Lecture notes on topological insulators

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CONTENTS

I. PERIODICAL TABLE: BASICS

The topological insulators/superconductors mentioned so far are summarized in Table I, in which we have specified the topological number in each spatial dimension. Three fundamental symmetries are considered: T for time-reversal symmetry, P for particle-hole symmetry, and S for chiral symmetry. If T^2 (or P^2) equals to ± 1, then ± 1 is indicated (0 if there is no symmetry).

Since $S = TP$, S^2 can be ± 1 if both TRS and PHS exist. However, we only keep $+1$, since the sign of S^2 can be changed at will by a phase shift $S \to S' = \pm iS$.

A. Altland-Zirnbauer classes

There are 9 possible combinations, $\{-1, 0, 1\} \times \{-1, 0, 1\}$, for the values of T^2 and P^2. They determine the value of S^2. However, when both T^2 and P^2 are zero, there could still be chiral symmetry ($S^2 = 1$). So a complete list has 10 symmetry classes (see Table II). They are often called as the **Altland-Zirnbauer classes** (Schnyder *et al.*, 2008), which first appeared in the universality classes of disordered systems (Altland and Zirnbauer, 1997).

TABLE I Topological insulators and superconductors

	1d	2d	3d	T	P	S	Lect
Quantum Hall insulator	0	Z	0	0	0	0	**??**
Topological insulator	0	Z_2	Z_2	-1	0	0	**??**,**??**
Chiral superconductor	Z_2	Z	0	0	1	0	**??**,**??**
Helical superconductor	Z_2	Z_2	Z	-1	1	1	**??**,**??**

In Table II, one can see that there are 5 non-trivial topological classes in each dimension. We also show the constraint on the *spectral-flattened* Hamiltonians Q_k due to the 3 fundamental symmetries. For example, in class A, the Hamiltonian is not constrained by any of the symmetries. We assume that there are m filled levels separated from n empty levels. The unitary rotations $U(m)$ and $U(n)$ within the filled block and the empty block do not alter the topology. Thus the space of the Hamiltonian matrix is the complex **Grassmanian**,

$$G_{m+n,m}(\mathbb{C}) \equiv \frac{U(m+n)}{U(m)\times U(n)}. \tag{1.1}$$

Like a Lie group, the Grassmannian is a manifold. For example, since $U(2) = U(1) \times SU(2)$,

$$G_{2,1}(\mathbb{C}) = SU(2)/U(1). \tag{1.2}$$

This is the **Hopf fibration**, and it can be shown that $SU(2)/U(1)$ is diffeomorphic to S^2.

In Table II, q_k is the off-diagonal block in

$$\mathsf{Q}_k = \begin{pmatrix} 0 & \mathsf{q}_k \\ \mathsf{q}_k^\dagger & 0 \end{pmatrix}. \tag{1.3}$$

For example, in class AIII, there is no symmetry other than the chiral symmetry. Thus q_k is only required to be unitary.

It worths emphasizing that such a classification is for quadratic (non-interacting) fermionic systems with an energy gap. If there is additional symmetry, such as the reflection symmetry, beyond the 3 fundamental symmetries, then a class could be further divided to several sub-classes. The classification of interacting systems is a subject that is still under progress and is beyond the scope of this Lecture.

The AZ classification can be reached via at least 4 different routes:

1. For continuous systems, one can rely on Dirac Hamiltonian representatives, and classify them with the mathematical tools of **Clifford algebra**, **symmetric spaces**, and **homotopy theory**.

2. For lattice systems, one can use the homotopy theory, or the more advanced tool of **K-theory**.

3. For disordered systems, by extending the works of Wigner and Dyson, Altland and Zirnbauer showed that

TABLE II Altland-Zirnbauer classes

	Cartan's label	T	P	S	1d	2d	3d	Space of Hamiltonian matrix	Example
Standard	A (unitary)	0	0	0	0	Z	0	$\{\mathsf{Q}_k \in G_{m+n,m}(\mathbb{C})\}$	IQHE, QAHE
(Wigner-Dyson)	AI (orthogonal)	+1	0	0	0	0	0	$\{\mathsf{Q}_k \in G_{m+n,m}(\mathbb{C})\|\mathsf{Q}_k^* = \mathsf{Q}_{-k}\}$	
	AII (symplectic)	−1	0	0	0	Z_2	Z_2	$\{\mathsf{Q}_k \in G_{2m+2n,2m}(\mathbb{C})\|i\sigma_y\mathsf{Q}_k^*(-i\sigma_y) = \mathsf{Q}_{-k}\}$	TI
Chiral	AIII (chiral unitary)	0	0	1	Z	0	Z	$\{\mathsf{q}_k \in U(m)\}$	π-flux state
(sublattice)	BDI (chiral orthogonal)	+1	+1	1	Z	0	0	$\{\mathsf{q}_k \in U(m)\|\mathsf{q}_k^* = \mathsf{q}_{-k}\}$	SSH model
	CII (chiral symplectic)	−1	−1	1	Z	0	Z_2	$\{\mathsf{q}_k \in U(2m)\|i\sigma_y\mathsf{q}_k^*(-i\sigma_y) = \mathsf{q}_{-k}\}$	
BdG	D	0	+1	0	Z_2	Z	0	$\{\mathsf{Q}_k \in G_{2m,m}(\mathbb{C})\|\tau_x\mathsf{Q}_k^*\tau_x = -\mathsf{Q}_{-k}\}$	chiral p-wave
(superconductor)	C	0	−1	0	0	Z	0	$\{\mathsf{Q}_k \in G_{2m,m}(\mathbb{C})\|\tau_y\mathsf{Q}_k^*\tau_y = -\mathsf{Q}_{-k}\}$	$d_x \pm id_y$-wave
	DIII	−1	+1	1	Z_2	Z_2	Z	$\{\mathsf{q}_k \in U(2m)\|\mathsf{q}_k^T = -\mathsf{q}_{-k}\}$	helical p-wave, He-3
	CI	+1	−1	1	0	0	Z	$\{\mathsf{q}_k \in U(m)\|\mathsf{q}_k^T = \mathsf{q}_{-k}\}$	$d_{xy}, d_{x^2-y^2}$-wave

Note: $G_{m+n,m}(\mathbb{C})$ is the Grassmanian; $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ operate on spin and particle-hole degrees of freedom respectively.

TABLE III Kitaev's periodic table

	T	P	S	1d	2d	3d	4d	5d	6d	7d	8d
A	0	0	0	0	Z	0	Z	0	Z	0	Z
AIII	0	0	1	Z	0	Z	0	Z	0	Z	0
AI	1	0	0	0	0	0	Z	0	Z_2	Z_2	Z
BDI	1	1	1	Z	0	0	0	Z	0	Z_2	Z_2
D	0	1	0	Z_2	Z	0	0	0	Z	0	Z_2
DIII	−1	1	1	Z_2	Z_2	Z	0	0	0	Z	0
AII	−1	0	0	0	Z_2	Z_2	Z	0	0	0	Z
CII	−1	−1	1	Z	0	Z_2	Z_2	Z	0	0	0
C	0	−1	0	0	Z	0	Z_2	Z_2	Z	0	0
CI	1	−1	1	0	0	Z	0	Z_2	Z_2	Z	0

there are 10 universality classes. Since the topology of the bulk states is closely related to the universal disordered classes of the surface states, we can use the latter to distinguish the former. This approach is also related to the subject of **random matrix theory**.

4. Different topological systems have different responses to external perturbations (electromagnetic or others). Therefore, the response function also provides a handle for the classification. This is closely related to the subject of **quantum anomaly** in field theory.

B. Kitaev's periodic table

In 2009, Kitaev recognized the **Bott periodicity** hidden in AZ's table. He thus is able to generalize the classification to all spatial dimensions, which is a culmination of the study of topological materials (see Table III). The first 2 rows (A and AIII) belong to the so-called **complex classes**, which has a period of 2. The remaining 8 rows belong to the **real classes**, which has a period of 8. That is, the topological numbers in dimension $d+8$ are the same as those in dimension d.

The topological number counts the disconnected pieces of the mapping from a d-dimensional torus (Brillouin zone) to the space of Hamiltonian matrix, $T^d \to X$. One can start from studying the mapping $S^d \to X$, which is characterized by the homotopy group $\pi_d(X)$. Rigorously speaking, the base space T^d can be replaced by S^d only if $\pi_i(X) = 0$, for all $i < d$ (Avron *et al.*, 1983). So some information could be lost by such a simplification (which means that a lattice system is replaced by a continuous one).

It is relatively easy to understand the topology of the complex classes. First, consider a huge number of Bloch bands, so that the homotopy group can be *stabilized*. That is, $\pi_k(G)$ become independent of the size of the Lie group G. For example, $\pi_k(O(n))$ is independent of n if $n \gg 1$. For class A ($m, n \gg 1$), the relevant homotopy groups are,

$$\pi_1(G_{m+n,m}(\mathbb{C})) = 0, \quad (1.4)$$
$$\pi_2(G_{m+n,m}(\mathbb{C})) = Z, \quad (1.5)$$
$$\pi_3(G_{m+n,m}(\mathbb{C})) = 0 \cdots \text{etc}, \quad (1.6)$$

which give the topological numbers in the table.

A note on *stabilization*: the result above may need to be revised if m, n are not large enough. For example,

$$\pi_3(G_{2,1}(\mathbb{C})) = Z, \text{ instead of } 0. \quad (1.7)$$

This is the homotopy group of the Hopf fibration.

Similarly, for the complex class AIII, it is known that for large n,

$$\pi_{d\in\text{odd}}((U(n)) = Z, \quad (1.8)$$
$$\pi_{d\in\text{even}}((U(n)) = 0, \quad (1.9)$$

which give the topological numbers in the table.

The presence of other symmetry (such as the DIII class) would alter the conclusion above. Even though intuitive, the homotopy groups are often difficult to calculate. Therefore, for the rest of the 8 real classes, instead of the homotopy approach, we will use an alternative approach based on the Clifford algebra (next Lect).

C. Analytic formula of topological number

1. Complex classes

The topology of class A is nontrivial in even dimensions. A famous example is the quantum Hall effect in $d = 2$. The topological numbers in $d = 2n$ are given by the Chern numbers,

$$C_n = \frac{1}{n!} \int_{T^d} \mathrm{tr} \left(\frac{i\mathsf{F}}{2\pi} \right)^n, \tag{1.10}$$

which are defined only in even dimensions. We have replaced the domain of integration S^d in Dirac models with T^d for lattice systems.

The topology of class AIII is nontrivial in odd dimensions. The topological numbers in $(2n + 1)$-dimensions are given by the winding numbers (Ryu *et al.*, 2010),

$$\nu_{2n+1} = \frac{(-1)^n n!}{(2n+1)!} \int_{T^{2n+1}} \left(\frac{i}{2\pi} \right)^{n+1} \mathrm{tr} \left[(\mathsf{q}^{-1} d\mathsf{q})^{2n+1} \right] \tag{1.11}$$

2. Real classes

First, the topological numbers of complex classes and real classes are related. In Table **??**, let's first focus on the central diagonal line of $Z's$. The Z of class D (chiral p-wave) in 2d, as that of class A in 2d (QHE), is given by the Chern number. The Z of class DIII (helical p-wave) in 3d, as that of class AIII in 3d, is given by the winding number, and so on. Following this regularity, one could expect that the topological number of class BDI in 1d is given by a winding number.

Note that the parent states described by Chern numbers are all non-chiral, while those described by winding numbers are all chiral.

Second, the Z_2-Z_2-Z triplet in the same row are closely related. The state with topological number Z can be considered as the *parent* state of lower dimensional ones in the same symmetry class. For example, class AII in 4d is the QHE with topological number Z. By dimensional reduction, one can get the 3d and 2d TIs with topological numbers Z_2 (see Lect **??**). Similarly, one expects that the 3d helical p-wave in class DIII could be linked with the topological SCs with the same symmetry in 2d and 1d. See Sec. 4 of Ryu *et al.*, 2010 for more details.

The dimension d is not necessarily the spatial dimension. It is the dimension of parameter space, which includes the Bloch momentum, plus possible extra parameters of the Hamiltonian (such as deformation).

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