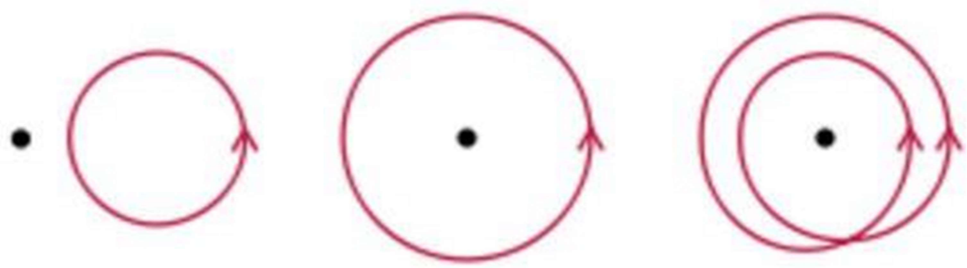
Basics of Topology

- Winding number
- Euler characteristics
- Intrinsic and extrinsic curvatures
- Parallel transport
- Gauss-Bonnet theorem
- Hopf-Poincare theorem

Topology: the property of an object that is invariant under *continuous deformation*

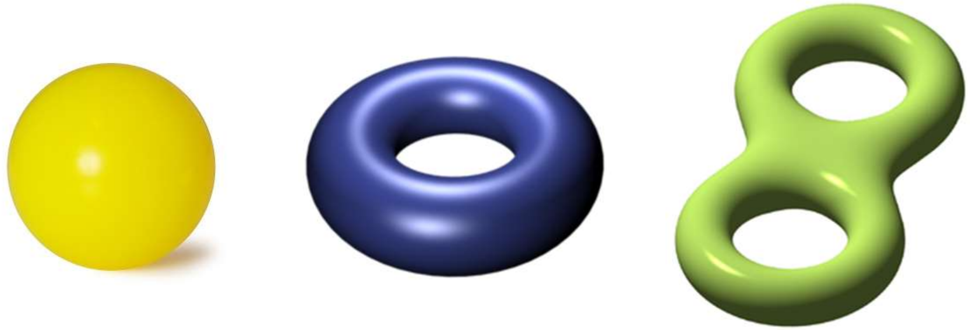
Examples of
Topological invariant:

- winding number



0 1 2

- Euler characteristics
  (number of holes)



- Chern number . . .

If a physics system has topological property, then it is stable against small change of physical condition.

Topology in vector field (Fluid flow, EM field …)

Patterns of flow near a "zero" (a nodal point) can be related to the winding number

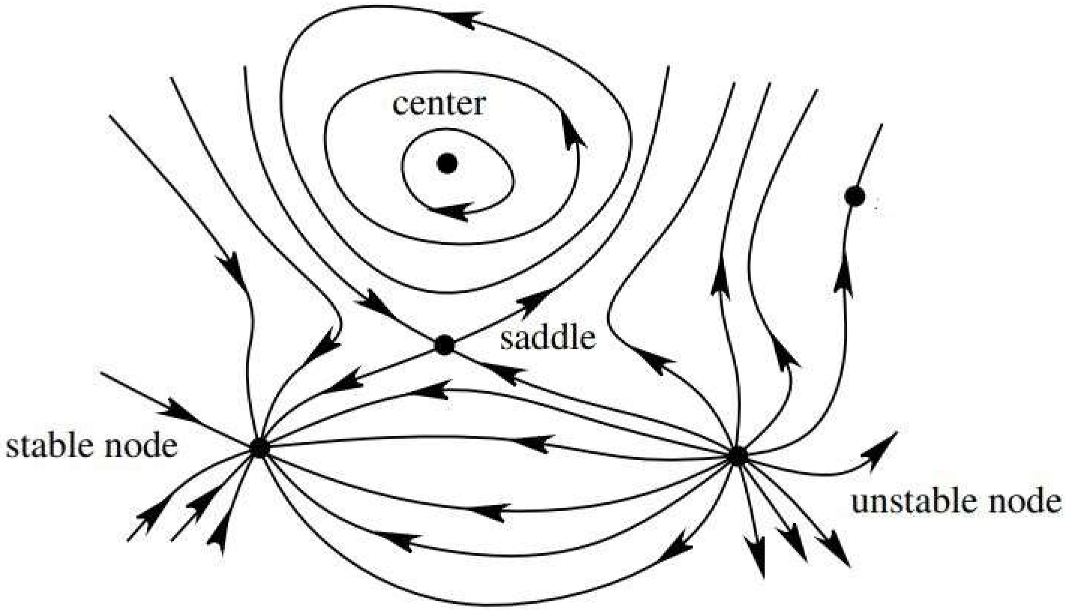
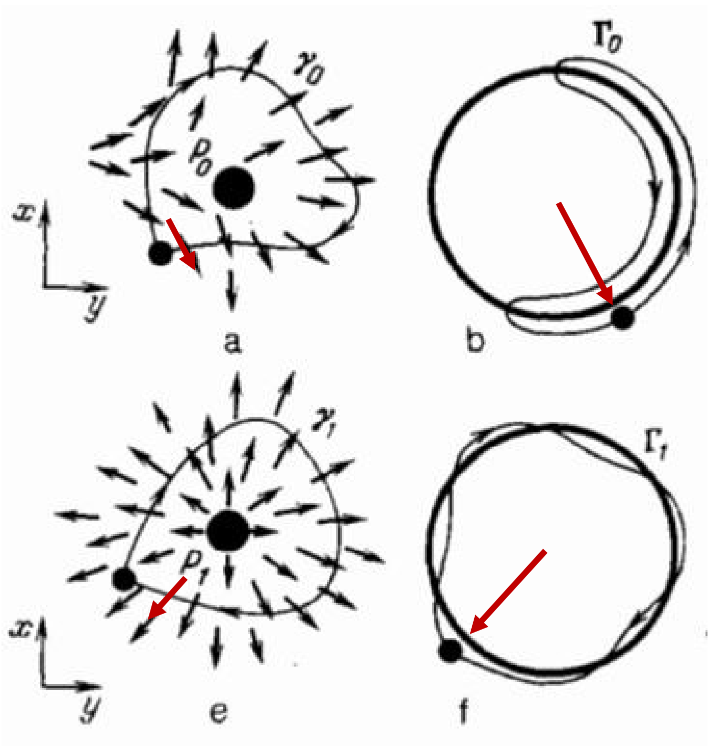


Fig from Belyaev's article

Consider a map from a closed path to the circle depicted by the direction of vectors $f : S^1 \to S^1$



Winding number

w=0

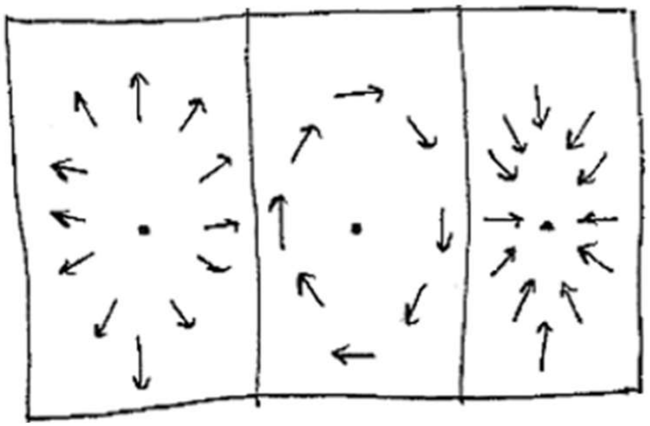
w=1

Fig from Kurik and Lavrentovich, 1988

Example:

Source,

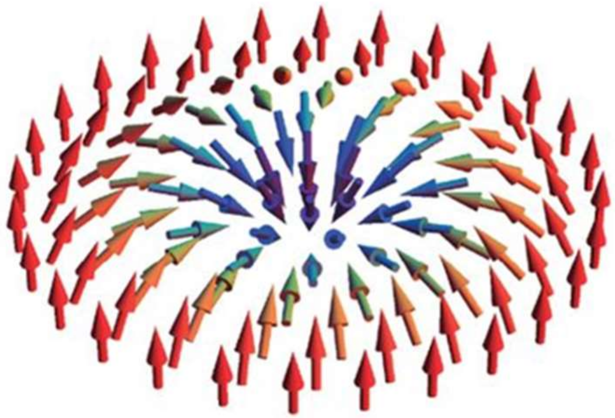
vortex, drain



They all have w=1 and are continuously deformable to each other (in 2D).

Topology in vector field

Now the vectors on a plane can point out of the plane

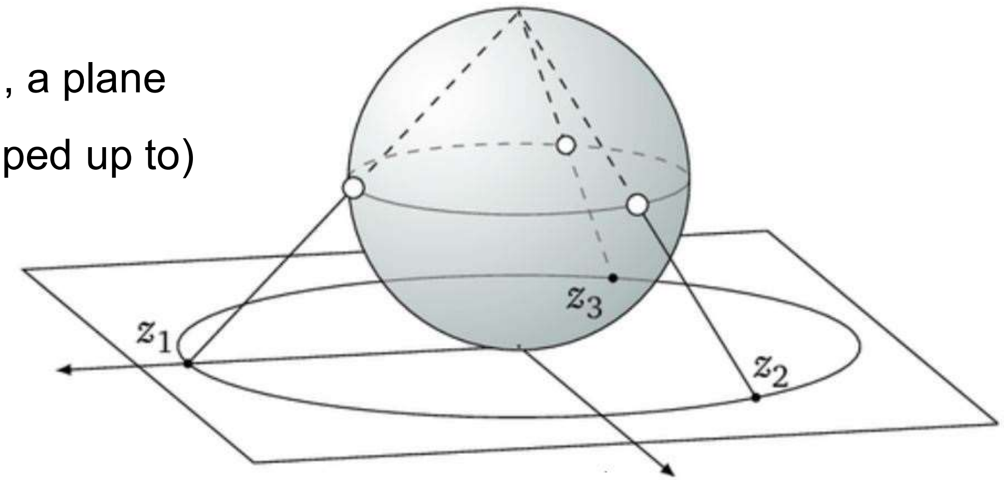


Such a localized texture of vectors is called a skyrmion 史科子

Hypothetical structure of nucleons proposed by Skyrme, 1962

Fig from Karin and Matthias Sitte, JAP 2014

By stereographic projection, a plane can be identified with (wrapped up to) a sphere



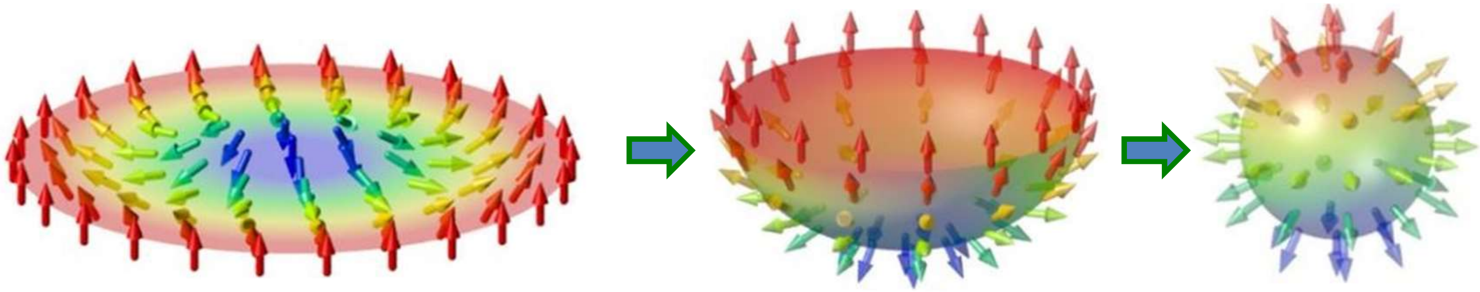
A map from this sphere to the direction of vectors $f : S^2 \to S^2$

Winding number

(details later)

$$w = \frac{1}{4\pi}\int_A \mathbf{m} \cdot \left(\frac{\partial \mathbf{m}}{\partial x} \times \frac{\partial \mathbf{m}}{\partial y} \right) \mathrm{d}A.$$

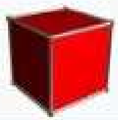
w=1



I. Kézsmárki et al, Nature Material 2015

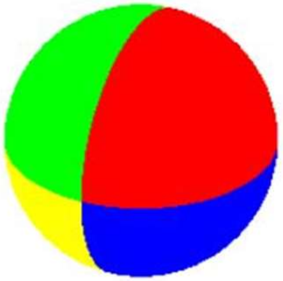
Euler characteristics

Platonic solids, discovered by F. Maurolico (1537) 正多面體

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V-E+F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

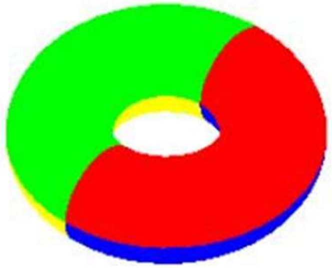
Beyond regular polyhedron, Euler (1758)

sphere



$$\chi = V - E + F = 2 - 4 + 4 = 2$$

torus

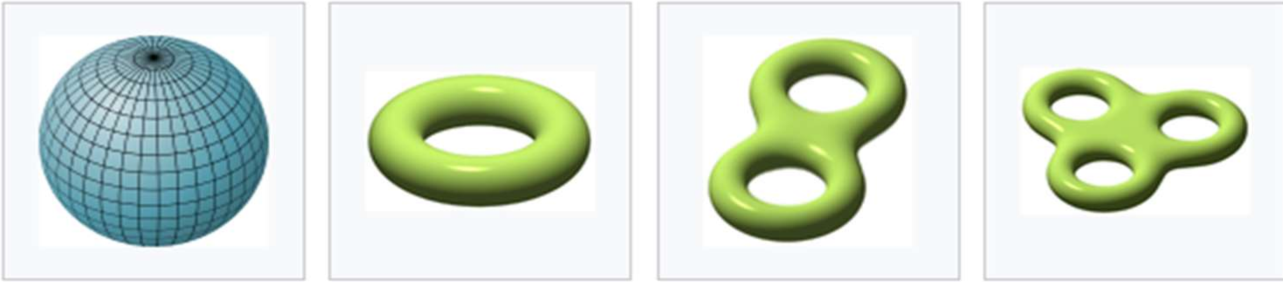


$$\chi = V - E + F = 4 - 8 + 4 = 0$$

- Divide a surface into patches.
- This number χ is independent of the ways of division, so it's a property of the surface itself.
- Furthermore, it does not change under continuous deformation, so it's a topological invariant.

Euler characteristic of a surface $\chi(M) = 2(1-g)$

# of holes



$\chi = 2$ $\chi = 0$ $\chi = -2$ $\chi = -4$

In general, for a surface M with dimension D, we can divide it into a patchwork of cells, and define

$$\chi(M) = \sum_{k=0}^{D} (-1)^k \beta_k, \qquad \text{(B8)}$$

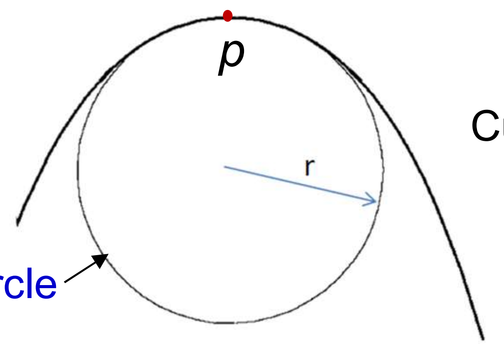
where β_k is the number of k-simplexes. *k*-單體

k=0, 1, 2, 3, … =  , …

For a surface (D=2), $\chi(M) = \beta_0 - \beta_1 + \beta_2$

Curvature of a surface

- First, how do we quantfy the curvature of a line (at point *p*)?

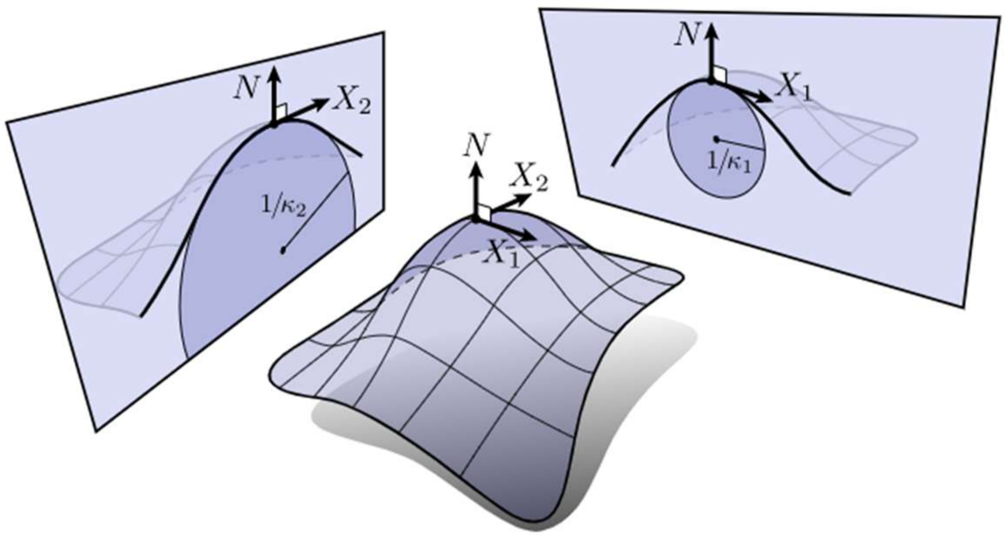


Curvature k at *p*:

$$k = \frac{1}{r}$$

- How do we quantify the curvature of a surface?

One can fit the surface near *p* by a quadratic surface (ellipsoid, paraboloid, hyperboloid)



A quadratic surface must have two principal directions with maximum and minimum radii r_1, r_2. They correspond to two **principle curvatures** $k_1 = 1/r_1, k_2 = 1/r_2$ (up to a sign). 主曲率

Two kinds of curvature

- Mean curvature
  平均曲率

$$H = k_1 + k_2 = \frac{1}{r_1} + \frac{1}{r_2}$$

Extrinsic 外在

- Gaussian curvature

$$G = k_1 k_2 = \frac{1}{r_1 r_2}$$

Intrinsic 内在

- *Without* stretching/squeezing a surface (i.e., the shortest distance between any 2 points remain the same), its G will not change.

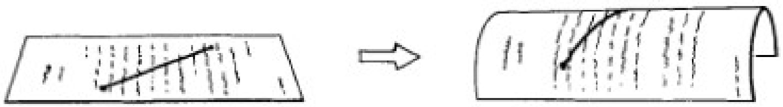
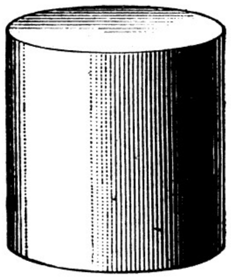


Figure 3.6 Bending a sheet of paper changes its extrinsic—but not its intrinsic—geometry.

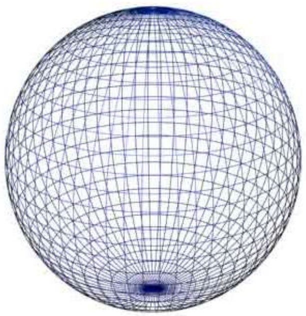
H≠0

G=0



H≠0

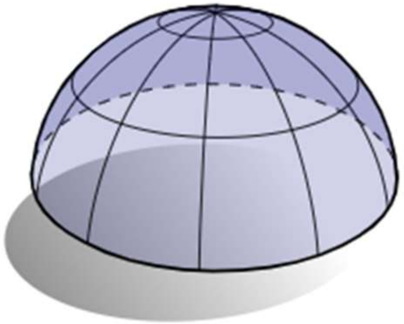
G=0



H≠0

G≠0

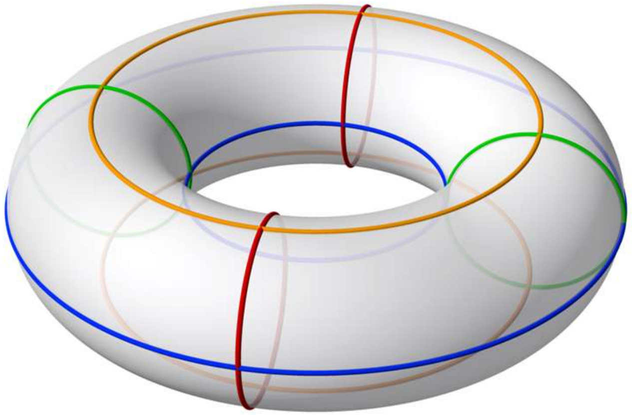
Positive and *negative* Gaussian curvature

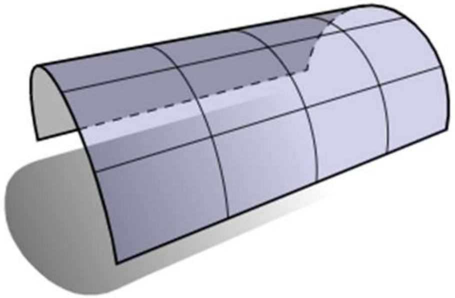


$k_1, k_2 > 0$

G>0

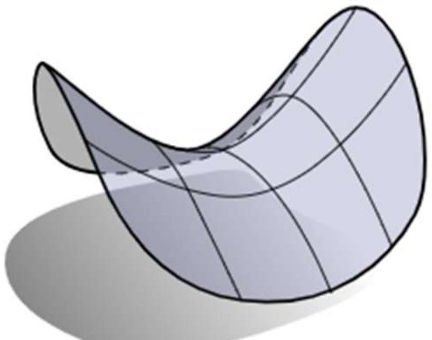
A torus　環面





$k_1 > 0, k_2 = 0$

G=0



$k_1 > 0, k_2 < 0$

G<0

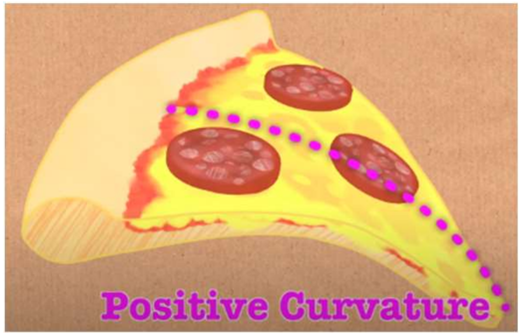
Saddle point

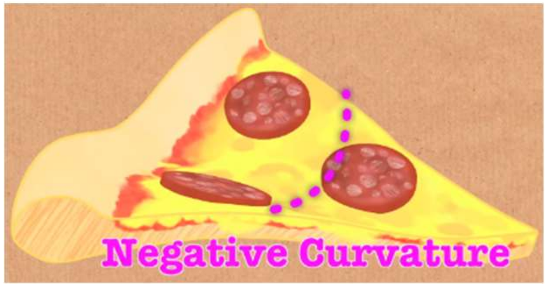
wordpress.discretization.de/geometryprocessingandapplicationsws19/a-quick-and-dirty-introduction-to-the-curvature-of-surfaces/

The Remarkable Way We Eat Pizza -

Youtube: *Numberphile*

- You cannot change Gaussian curvature without stretching/squeezing the surface.
- That is, without stretching your pizza, its G must remain zero, and one of the $k_{1,2}$ must be zero.





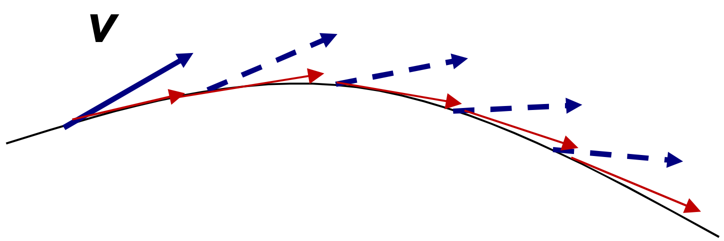
Theorema Egregium (Gauss, 1827)

i.e. the most remarkable theorem

Gaussian curvature can be determined entirely by measuring angles and distances on a surface.

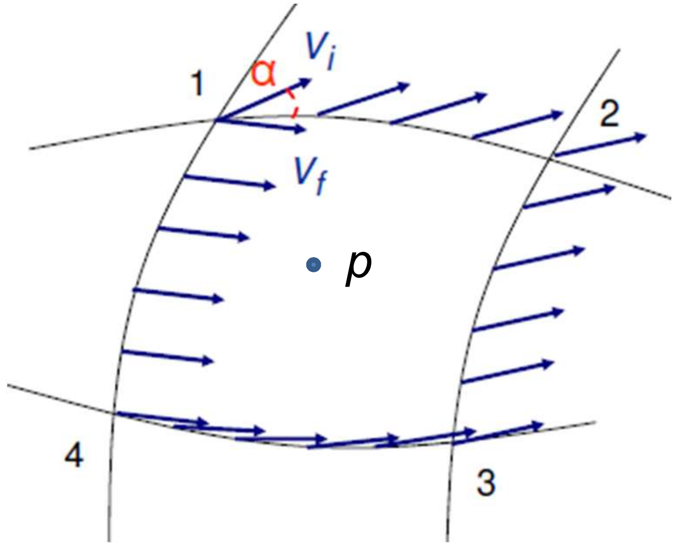
- First, how do we compare two vectors at different locations on a curved surface?
- Parallel transport a vector ***v*** along a geodesic curve

That is, keep the angles between ***v*** and the tangent vectors of the curve fixed.



Intrinsic definition of Gaussian curvature

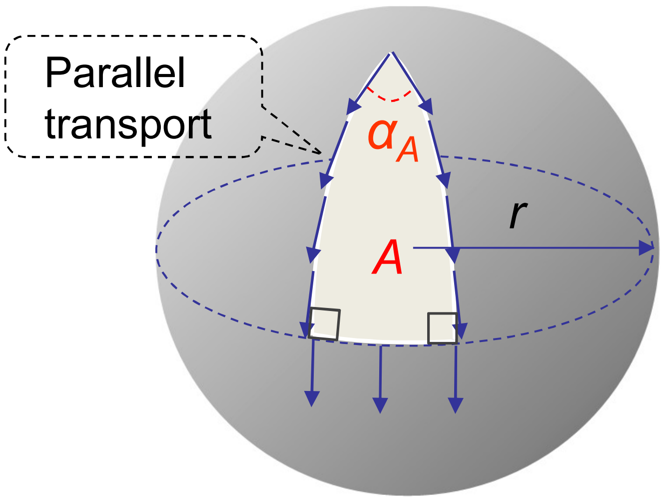
- Parallel transport a vector around a loop of geodesics.
- After circling a loop on a curved surface, ***v*** would not come back to its initial state, but is rotates by an angle α
- This kind of behavior is called anholonomy (incomplete), and the angle α is called an anholonomy angle (or defect angle) 虧角

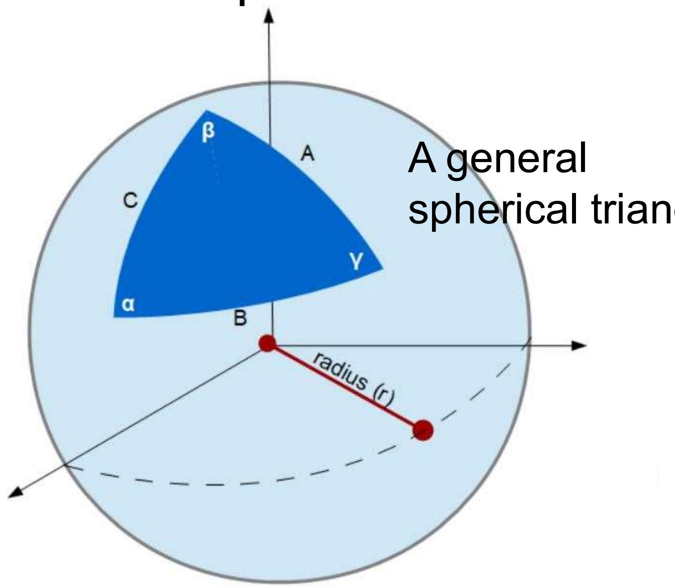


Gaussian curvature at p can be defined as

$$G \equiv \lim_{A \to 0} \frac{\alpha_A}{A}$$

Anholonomy angle and curvature of a sphere





A general spherical triangle,

Girard theorem (1626)

area $A = r^2(\alpha + \beta + \gamma - \pi)$

The gravity acts downward, so it cannot affect the orientation of the pendulum. Hence, only the angles between two segments contribute to the anholonomy.

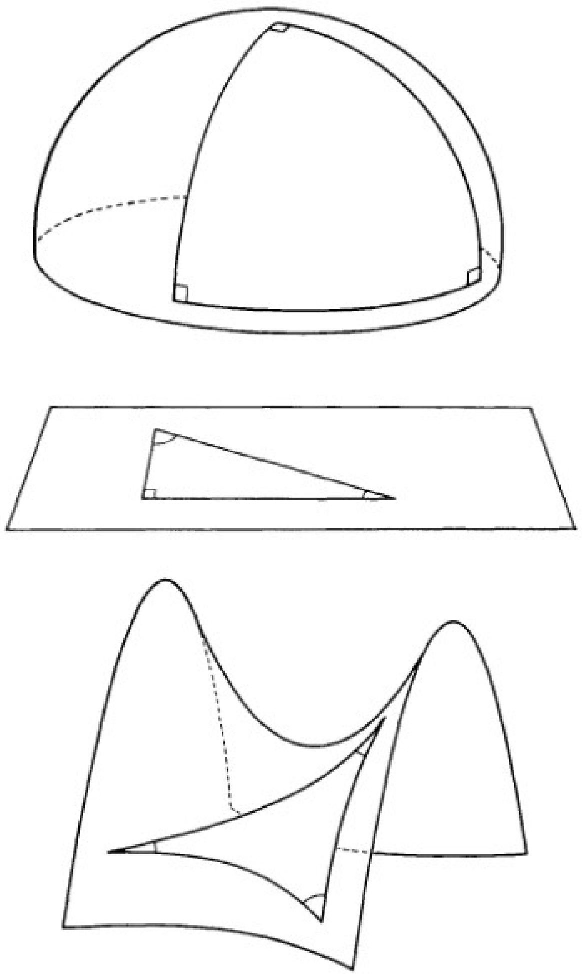
(From Satija's note)

Imagine you're carrying a pendulum walking slowly around the triangle, then

$$\alpha_A = \alpha + \beta + \gamma - \pi$$

$$\Rightarrow \quad G \equiv \lim_{A \to 0} \frac{\alpha_A}{A} = \frac{1}{r^2}$$

Anholonomy angle and curvature



$\alpha_A > 0$

G>0

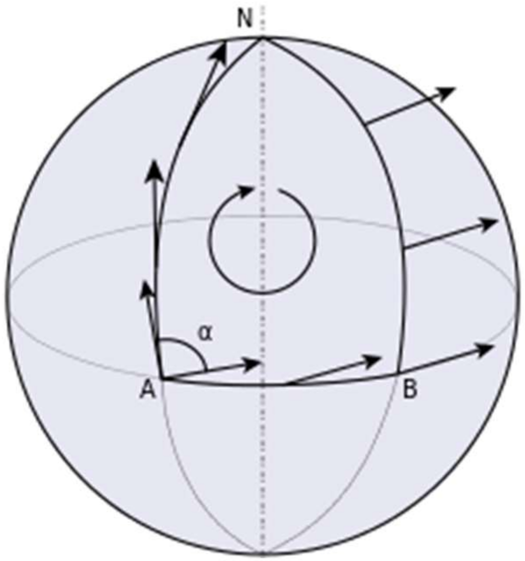
$\alpha_A = 0$

G=0

$\alpha_A < 0$

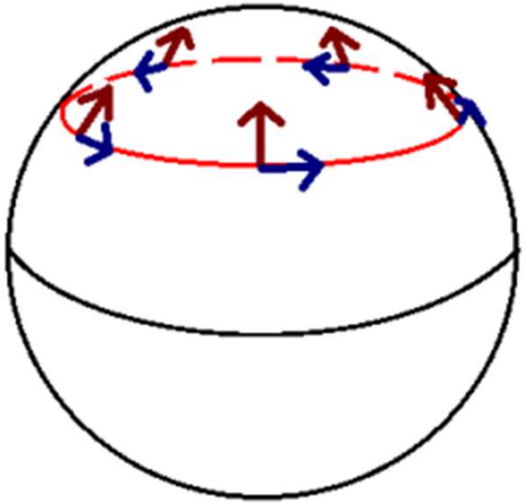
G<0

Great circle



$$\alpha_A = \frac{A}{r^2}$$

Small circle



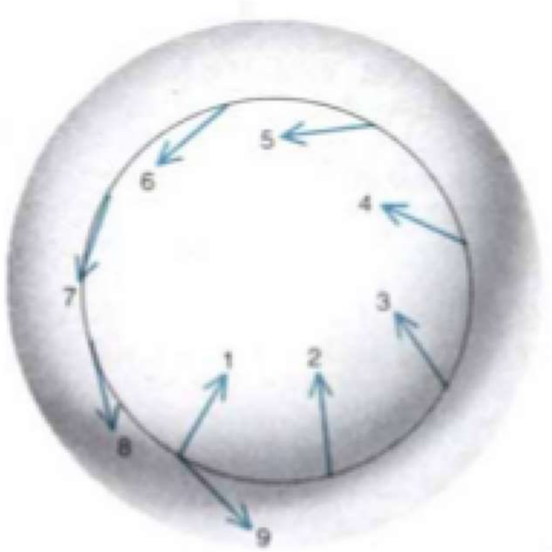
$\alpha_A = 0$?

PT condition along a general curve:
The earlier definition of PT cannot be right (e.g., transporting a vector along a curve on a flat surface).

New definition:
v does not twist around the local vertical axis (normal vector **n**) as we move along a curve *C.*

Parallel transport around a *small* circle on a sphere

v does not twist around the local vertical axis (normal vector **n**) as we move along a curve *C.*



What is the anholonomy angle?

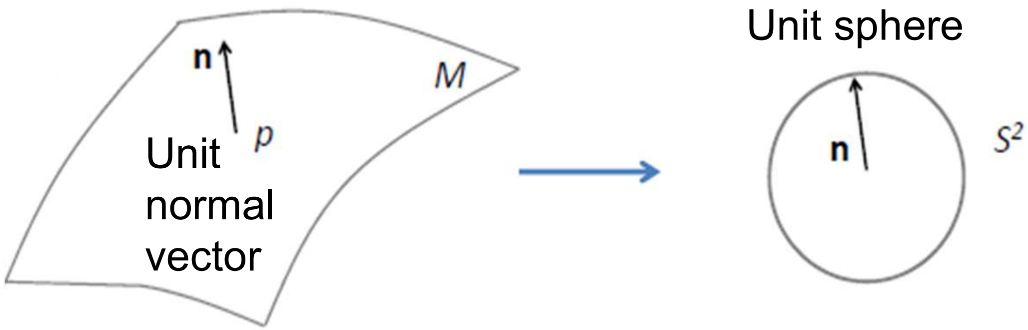
p.234, Intro Diff geometry and Riemannian geometry, by Kreyszig

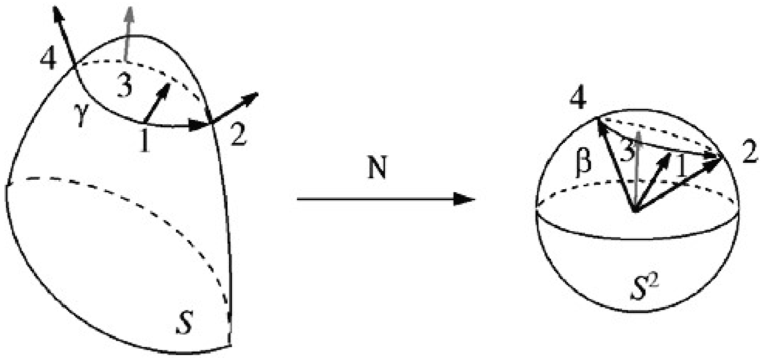
Gauss map and Gaussian curvature

Gauss map

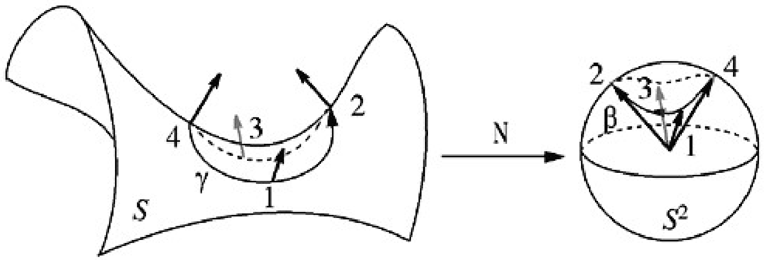
n: M → S^2

From normal vector to a sphere





(a)



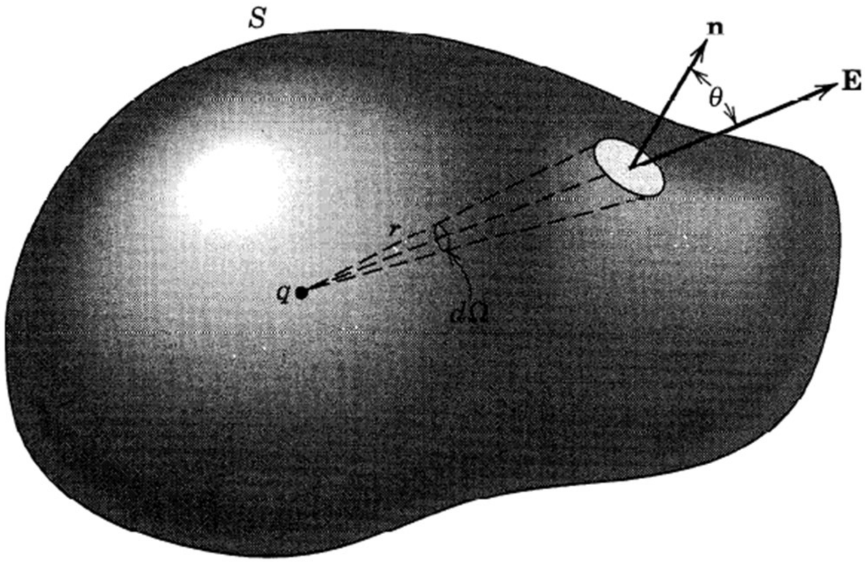
(b)

$$G \equiv \lim_{A \to 0} \frac{S_A}{A}$$

(Ratio between two areas)

Elli Angelopoulou and Lawrence B. Wolff, IEEE Transactions 1998

Total curvature of a closed surface



Total curvature of a closed surface is 4π (=solid angle of the unit sphere), *no matter how the surface is deformed*

$$\int_M da\, G = \int_M da\, \frac{dS_a}{da} = 4\pi$$

Total curvature is a topological invariant

Gauss-Bonnet theorem (for 2D surface)

– connecting *local curvature* with *global topology*

- Closed surface

$$\frac{1}{2\pi}\int_M da\, G = \chi(M)$$

The most beautiful theorem in differential topology

- Open surface

$$\frac{1}{2\pi}\left[\int_M da\, G + \int_{\partial M} d\ell\, \kappa_g\right] = \chi(M, \partial M)$$

p.211, Intro Diff geometry and Riemannian geometry, by Kreyszig

$\chi = 1$

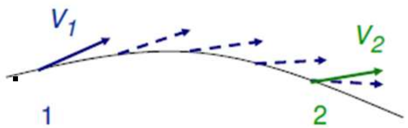
Q: Verify that the Euler characteristics of a disk is 1.

How would χ change if you punch a hole in the disk?

https://math.stackexchange.com/questions/2270687/variation-on-gauss-bonnet-theorem-disjoint-discs

p.212, Intro Diff geometry and Riemannian geometry, by Kreyszig

Anholonomy in geometry and quantum state

	Geometry	Quantum state
PT condition		$i\langle\psi\|\dot{\psi}\rangle = 0$
anholonomy	Anholonomy angle	Berry phase
curvature	Gaussian curvature	Berry curvature
Topo number	Euler characteristic $\chi = \frac{1}{2\pi}\int_S da\, G$	Chern number $C = \frac{1}{2\pi}\int_M da\, \Omega$

- Chern number refers to the topological number of *fiber bundle space*

陳省身

Winding number again

Index of a point defect in a vector field

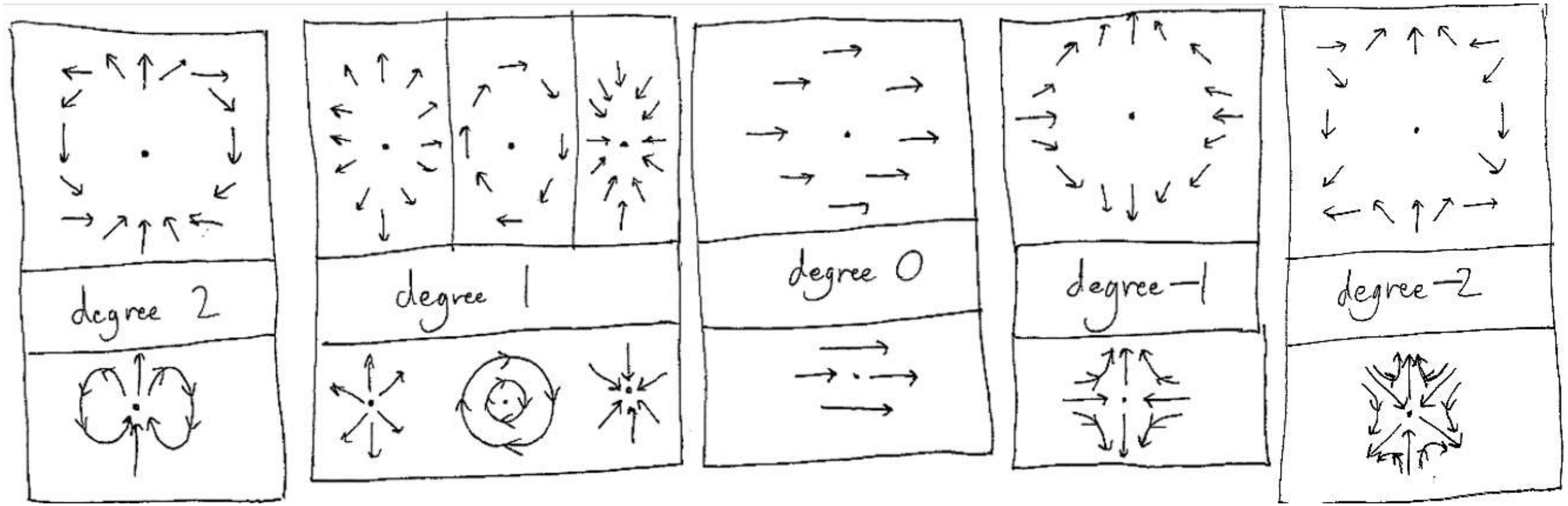


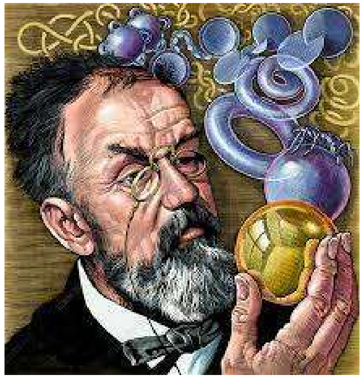
Fig from Jonas Kibelbek

Hopf-Poincare theorem

- Connecting index of point defect with topology

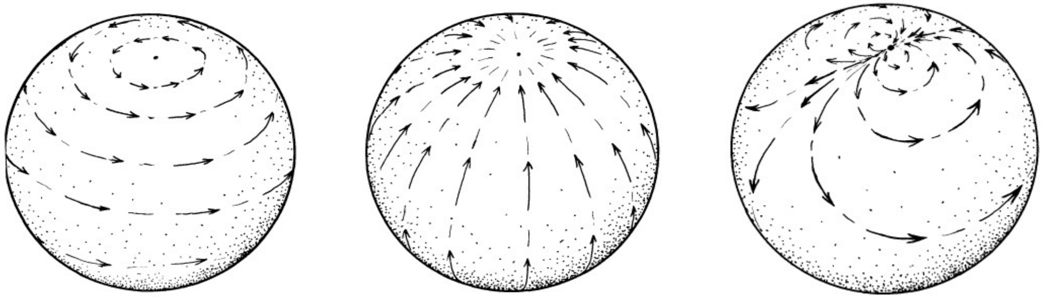
$$\sum_i ind(v_i) = \chi(M)$$

Winding number

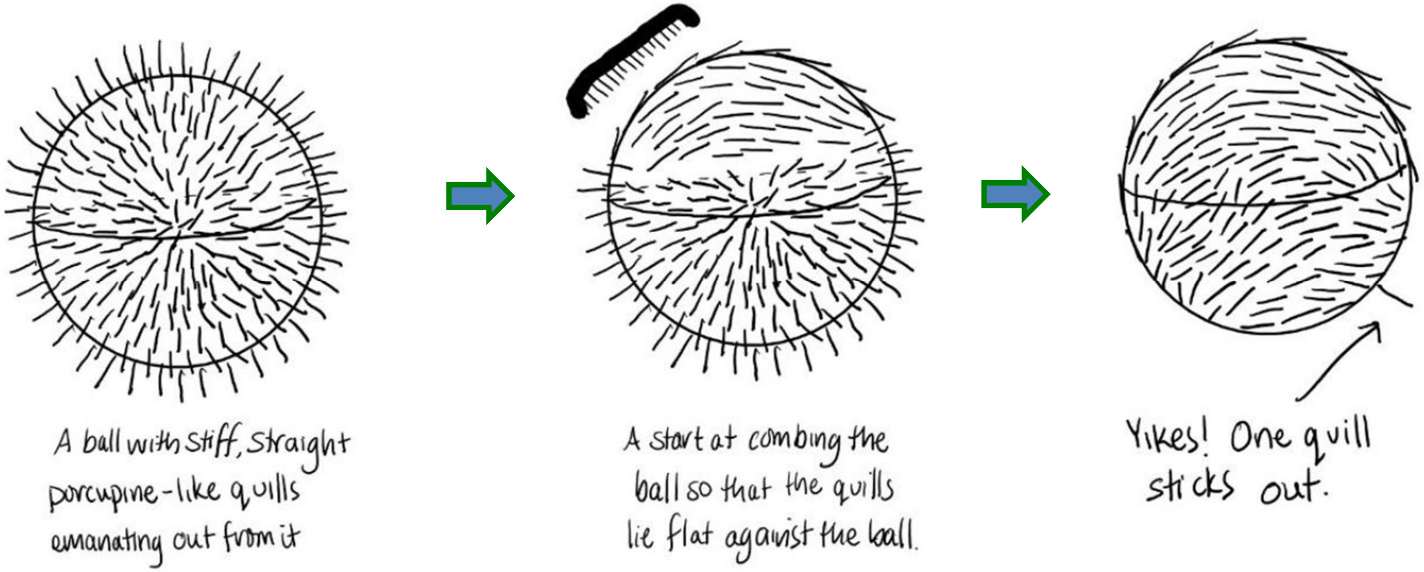


On a sphere

$$\sum_i ind(v_i) = 2$$



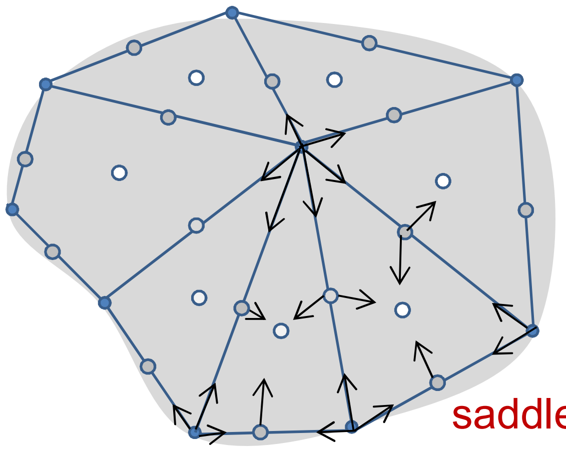
Hairy ball theorem



A "proof" of Hopf-Poincare theorem

Youtube course: *Topology & Geometry*, by Tadashi Tokieda
時枝正

Put a source on a vertex, a saddle point on an edge,
and a sink on a face



$$\sum_i \mathrm{ind}(\mathbf{v}_i) = (+1)\beta_0 + (-1)\beta_1 + (+1)\beta_2$$
$$= \chi(M)$$

$$\chi(M) = \sum_{k=0}^{D} (-1)^k \beta_k$$

Vector field on a torus

$$\sum_i ind(v_i) = \chi(T^2) = 0$$

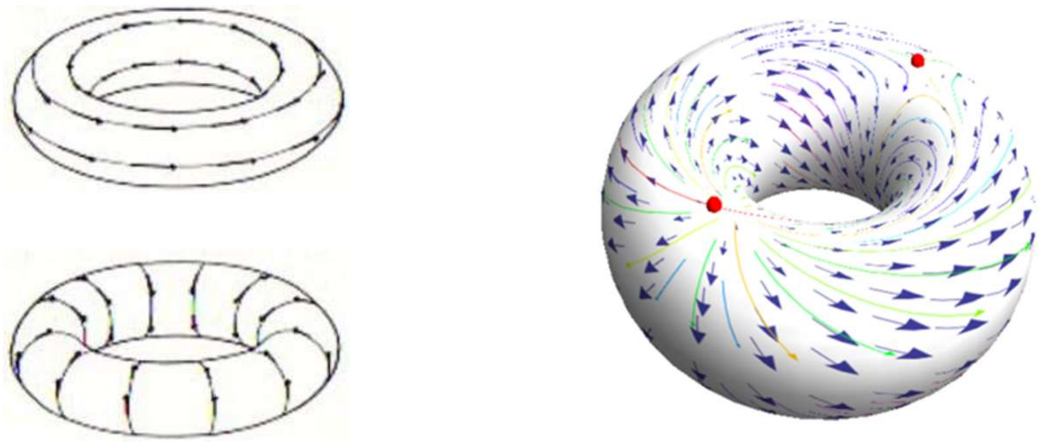
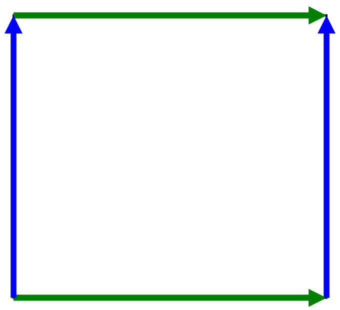
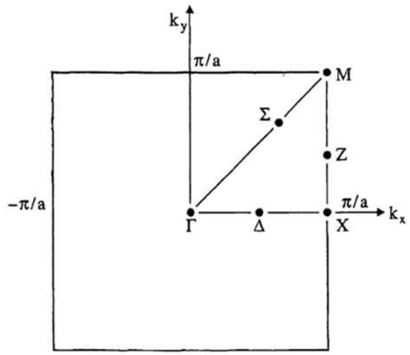


Fig from arxiv.org/pdf/1201.1162.pdf

Application: Brillouin zone as a torus (1D, 2D, 3D)



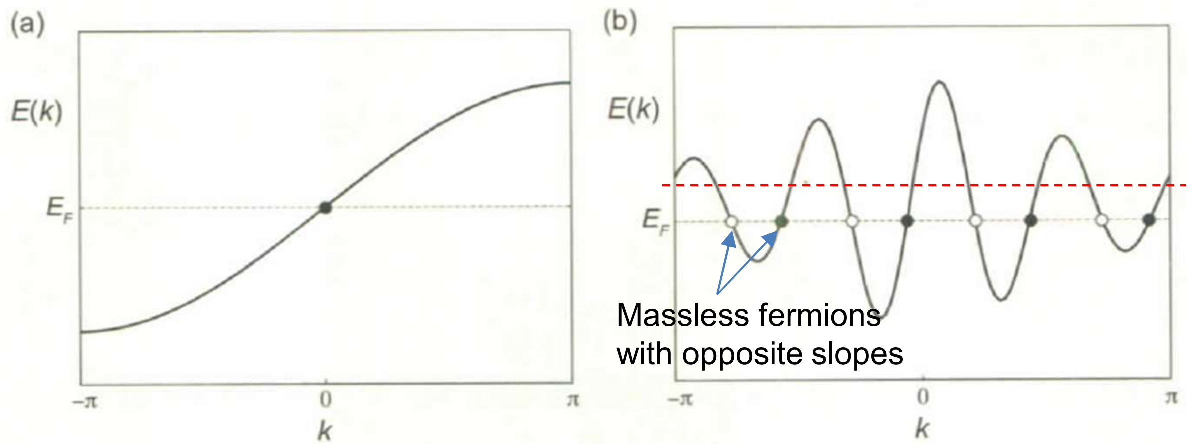


Berry connection **A**(***k***) as a vector field in BZ

Nielsen-Ninomiya theorem: 二宮正夫

(non-interacting) massless lattice fermions must appear in pairs

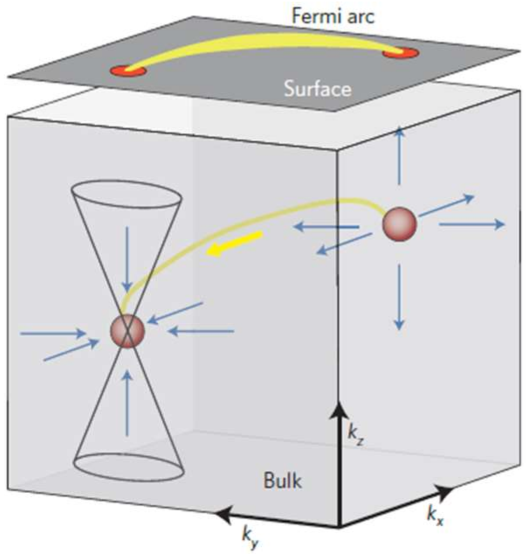
1D



“ 2D ”

Chirality not
defined in 2D

3D



Massless Weyl fermions appear in pairs
(no isolated monopole)