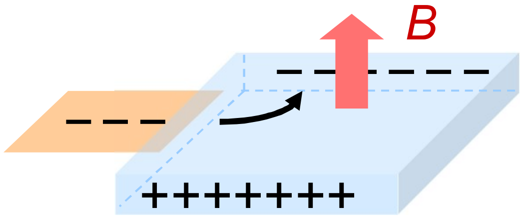
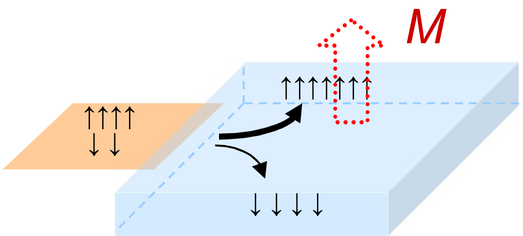
• classical Hall effect (E. Hall 1879)



✓ Lorentz force

• anomalous Hall effect (E. Hall, 1881)



Ingredients:

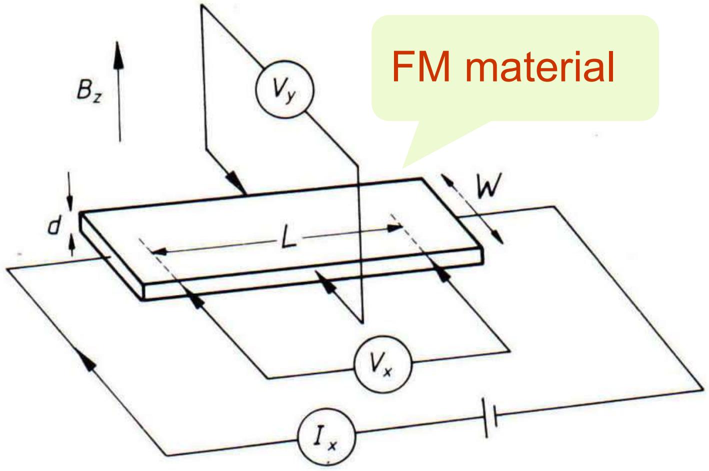
• magnetization (majority spin)

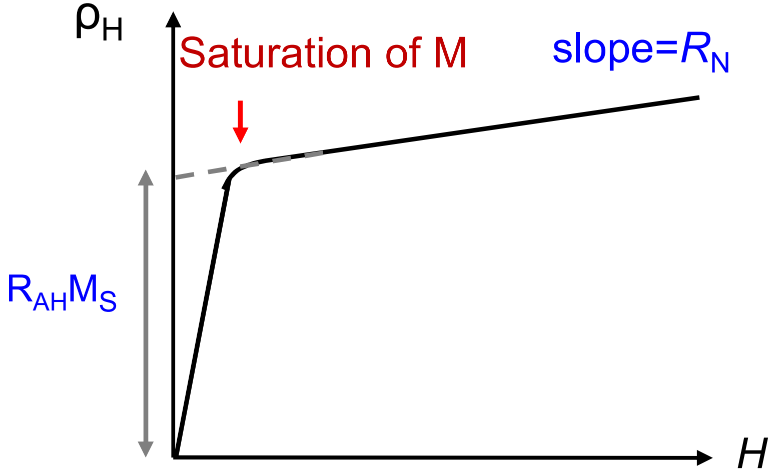
• spin-orbit coupling

(to couple the *majority-spin* direction to transverse motion)

Note: An example that requires no magnetization is provided by Haldane's graphene model

Anomalous Hall effect
(Hall 1880)





Recall that $\rho_H = B/ne$ for free electron gas

The usual term

$$\rho_H = R_N H + \rho_{AH}(H),$$

Anomalous term

$$\rho_{AH}(H) \equiv R_{AH} M(H)$$

Theory: Intrinsic mechanism (ideal lattice without impurity)

PHYSICAL REVIEW VOLUME 95, NUMBER 5 SEPTEMBER 1, 1954

Hall Effect in Ferromagnetics*

ROBERT KARPLUS,† *Department of Physics, University of California, Berkeley, California*

AND

J. M. LUTTINGER, *Department of Physics, University of Michigan, Ann Arbor, Michigan*

(Received May 21, 1954)

Both the unusually large magnitude and strong temperature dependence of the extraordinary Hall effect in ferromagnetic materials can be understood as effects of the spin-orbit interaction of polarized conduction electrons. It is shown that the interband matrix elements of the applied electric potential energy combine with the spin-orbit perturbation to give a current perpendicular to both the field and the magnetization. Since the net effect of the spin-orbit interaction is proportional to the extent to which the electron spins are aligned, this current is proportional to the magnetization. The magnitude of the Hall constant is equal to the square of the ordinary resistivity multiplied by functions that are not very sensitive to temperature and impurity content. The experimental results behave in such a way also.

- Linear response theory, with

  Spin-orbit coupling

  magnetization

- They find a transverse electron velocity (aka anomalous velocity) that depends only on band structure
- Gives correct order of magnitude of ρ_H for Fe

  also explains $\rho_{AH} \propto \rho_L^2$

  that's observed in some data

Old wine in new bottle

Karplus-Luttinger theory (1954)

= Berry curvature theory (2001)

Anomalous velocity is essentially this term

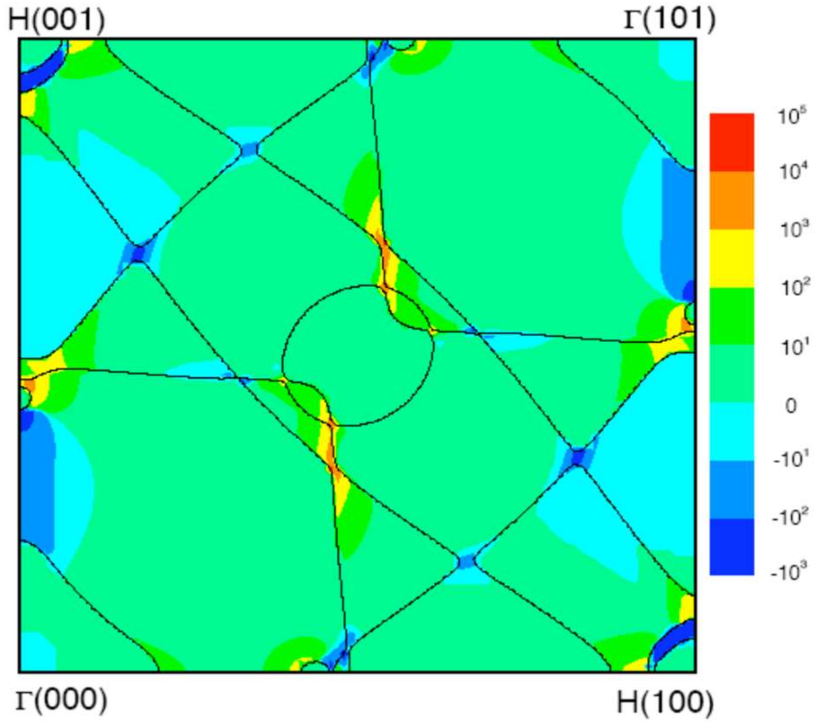
$$\mathbf{v}_n(\mathbf{k}) = \frac{1}{\hbar}\frac{\partial \varepsilon_{n\mathbf{k}}}{\partial \mathbf{k}} + \frac{e}{\hbar}\mathbf{E} \times \mathbf{F}_n(\mathbf{k})$$

→ AHE

$$\sigma_{xy} = \frac{e^2}{\hbar}\int_{filled} \frac{d^3k}{(2\pi)^3}\, F_z(\vec{k})$$

Berry curvature of fcc Fe

(Yao et al, PRL 2004)



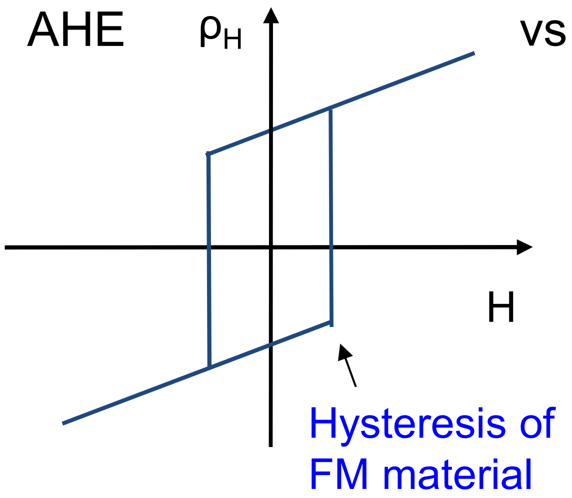
Ideal lattice without impurity

With this mechanism working, it’s possible to have Quantum AHE (in 2D)

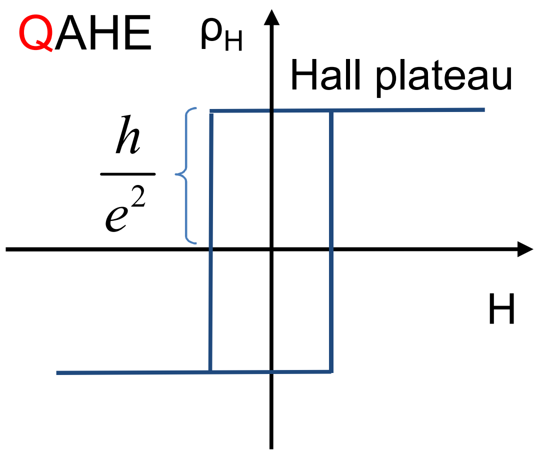
Experimental Observation of the Quantum Anomalous Hall Effect in a Magnetic Topological Insulator

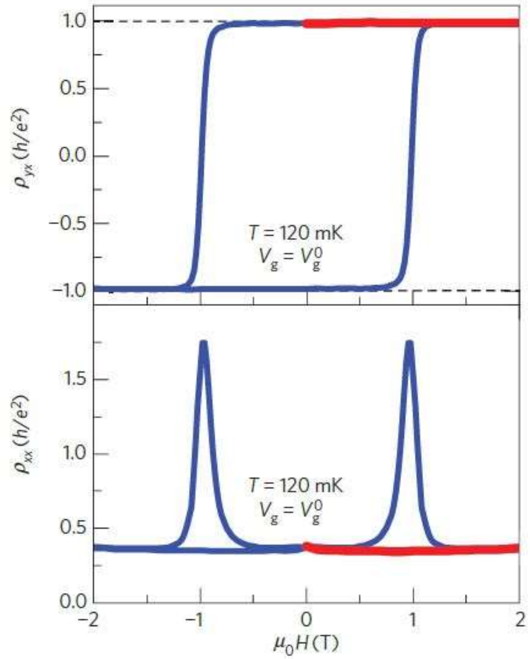
Cui-Zu Chang,[1,2*] Jinsong Zhang,[1*] Xiao Feng,[1,2*] Jie Shen,[2*] Zuocheng Zhang,[1] Minghua Guo,[1] Kang Li,[2] Yunbo Ou,[2] Pang Wei,[2] Li-Li Wang,[2] Zhong-Qing Ji,[2] Yang Feng,[1] Shuaihua Ji,[1] Xi Chen,[1] Jinfeng Jia,[1] Xi Dai,[2] Zhong Fang,[2] Shou-Cheng Zhang,[3] Ke He,[2†] Yayu Wang,[1†] Li Lu,[2] Xu-Cun Ma,[2] Qi-Kun Xue[1†]

Science, 2013



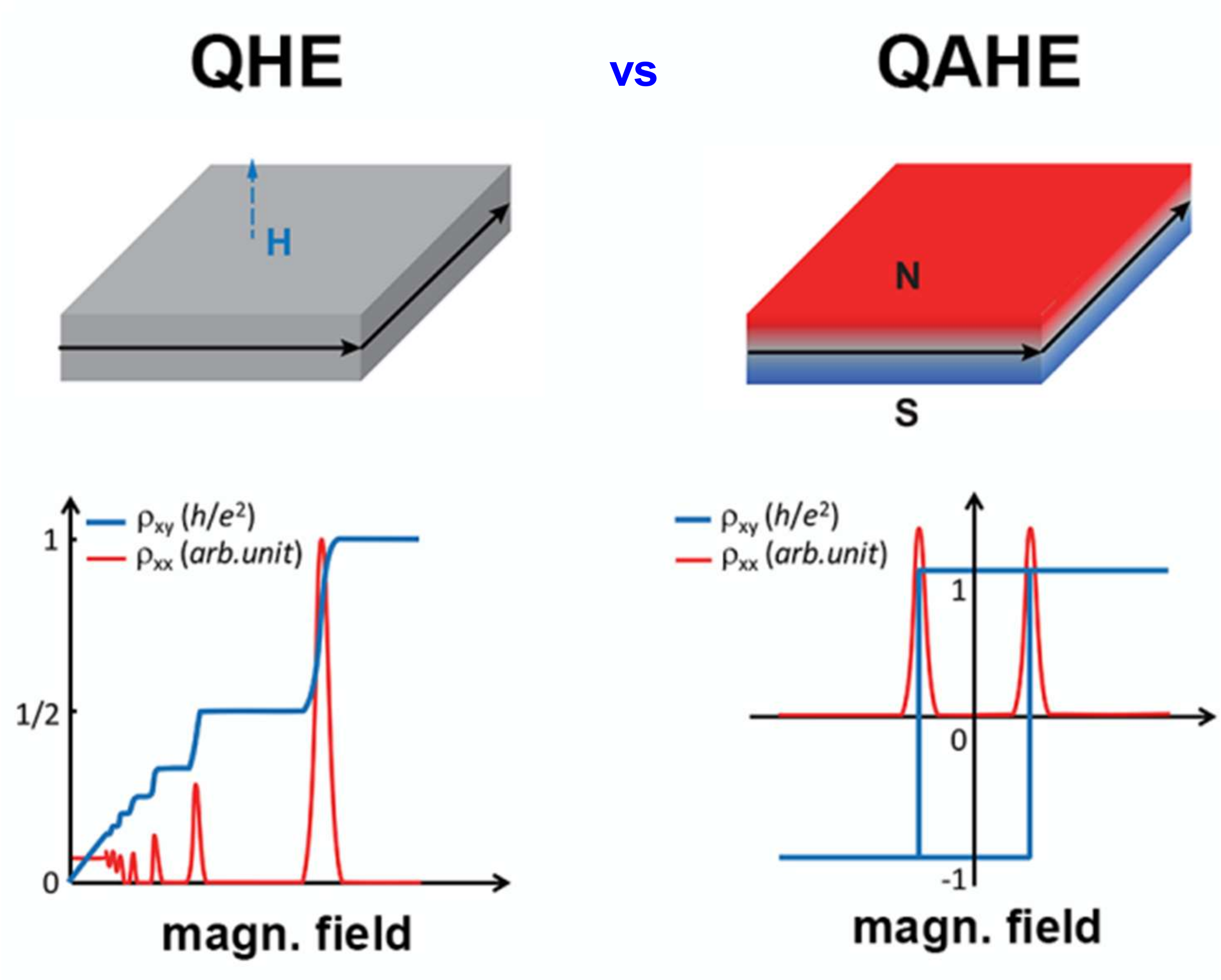
vs





C.Z. Chang et al, Nat Phys 2015

(accurate to 10^{-4})


QHE
vs
QAHE
H
N
S
ρxy (h/e2)
ρxx (arb.unit)
1
1/2
0
-1
magn. field
magn. field

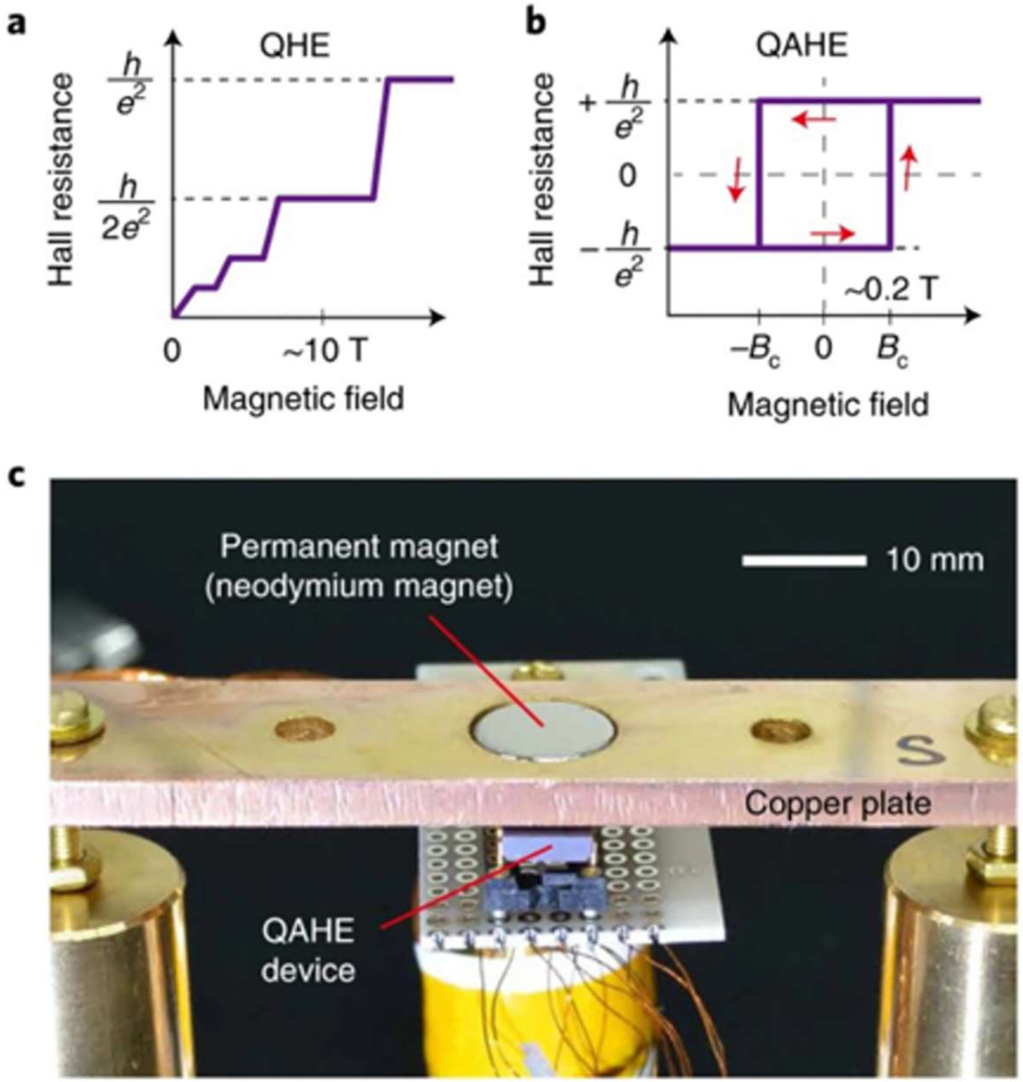
Reports on Quantum anomalous Hall effect

- Bi_2Te_3 theory (Yu et al, Science 2010)
- Bi_2Te_3 experiment (Chang et al, Nat Material 2015)
- manganese bismuth telluride ($MnBi_2Te_4$) (Deng et al, Science 2020)
- Twisted bilayer graphene (Serlin et al, Science 2020) **Orbital magnetism**
- $MoTe_2/WSe_2$ heterobilayers (Li et al, Nature 2021)
- $Cr_{1-x}(Bi_{1-y}Sb_y)_{2-x}Te_3$ (Okazaki et al, Nat Phys 2022)
- Twisted Bilayer $MoTe_2$ (Cai et al, Nature 2023)

…

Nature Phys, Okazaki et al 2022

QAHE with a permanent magnet defines a quantum resistance standard



a precision of 10 parts per billion (at mK)

Quantum anomalous Hall effect

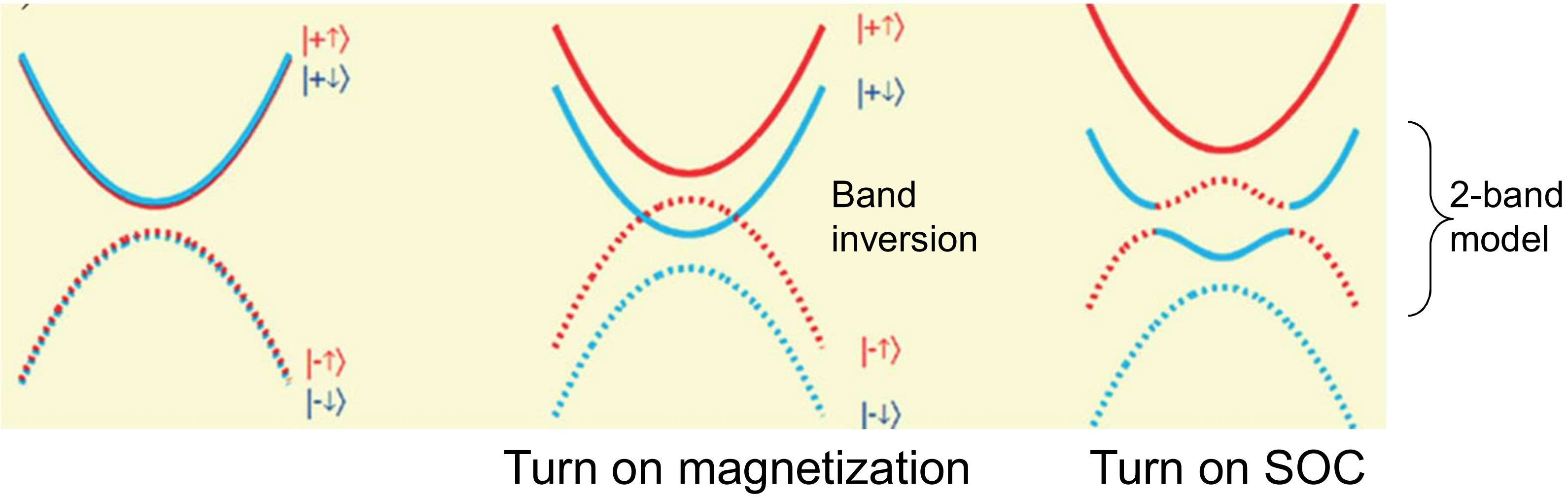
A. Qi-Wu-Zhang model

B. Edge state in Qi-Wu-Zhang model

Engineering a topological band by level crossing

(Qi-Wu-Zhang model, 2006; Yu et al, Science 2010)

Using 2D surface states of magnetic topological insulator



A. Qi-Wu-Zhang model - a toy model of QAHE (2006)

Can be realized using ultracold fermions, see Liang et al, Phys Rev Res 2023

$$\mathsf{H}(\mathbf{k}) = \mathsf{H}_0 + \mathsf{H}_m + \mathsf{H}_{so}, \tag{1.1}$$

$$\mathsf{H}_0 = \varepsilon_0(\mathbf{k}) +$$

$$t \begin{pmatrix} 2 - \cos k_x a - \cos k_y a & 0 \\ 0 & -(2 - \cos k_x a - \cos k_y a) \end{pmatrix}, \quad \begin{matrix} \uparrow \\ \downarrow \end{matrix}$$

$$\mathsf{H}_m = m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathsf{H}_{so} = \lambda \begin{pmatrix} 0 & \sin k_x a - i \sin k_y a \\ \sin k_x a + i \sin k_y a & 0 \end{pmatrix}.$$

$$\mathsf{H}(\mathbf{k}) = \varepsilon_0(\mathbf{k}) + \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}, \tag{1.2}$$

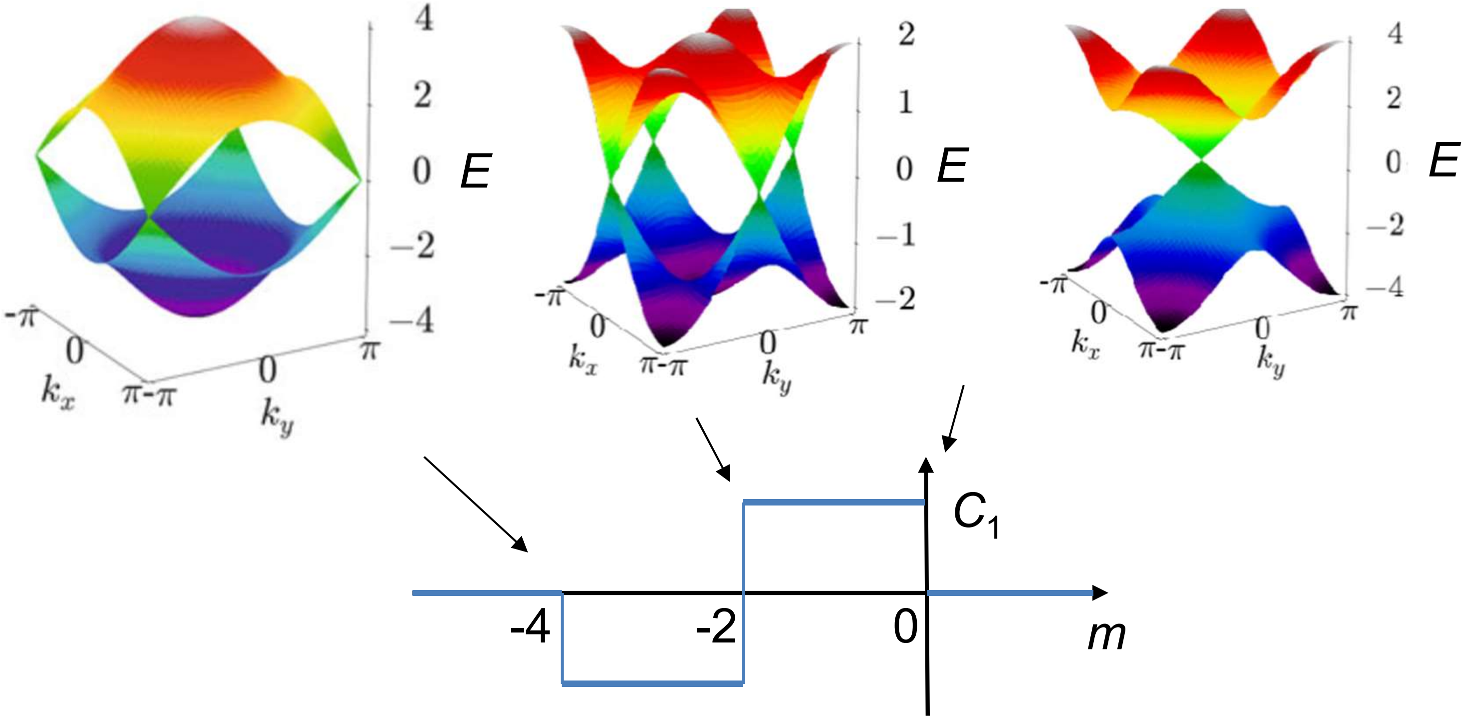
where

$$\mathbf{h}(\mathbf{k}) = \left(\lambda \sin k_x a, \lambda \sin k_y a, m + t \sum_{j=1}^{2} (1 - \cos k_j a) \right).$$

$$\varepsilon_\pm(\mathbf{k}) = \varepsilon_0(\mathbf{k}) \pm |\mathbf{h}(\mathbf{k})|$$

Band gap could close at

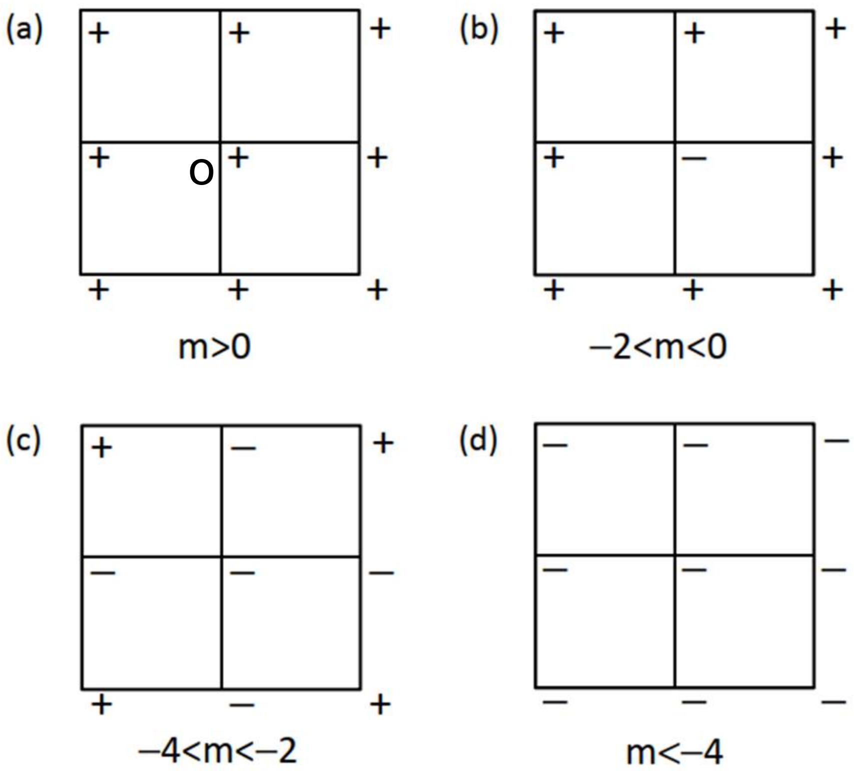
$$\begin{cases} \mathbf{k}_0 = 0 \rightarrow \varepsilon_\pm(\mathbf{k}_0) = \varepsilon_0 \pm m, \\ \mathbf{k}_0 = (\pi, 0), (0, \pi) \rightarrow \varepsilon_\pm(\mathbf{k}_0) = \varepsilon_0 \pm |m+2|, \\ \mathbf{k}_0 = (\pi, \pi) \rightarrow \varepsilon_\pm(\mathbf{k}_0) = \varepsilon_0 \pm |m+4|. \end{cases}$$



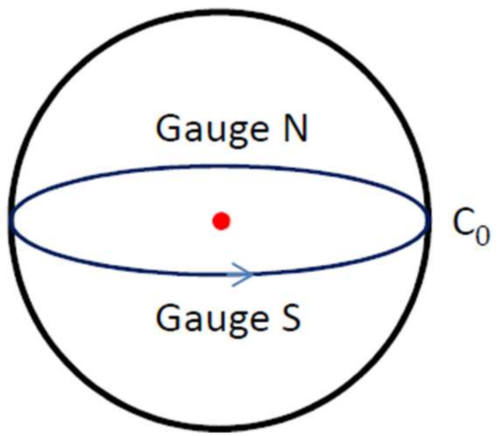
Figs. From Asboth et al, A short course on TI

Hint of topology from the distribution of of $h_z(\boldsymbol{k})$:

1) $m > 0 : h_z(\mathbf{k}) > 0$ over the whole BZ.
2) $-2 < m < 0 : h_z(\mathbf{k}) < 0$ near $\mathbf{k} = 0$.
3) $-4 < m < -2 : h_z(\mathbf{k}) > 0$ near $\mathbf{k} = (\pi, \pi)$ (and its equivalent points).
4) $m < -4 : H_z(\mathbf{k}) < 0$ over the whole BZ.



$$\mathrm{H}(\mathbf{k}) = \varepsilon_0(\mathbf{k}) + \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma},$$



(a),(d) would be trivial
(b),(c) may be topological

Berry curvature of 2-band model

$$F_z^{\pm}(\mathbf{k}) = \mp\frac{1}{2h^3}\mathbf{h}\cdot\frac{\partial\mathbf{h}}{\partial k_x}\times\frac{\partial\mathbf{h}}{\partial k_y}.$$

Pf: $H\left(\vec{k}\right)\left|\vec{h},\pm\right\rangle = \varepsilon_{\pm}|\vec{h},\pm\rangle$

Berry connection

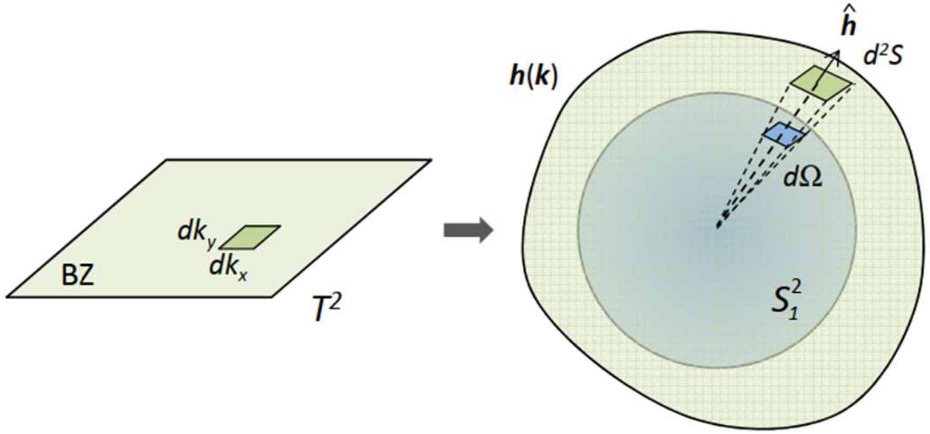
$$\begin{aligned} A_\ell^{\pm}(\mathbf{k}) &= i\langle\mathbf{h},\pm|\frac{\partial}{\partial k_\ell}|\mathbf{h},\pm\rangle \\ &= \frac{\partial h_\alpha}{\partial k_\ell} i\langle\mathbf{h},\pm|\frac{\partial}{\partial h_\alpha}|\mathbf{h},\pm\rangle \\ &= \frac{\partial h_\alpha}{\partial k_\ell} a_\alpha^{\pm}(\mathbf{h}), \end{aligned}$$

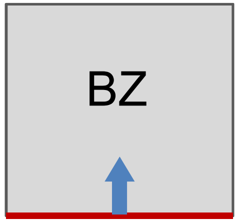
$$a_\alpha^{\pm}\left(\vec{h}\right) = i\langle\vec{h},\pm\left|\frac{\partial}{\partial h_\alpha}\right|\vec{h},\pm\rangle$$

$$\begin{aligned} F_z^{\pm}(\mathbf{k}) &= \frac{\partial A_y^{\pm}}{\partial k_x} - \frac{\partial A_x^{\pm}}{\partial k_y} \\ &= \frac{\partial}{\partial k_x}\left(\frac{\partial h_\beta}{\partial k_y}a_\beta^{\pm}\right) - \frac{\partial}{\partial k_y}\left(\frac{\partial h_\alpha}{\partial k_x}a_\alpha^{\pm}\right) \\ &= \frac{\partial h_\alpha}{\partial k_x}\frac{\partial h_\beta}{\partial k_y}\left(\frac{\partial a_\beta^{\pm}}{\partial h_\alpha} - \frac{\partial a_\alpha^{\pm}}{\partial h_\beta}\right) \\ &= \frac{\partial h_\alpha}{\partial k_x}\frac{\partial h_\beta}{\partial k_y}\varepsilon_{\alpha\beta\gamma}f_\gamma^{\pm} \\ &= \mp\frac{1}{2h^3}\mathbf{h}\cdot\frac{\partial\mathbf{h}}{\partial k_x}\times\frac{\partial\mathbf{h}}{\partial k_y}, \end{aligned}$$

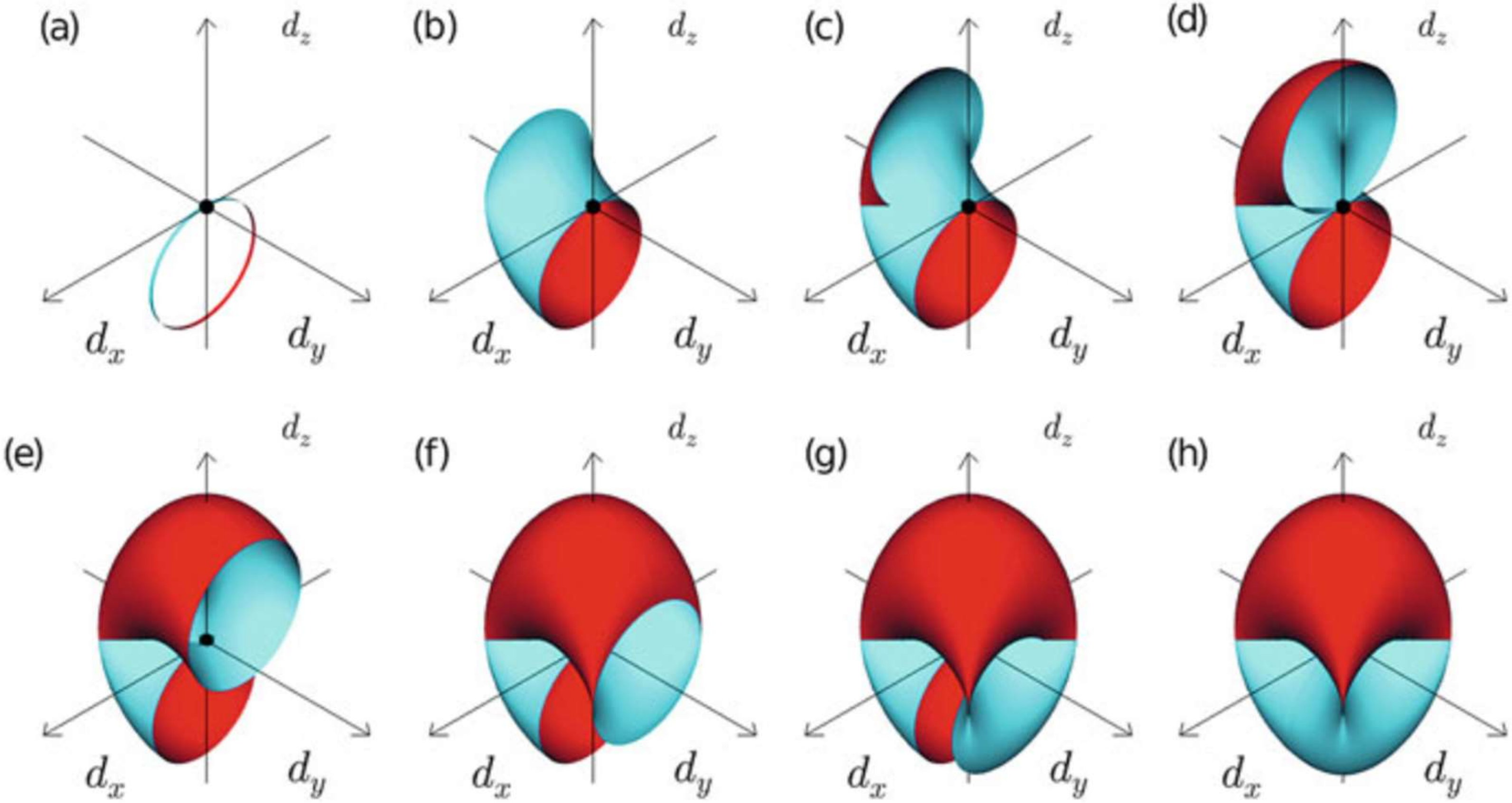
Ch 2 $\quad \mathbf{F}_{\pm}(\mathbf{B}) = \nabla_{\mathbf{B}}\times\mathbf{A}_{\pm}(\mathbf{B}) = \mp\frac{1}{2}\frac{\hat{B}}{B^2}$

$$\Rightarrow \quad f_\gamma^{\pm} = \mp h_\gamma/2h^3$$





$\underline{m = -2}$



Figs. From Asboth et al, A short course on TI

Hall conductivity from the valence band

$$\sigma_H = \frac{e^2}{h}\frac{1}{2\pi}\int_{BZ} d^2k F_z^-(\mathbf{k}) \quad \text{(for filled valence and)}$$
$$= \frac{e^2}{h}\frac{1}{4\pi}\int_{BZ} d^2k \frac{1}{h^3}\mathbf{h}\cdot\frac{\partial \mathbf{h}}{\partial k_x}\times\frac{\partial \mathbf{h}}{\partial k_y}$$

$$\Rightarrow \quad \sigma_H = w\frac{e^2}{h}, w \in Z.$$

QAHE

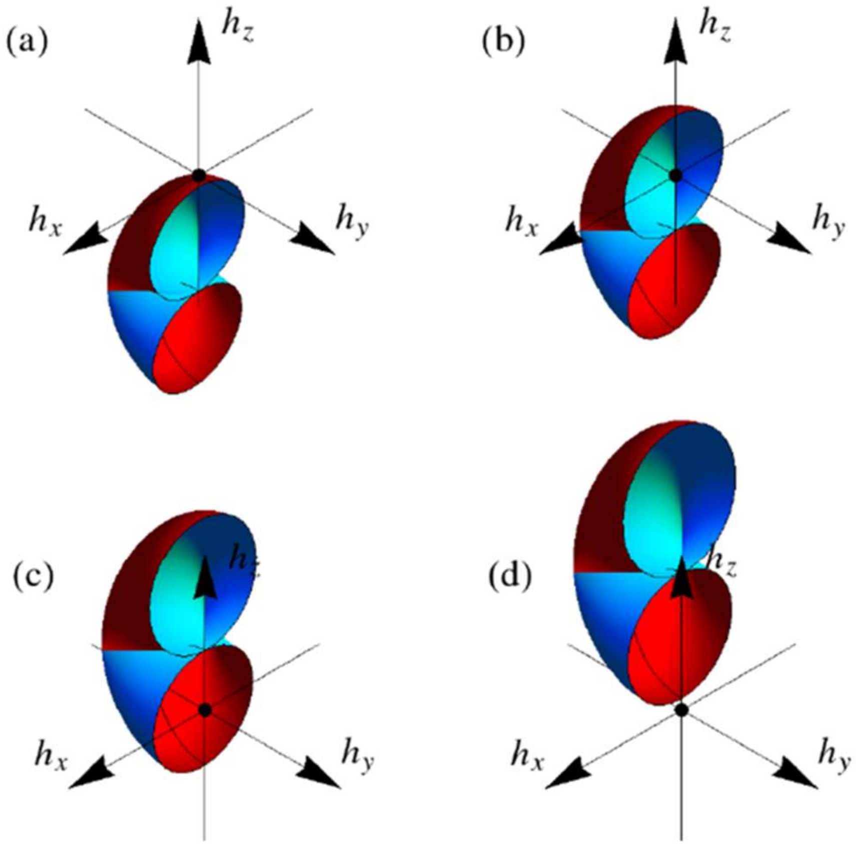
wrapping number of the ***h***-surface around the origin

m > 0

-2 < m < 0

-4 < m < -2

m < -4



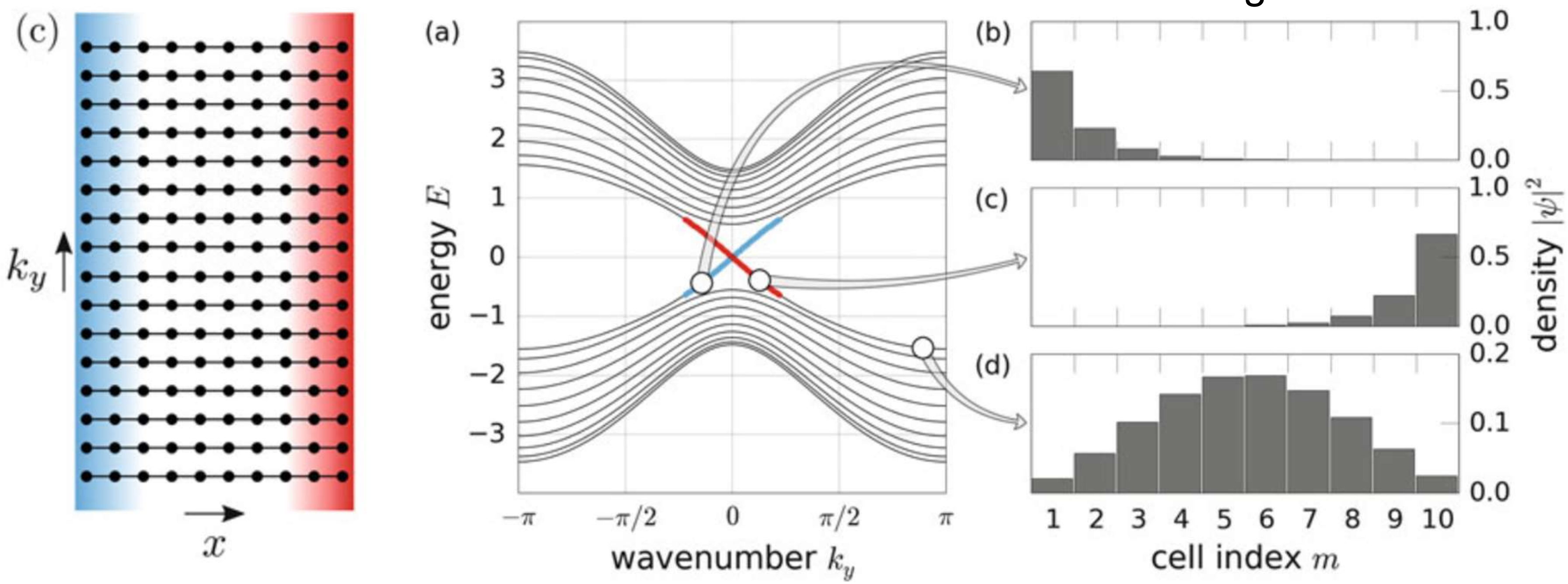
Figs. From Asboth et al, A short course on TI

Bulk-edge correspondence

B. Edge state in Qi-Wu-Zhang model

Numerical calculation based on lattice QWZ model

Chiral edge states



Figs. From Asboth et al,
A short course on TI

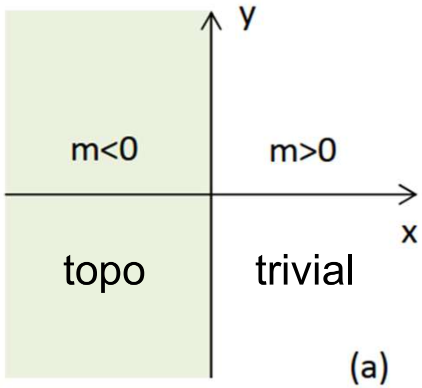
Low-energy continuum theory

$$\mathsf{H}(\mathbf{k}) = \mathsf{H}_0 + \mathsf{H}_m + \mathsf{H}_{so}, \qquad (1.1)$$

$$\mathsf{H}_0 = \varepsilon_0(\mathbf{k}) + t\begin{pmatrix} 2-\cos k_x a - \cos k_y a & 0 \\ 0 & -(2-\cos k_x a - \cos k_y a) \end{pmatrix},$$

$$\mathsf{H}_m = m\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathsf{H}_{so} = \lambda\begin{pmatrix} 0 & \sin k_x a - i\sin k_y a \\ \sin k_x a + i\sin k_y a & 0 \end{pmatrix}.$$



$$m(x)\begin{cases} >0 \text{ for } x>0 \\ <0 \text{ for } x<0 \end{cases}$$

sssume m(x) varies smoothly

Small k limit:

$$\mathsf{H}(\mathbf{k}) = \varepsilon_0 + \begin{pmatrix} m & \lambda(k_x - ik_y) \\ \lambda(k_x + ik_y) & -m \end{pmatrix} + O(k^2).$$

Re-quantize,

$$\Rightarrow \mathsf{H}(\mathbf{p}) = \varepsilon_0 + \begin{pmatrix} m(x) & \lambda\left(\frac{1}{i}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \\ \lambda\left(\frac{1}{i}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) & -m(x) \end{pmatrix}$$

$$H(\mathbf{p})\psi(x,y) = \varepsilon\psi(x,y)$$

$$\psi(x,y) = \phi_1(x)\phi_2(y) \qquad \phi_2(y) = e^{ik_y y}$$

$$\begin{pmatrix} m(x) & \frac{\lambda}{i}\left(\frac{\partial}{\partial x} + k_y\right) \\ \frac{\lambda}{i}\left(\frac{\partial}{\partial x} - k_y\right) & -m(x) \end{pmatrix} \phi_1(x) = \varepsilon_e(k_y)\phi_1(x)$$

$$\phi_1(x) = e^{-\frac{1}{\lambda}\int_0^x dx' m(x')} \begin{pmatrix} a \\ b \end{pmatrix}$$

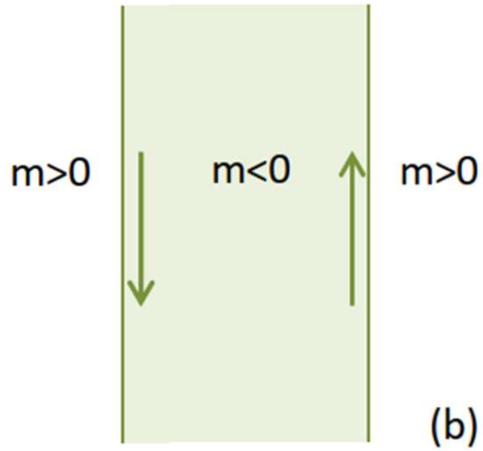
It can be verified as an eigenstate with eigenvalue $\varepsilon_e(k_y) = \lambda k_y$ if $(a,b) = (1,i)$.

$$\phi_1(x) = e^{-\frac{1}{\lambda}\int_0^x dx' m(x')} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Or, $m(x) \begin{cases} > 0 \text{ for } x < 0 \\ < 0 \text{ for } x > 0 \end{cases}$

$$\phi_1(x) = e^{\frac{1}{\lambda}\int_0^x dx' m(x')} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

and $\varepsilon_e(k_y) = -\lambda k_y$



Chiral edge states