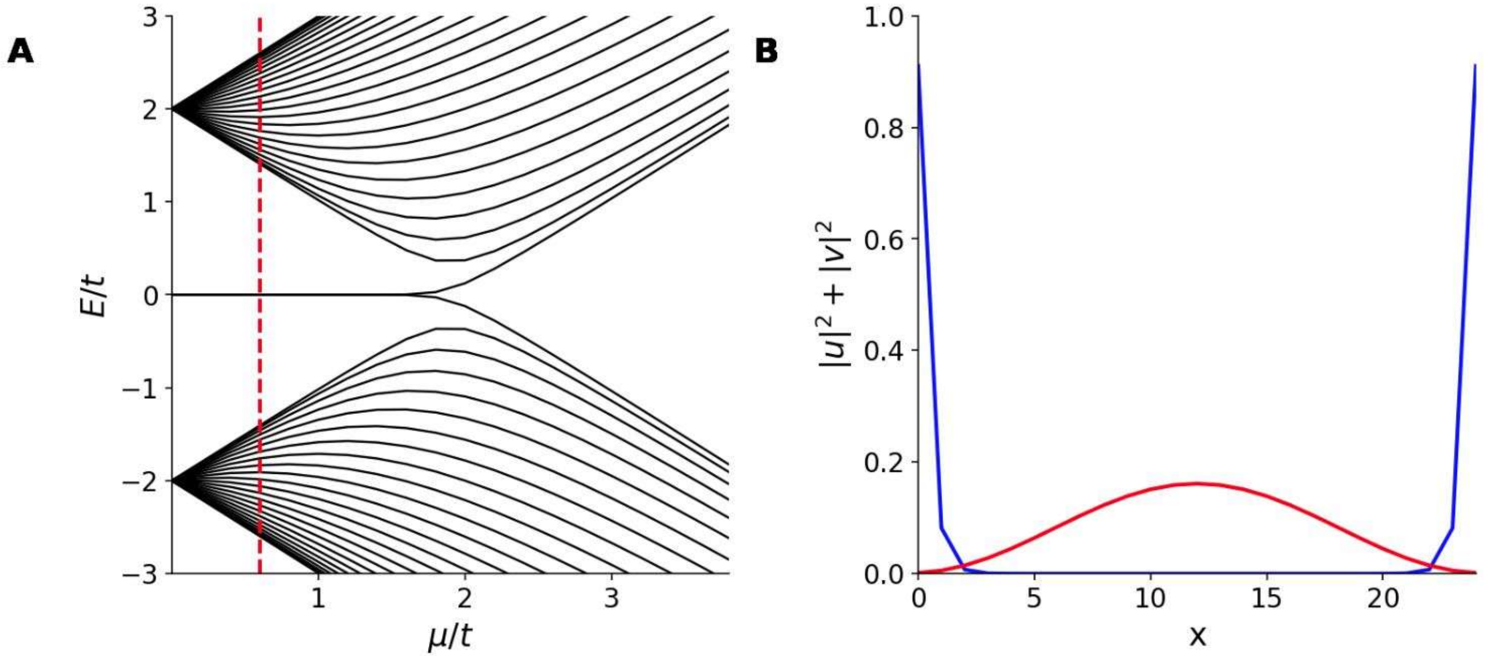


Majorana zero mode

- For a chain with periodic BC

$$H = \frac{1}{2}\sum_k (\gamma_k^\dagger, \gamma_{-k}) \begin{pmatrix} E(k) & 0 \\ 0 & -E(k) \end{pmatrix} \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix}$$
$$= \sum_k E(k) \gamma_k^\dagger \gamma_k.$$

- For a chain with open ends $H = \sum_{\alpha=1}^{N} E_\alpha \gamma_\alpha^+ \gamma_\alpha$



https://topocondmat.org/w1_topointro/1D.html

$$\gamma_\alpha = \sum_{j=1}^{N} \left(u^*_{\alpha j} c_j + v^*_{\alpha j} c_j^\dagger \right)$$

For the zero mode, $\mathbf{u}_0 = \frac{1}{2}(1, 0, \cdots, 0, 1)$

$$\mathbf{v}_0 = \frac{1}{2}(1, 0, \cdots, 0, -1)$$

$$\begin{aligned} \Rightarrow \quad \gamma_1 &= \frac{1}{2}(c_1 + c_N) + \frac{1}{2}(c_1^\dagger - c_N^\dagger) \\ &= \frac{1}{2}(a_1 + i a_{2N}). \end{aligned}$$

a highly nonlocal fermion

It annihilates the ground state

$$\gamma_1 |\Psi_0\rangle = 0$$

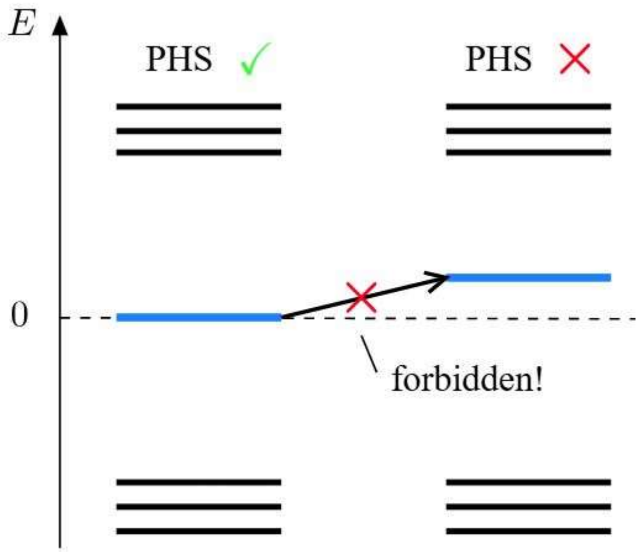
- Degenerate ground state

$|\Psi_0\rangle$ or $\gamma_1^\dagger |\Psi_0\rangle$ (or $|0_+\rangle, |0_-\rangle$)

both have zero energy

- Fermion occupation, $$\begin{aligned} \gamma_1^\dagger \gamma_1 |0_+\rangle &= 0|0_+\rangle, \\ \gamma_1^\dagger \gamma_1 |0_-\rangle &= 1|0_-\rangle. \end{aligned}$$

Stability of Majorana fermions



Fermion parity of the ground state (even/odd-ness of the number of electrons)

the topological ground states of the Kitaev chain is two-fold degenerate, and can be distinguished by fermion parity,

- Fermion parity of site-j with fermion number n_j is defined as

$$(-1)^{n_j} = \begin{cases} +1 \text{ if } n_j = 0, \\ -1 \text{ if } n_j = 1. \end{cases}$$

operator

$n_j = c_j^\dagger c_j$ ➡

$$\begin{aligned}(-1)^{n_j} &= e^{i\pi n_j} \quad (n_j^2 = n_j) \\ &= 1 - 2n_j \\ &= -ia_{2j-1}a_{2j}.\end{aligned}$$

- Fermion parity for the system

$$\begin{aligned}P_F &= \prod_{j=1}^{N}(1 - 2c_j^\dagger c_j) \\ &= \prod_{j=1}^{N}(-ia_{2j-1}a_{2j}), \quad P_F^2 = 1.\end{aligned}$$

$\rightarrow P_F = \pm 1$

- For the trivial ground state

Let $\Delta_0 = t = 0, \mu < 0$

$$\Rightarrow \quad H = |\mu| \sum_{j=1}^{N} c_j^\dagger c_j$$

Ground state

$$c_j |\Psi_0\rangle = 0 \quad (j=1,2,\ldots,N)$$

$$\Rightarrow \quad P_F |\Psi_0\rangle = \prod_{j=1}^{N} (1 - 2c_j^\dagger c_j)|\Psi_0\rangle = |\Psi_0\rangle.$$

- For the non-trivial ground state

Let $\Delta_0 = t, \mu = 0$

$$\Rightarrow \quad H = \frac{i}{2} \sum_{j=1}^{N-1} t a_{2j} a_{2j+1}$$

define $$d_j = \frac{1}{2}(a_{2j} + i a_{2j+1}),$$

then $$d_j^\dagger = \frac{1}{2}(a_{2j} - i a_{2j+1}).$$

$$i a_{2j} a_{2j+1} = 2 d_j^\dagger d_j - 1$$

$$\Rightarrow \quad H = t \sum_{j=1}^{N-1} \left(d_j^\dagger d_j - \frac{1}{2} \right)$$

$$P_F = -i a_1 \prod_{j=1}^{N-1} (-i a_{2j} a_{2j+1}) a_{2N} = -i a_1 a_{2N} \prod_{j=1}^{N-1} (1 - d_j^\dagger d_j).$$

$$d_j |\Psi_0\rangle = 0 \quad \Rightarrow \quad \underline{P_F |\Psi_0\rangle = (-i a_1 a_{2N}) |\Psi_0\rangle}$$

- For the non-trivial ground state (cont'd)

$$
\begin{aligned}
\gamma_1 &= \frac{1}{2}(a_1 + i a_{2N}), \\
\gamma_1^{+} &= \frac{1}{2}(a_1 - i a_{2N}).
\end{aligned}
$$

➡ $$-i a_1 a_{2N} = 1 - 2\gamma_1^{\dagger}\gamma_1$$

➡ $$
\begin{aligned}
P_F|0_\pm\rangle &= (-i a_1 a_{2N}|0_\pm\rangle \\
&= (1 - 2\gamma_1^{\dagger}\gamma_1)|0_\pm\rangle \\
&= \pm|0_\pm\rangle.
\end{aligned}
$$

The ground states is a 2-state system that can store 1 qubit of information. Such a qubit is robust against local perturbation since the Majorana fermions store info nonlocally.