I. PERIODICAL TABLE: BASICS

The topological insulators and superconductors mentioned so far are summarized in Table I, in which we have specified the topological number in each spatial dimension. Three fundamental symmetries are considered: $T$ for time-reversal symmetry, $P$ for particle-hole symmetry, and $S$ for chiral symmetry. If $T^2$ (or $P^2$) equals to $\pm 1$, then $\pm 1$ is indicated (0 if there is no symmetry).

Since $S = TP$, $S^2$ can be $\pm 1$ if both TRS and PHS exist. However, we only keep $+1$, since the sign of $S^2$ can be changed at will by a phase shift $S \rightarrow S' = \pm iS$.

### A. Altland-Zirnbauer classes

There are 9 possible combinations, $\{-1, 0, 1\} \times \{-1, 0, 1\}$, for the values of $T^2$ and $P^2$. They determine the value of $S^2$. However, when both $T^2$ and $P^2$ are zero, there could still be chiral symmetry ($S^2 = 1$). So a complete list has 10 symmetry classes (see Table II). They are often called as the Altland-Zirnbauer classes (Schnyder et al., 2008), which first appeared in classifying the universality classes of disordered systems (Altland and Zirnbauer, 1997).

In Table II, one can see that there are 5 non-trivial topological classes in each dimension. We also show the constraint on the spectral-flattened Hamiltonians $Q_k$ due to the 3 fundamental symmetries. For example, in class A, the Hamiltonian is not constrained by any of the symmetries. We assume that there are $m$ filled levels separated from $n$ empty levels. The unitary rotations $U(m)$ and $U(n)$ within the empty block and the filled block do not alter the topology. Thus the space of the Hamiltonian matrices is the complex Grassmanian,

$$G_{m+n,m}(\mathbb{C}) \equiv \frac{U(m+n)}{U(m) \times U(n)}.$$  \hfill (1.1)

Like a Lie group, the Grassmannian is a manifold. For example, consider $G_{2,1}(\mathbb{C})$. Since $U(2) = U(1) \times SU(2)$, $G_{2,1}(\mathbb{C}) = SU(2)/U(1)$. \hfill (1.2)

This is the Hopf fibration, and it can be shown that $SU(2)/U(1)$ is diffeomorphic to $S^2$.

For chiral systems in the table, $Q_k$ is the off-diagonal block in

$$Q_k = \begin{pmatrix} 0 & q_k \\ \bar{q}_k & 0 \end{pmatrix}.$$  \hfill (1.3)

For example, in class AIII, there is no symmetry other than the chiral symmetry. Thus $Q_k$ is only required to be unitary.

It worths emphasizing that such a classification is for quadratic (non-interacting) fermionic systems with an energy gap. If there is additional symmetry, such as the reflection symmetry, beyond the 3 fundamental symmetries, then a class could be further divided to several sub-classes. The classification of interacting systems is a subject that is still under progress and is beyond the scope of this Lecture.

Without exclusion principle, a non-interacting bosonic system cannot have an energy gap. Therefore, one cannot find topological invariant in free bosons.

The AZ classification can be reached from at least 4 different routes:

1. For continuous systems, one can rely on Dirac Hamiltonian representatives, and classify them with the mathematical tools of Clifford algebra, symmetric spaces, and homotopy theory.

2. For lattice systems, one can use the homotopy theory, or the more advanced tool of K-theory.

3. For disordered systems, by extending the works of Wigner and Dyson, Altland and Zirnbauer showed that there are 10 universality classes. Since the topology of the bulk states is closely related to the universal disordered classes of the surface states, we can use the latter to distinguish the former. This approach is closely related to the subject of random matrix theory.

4. Different topological systems have different responses to external perturbations (electromagnetic one or others). Therefore, the response function also provides a handle for the classification. This is closely related to the subject of quantum anomaly in field theory.
### TABLE II Atland-Zirnbauer classes

<table>
<thead>
<tr>
<th></th>
<th>Cartan’s label</th>
<th>$T$</th>
<th>$P$</th>
<th>$S$</th>
<th>1d</th>
<th>2d</th>
<th>3d</th>
<th>Space of Hamiltonians</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard (Wigner-Dyson)</td>
<td>A (unitary)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Z</td>
<td>0</td>
<td>${q_k \in G_{m+n,m}(\mathbb{C})}$</td>
<td>IQHE, QAHE</td>
</tr>
<tr>
<td></td>
<td>AI (orthogonal)</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>${q_k \in G_{m+n,m}(\mathbb{C})</td>
<td>q_k^* = q_{-k}}$</td>
</tr>
<tr>
<td></td>
<td>AII (symplectic)</td>
<td>−1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Z</td>
<td>Z</td>
<td>${q_k \in G_{2m+2n,2m}(\mathbb{C})</td>
<td>\sigma_y q_k^* (-i\sigma_y) = q_{-k}}$</td>
</tr>
<tr>
<td>Chiral (sublattice)</td>
<td>AIII (chiral unitary)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>Z</td>
<td>0</td>
<td>Z</td>
<td>${q_k \in U(m)}$</td>
<td>$\pi$-flux state</td>
</tr>
<tr>
<td></td>
<td>BDI (chiral orthogonal)</td>
<td>+1</td>
<td>+1</td>
<td>1</td>
<td>0</td>
<td>Z</td>
<td>Z</td>
<td>${q_k \in U(2m)</td>
<td>q_k^* = q_{-k}}$</td>
</tr>
<tr>
<td></td>
<td>CI (chiral symplectic)</td>
<td>−1</td>
<td>−1</td>
<td>1</td>
<td>Z</td>
<td>0</td>
<td>Z</td>
<td>${q_k \in U(2m)</td>
<td>\sigma_y q_k^*(-i\sigma_y) = q_{-k}}$</td>
</tr>
<tr>
<td>BdG (superconductor)</td>
<td>D</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>Z</td>
<td>Z</td>
<td>0</td>
<td>${q_k \in G_{2m+n,m}(\mathbb{C})</td>
<td>\tau_q q_k^* \tau_q = -q_{-k}}$</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>0</td>
<td>−1</td>
<td>0</td>
<td>Z</td>
<td>0</td>
<td>0</td>
<td>${q_k \in G_{2m,m}(\mathbb{C})</td>
<td>\tau_q q_k^*(-i\sigma_y) = q_{-k}}$</td>
</tr>
<tr>
<td></td>
<td>DIII</td>
<td>−1</td>
<td>+1</td>
<td>1</td>
<td>Z</td>
<td>Z</td>
<td>Z</td>
<td>${q_k \in U(2m)</td>
<td>q_k^* = -q_{-k}}$</td>
</tr>
<tr>
<td></td>
<td>CI</td>
<td>+1</td>
<td>−1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>Z</td>
<td>${q_k \in U(m)</td>
<td>q_k^* = q_{-k}}$</td>
</tr>
</tbody>
</table>

Note: $G_{m+n,m}(\mathbb{C})$ is the Grassmanian; $\sigma$ and $\tau$ operate on spin and particle-hole degrees of freedom, respectively.

### TABLE III Kitaev’s periodic table

| A | 0 | 0 | 0 | 0 | Z | Z | Z | Z | Z | Z |
| AIII | 0 | 0 | 1 | Z | 0 | Z | 0 | Z | 0 | Z |
| AI | 1 | 0 | 0 | 0 | Z | Z | Z | Z | Z | Z |
| BDI | 1 | 1 | 1 | Z | 0 | 0 | Z | Z | Z | Z |
| D | 0 | 1 | 0 | Z | Z | 0 | 0 | Z | Z | Z |
| DIII | −1 | 1 | 1 | Z | Z | Z | Z | Z | 0 | Z |
| AII | −1 | 0 | 0 | 0 | Z | Z | Z | 0 | 0 | Z |
| CII | −1 | −1 | 1 | Z | 0 | Z | Z | 0 | 0 | Z |
| C | 0 | −1 | 0 | 0 | Z | Z | Z | Z | 0 | 0 |
| CI | 1 | −1 | 1 | 0 | 0 | Z | Z | Z | Z | Z |

### B. Kitaev’s periodic table

In 2009, Kitaev recognized the Bott periodicity hidden in AII’s table. He thus is able to generalize the classification to all spatial dimensions, which is a culmination of the study of topological materials (see Table III). The first 2 rows (A and AIII) belong to the so-called complex classes, which has a period of 2. The remaining 8 rows belong to the real classes, which has a period of 8. That is, the topological numbers in dimension $d + 8$ are the same as those in dimension $d$.

The topology is a result of the disconnected pieces of the space $X$ of the Hamiltonian matrix. That is, the topological number counts the disconnected pieces of the mapping from $T^d \to X$. One can start from studying the mapping $S^d \to X$, which is characterized by the homotopy group $\pi_d(X)$. Rigorously speaking, the base space $T^d$ can be replaced by $S^d$ only if $\pi_i(X) = 0$, for all $i < d$ (Avron et al., 1983). So some information could be lost by such a simplification (which means that a lattice system is replaced by a continuous one).

It is relatively easy to understand the topology of the complex classes. First, consider a huge number of Bloch bands, so that the homotopy group can be stabilized. That is, $\pi_k(G)$ become independent of the size of the Lie group $G$. For example, $\pi_k(O(n))$ is independent of $n$ if $n \gg 1$. For class A ($m, n \gg 1$), the relevant homotopy groups are

$$\pi_1(G_{m+n,n,m}(\mathbb{C})) = 0, \quad (1.4)$$

$$\pi_2(G_{m+n,n,m}(\mathbb{C})) = Z, \quad (1.5)$$

$$\pi_3(G_{m+n,n,m}(\mathbb{C})) = Z, \quad (1.6)$$

which give the topological numbers in the table.

A note on the “stabilization”: the result above may need be revised if $m, n$ are not large enough. For example, $\pi_3(G_{2,1}(\mathbb{C})) = Z$, instead of 0. (1.7)

This is the homotopy group of the Hopf fibration.

Similarly, for the complex class AIII, it is known that for large $n$,

$$\pi_{d\in\mathbb{Z}}(U(n)) = Z, \quad (1.8)$$

$$\pi_{d\in\mathbb{Z}}(U(n)) = 0, \quad (1.9)$$

which are the topological numbers in the table.

The presence of other symmetry (such as the DIII class) would alter the conclusion above. Even though intuitive, the homotopy groups are often difficult to calculate. Therefore, for the rest of the 8 real classes, instead of the homotopy approach, we will use an alternative approach based on the Clifford algebra (next Lect).

### C. Analytic formula of topological number

1. Complex classes

The topology of class A is nontrivial in even dimensions. A famous example is the quantum Hall effect in
\( d = 2 \). The topological number in \( d = 2n \) is given by the Chern number,

\[
C_n = \frac{1}{n!} \int_{T^d} \text{tr} \left( \frac{i \mathbf{F}}{2\pi} \right)^n ,
\]

which is defined only in even dimension. We have replaced the domain of integration \( S^d \) in Dirac models with the \( T^d \) for lattice systems.

The topology of class AIII is nontrivial in odd dimensions. The topological number in \((2n + 1)\)-dimension is given by the winding number (Ryu et al., 2010),

\[
\nu_{2n+1} = \frac{(-1)^n n!}{(2n + 1)!} \int_{T^{2n+1}} \left( \frac{i}{2\pi} \right)^{n+1} \text{tr} \left[ (q^{-1} dq)^{2n+1} \right]
\]

2. Real classes

First, the topological numbers of complex classes and real classes are related. In Table ??, let’s first focus on the central diagonal line of \( Z' \)’s. The \( Z \) of class D (chiral \( p \)-wave) in 2d, as that of class A in 2d (QHE), is given by the Chern number. The \( Z \) of class DIII (helical \( p \)-wave) in 3d, as that of class AIII in 3d, is given by the winding number. And so on. Following this regularity, one could expect that the topological number of class BDI in 1d is given by a winding number.

Note that the parent states described by Chern numbers are all non-chiral, while those described by winding numbers are all chiral.

Second, the \( Z_2 \)-\( Z_2 \)-\( Z \) triplet in the same row are closely related. The state with topological number \( Z \) can be considered as the parent state of lower dimensional ones in the same symmetry class. For example, class AII in 4d is the QHE with topological number \( Z \). By dimensional reduction, one can get the 3d and 2d TIs with topological numbers \( Z_2 \) (see Lect ??). Similarly, one expects that the 3d helical \( p \)-wave in class DIII could be linked with the topological SCs with the same symmetry in 2d and 1d. See Sec. 4 of Ryu et al., 2010 for more details.

The dimension \( d \) is not necessarily the spatial dimension. It is the dimension of parameter space, which includes the Bloch momentum, plus possible extra parameters of the Hamiltonian (such as deformation).

References