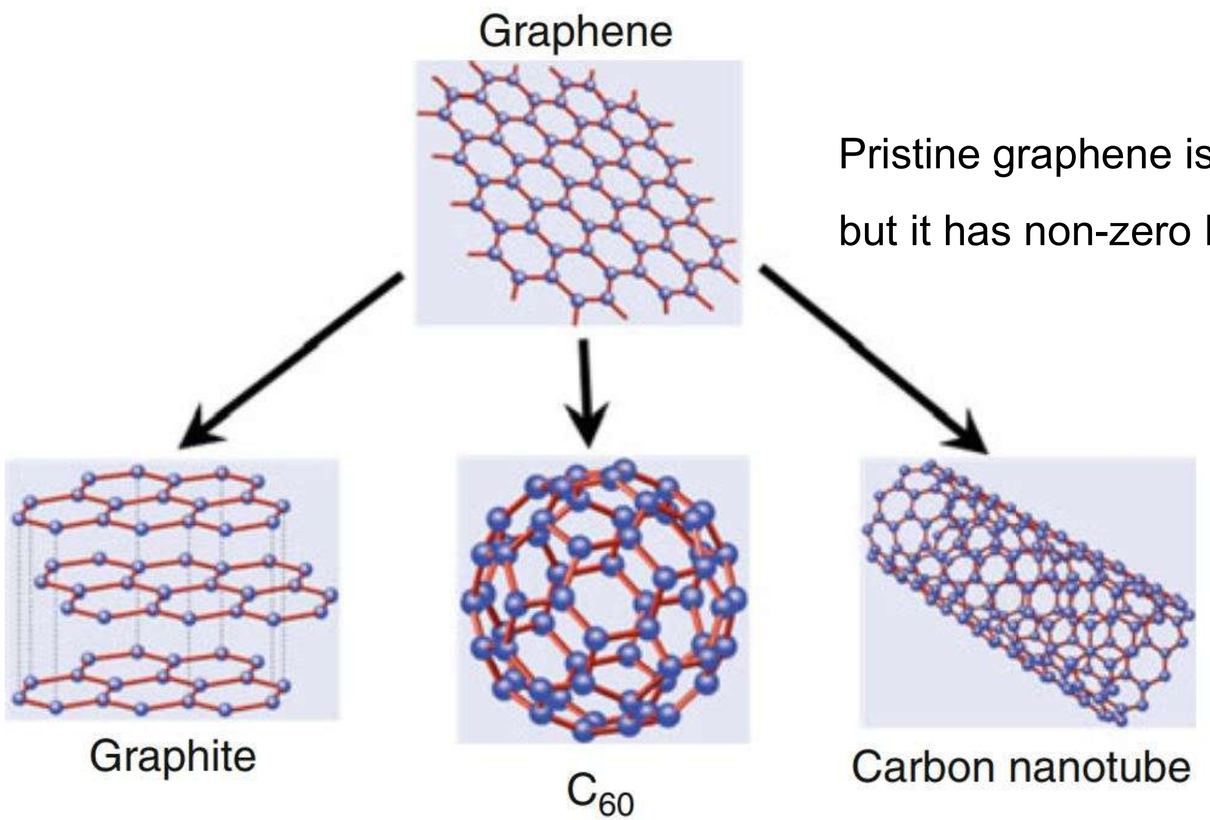
Two-dimensional material

- Graphene
  - symmetries
  - Effective Hamiltonian
- Transition metal dichalcogenide
  - Berry curvature
  - Optical transitions
- Haldane model
  - Haldane flux
  - Berry curvature

- 1947, theory of graphene by Wallace
- 1968, Mermin-Wagner theorem (no long-range order in 2D at finite T)
- 1985, discovery of bucky ball by Kroto, Smalley, and Curl (1996 Nobel prize)
- 1991, discovery of carbon nanotube by Iijima and Ichihashi.
- 2004, first graphene produced by Geim and Novoselov (2010 Nobel prize)



Pristine graphene is not topological,

but it has non-zero Berry curvature.

Extraordinary properties of graphene

- **Young's modulus of 1 TPa and intrinsic strength of 130 Gpa, the strongest materials ever tested.** (space elevator)

  Cu: 0.117 TPa — Phys. Rev. B 76, 064120 (2007).

- **A prediction in 2015 suggested a melting point at least 5000 K.**

  (in reality, sublimate near 3700 K)

- **Room-temperature electron mobility of 2.5x10^{5} cm^{2}V^{-1} s^{-1}**

  Nano Lett. 11, 2396–2399 (2011).

- **The electrical resistivity of graphene < 10^{-6} Ω·cm, less than silver, the lowest known at RT.** (better than copper)

- **High thermal conductivity: above 3,000 Wm^{-1}K^{-1}**

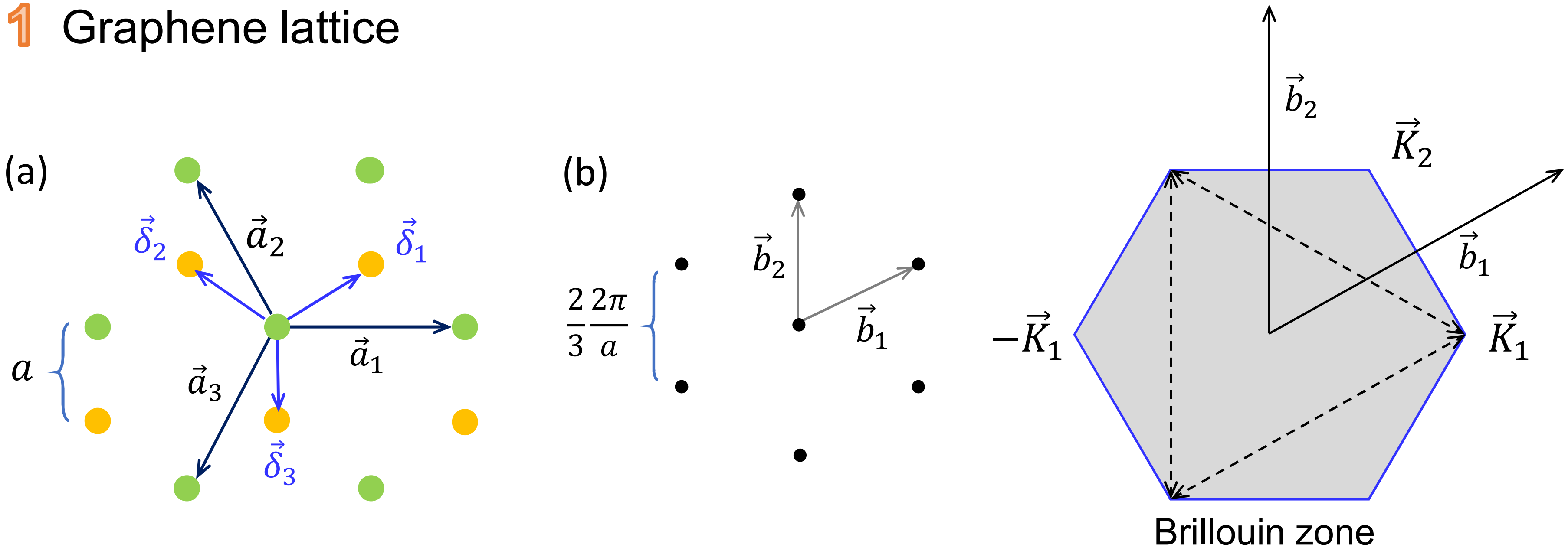
  Cu: 401 Wm^{-1}K^{-1} — Nature Mater. 10, 569–581 (2011).

- **Optical absorption of 2.3%** (related to fine structure constant)

  Science 320, 1308 (2008).

From Raynien Kwo's ppt

1 Graphene lattice



- Lattice vectors

$$\mathbf{a}_1 = \sqrt{3}a\hat{x},$$

$$\mathbf{a}_2 = -\frac{\sqrt{3}}{2}a\hat{x} + \frac{3}{2}a\hat{y},$$

$$\mathbf{a}_3 = -\frac{\sqrt{3}}{2}a\hat{x} - \frac{3}{2}a\hat{y}.$$

$$a = 1.42\ A$$

$$a_0 = \sqrt{3}a \simeq 2.46\text{Å}$$

- Reciprocal lattice vectors

$$\mathbf{b}_1 = \frac{2\pi}{a}\left(\frac{1}{\sqrt{3}}\hat{x} + \frac{1}{3}\hat{y}\right),$$

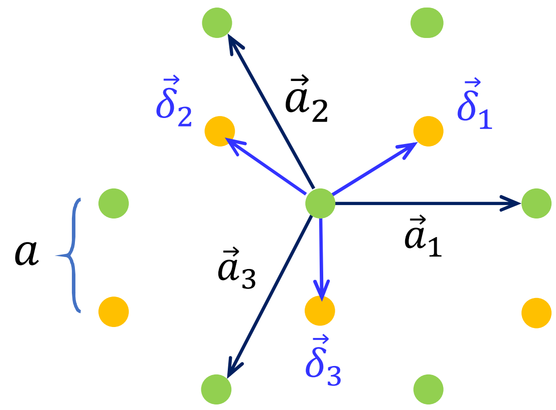
$$\mathbf{b}_2 = \frac{2\pi}{a}\frac{2}{3}\hat{y}.$$

Tight-binding model
(spin neglected)

$$\hat{H} = \hat{H}_{NN} + \hat{H}_{NNN} + \hat{H}_{on-site}.$$

(1)
$$\hat{H}_{NN} = t_1 \sum_{\mathbf{R}} \left(d^\dagger_{\mathbf{R}+\boldsymbol{\delta}_1} c_{\mathbf{R}} + d^\dagger_{\mathbf{R}+\boldsymbol{\delta}_2} c_{\mathbf{R}} + d^\dagger_{\mathbf{R}+\boldsymbol{\delta}_3} c_{\mathbf{R}} \right) + h.c.$$

$$\begin{aligned} \boldsymbol{\delta}_1 &= \frac{\sqrt{3}}{2} a\hat{x} + \frac{1}{2} a\hat{y}, \\ \boldsymbol{\delta}_2 &= -\frac{\sqrt{3}}{2} a\hat{x} + \frac{1}{2} a\hat{y}, \\ \boldsymbol{\delta}_3 &= -a\hat{y}. \end{aligned}$$



$t_1 \simeq 2.7$ eV

(2)
$$\begin{aligned} \hat{H}_{NNN} &= t_2 \sum_{\mathbf{R}} \left(c^\dagger_{\mathbf{R}+\mathbf{a}_1} c_{\mathbf{R}} + c^\dagger_{\mathbf{R}+\mathbf{a}_2} c_{\mathbf{R}} + c^\dagger_{\mathbf{R}+\mathbf{a}_3} c_{\mathbf{R}} \right) \\ &+ t_2 \sum_{\mathbf{R}} \left(d^\dagger_{\mathbf{R}+\boldsymbol{\delta}_1+\mathbf{a}_1} d_{\mathbf{R}+\boldsymbol{\delta}_1} + d^\dagger_{\mathbf{R}+\boldsymbol{\delta}_1+\mathbf{a}_2} d_{\mathbf{R}+\boldsymbol{\delta}_1} \right. \\ &\left. + d^\dagger_{\mathbf{R}+\boldsymbol{\delta}_1+\mathbf{a}_3} d_{\mathbf{R}+\boldsymbol{\delta}_1} \right) + h.c., \end{aligned}$$

(3)
$$\hat{H}_{on-site} = \Delta \sum_{\mathbf{R}} c^\dagger_{\mathbf{R}} c_{\mathbf{R}} - \Delta \sum_{\mathbf{R}} d^\dagger_{\mathbf{R}+\boldsymbol{\delta}_1} d_{\mathbf{R}+\boldsymbol{\delta}_1}$$

Fourier transform

$$c_{\mathbf{R}} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}} c_{\mathbf{k}}$$

$$d_{\mathbf{R}+\boldsymbol{\delta}_1} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{R}+\boldsymbol{\delta}_1)} d_{\mathbf{k}}$$

e.g.,

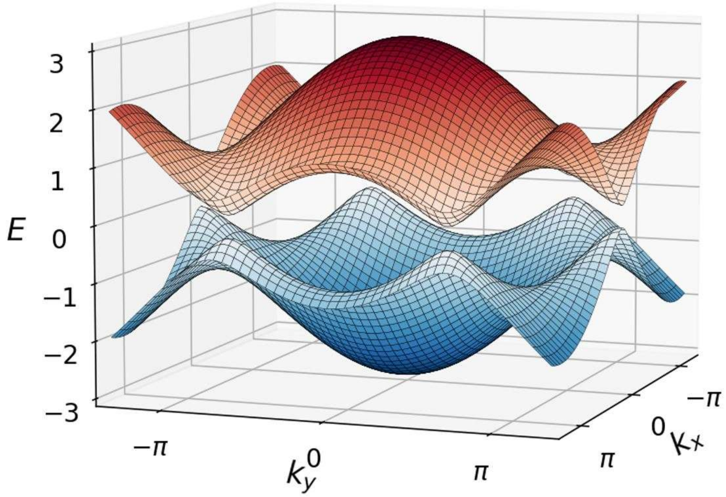
$$\sum_{\mathbf{R}} d^{\dagger}_{\mathbf{R}+\boldsymbol{\delta}_1} c_{\mathbf{R}} = \frac{1}{N} \sum_{\mathbf{k},\mathbf{k}'} \sum_{\mathbf{R}} e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}} e^{-i\mathbf{k}'\cdot\boldsymbol{\delta}_1} d^{\dagger}_{\mathbf{k}'} c_{\mathbf{k}} = \sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_1} d^{\dagger}_{\mathbf{k}} c_{\mathbf{k}},$$

$$\sum_{\mathbf{R}} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}} = N\delta_{\mathbf{k}'\mathbf{k}}.$$

$$\begin{aligned}
\Rightarrow \quad \hat{H} &= t_1 \sum_{\mathbf{k}} \left(e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_1} + e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_2} + e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_3}\right) d^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} \\
&+ t_2 \sum_{\mathbf{k}} \left(e^{-i\mathbf{k}\cdot\mathbf{a}_1} + e^{-i\mathbf{k}\cdot\mathbf{a}_2} + e^{-i\mathbf{k}\cdot\mathbf{a}_3}\right) c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} \\
&+ t_2 \sum_{\mathbf{k}} \left(e^{-i\mathbf{k}\cdot\mathbf{a}_1} + e^{-i\mathbf{k}\cdot\mathbf{a}_2} + e^{-i\mathbf{k}\cdot\mathbf{a}_3}\right) d^{\dagger}_{\mathbf{k}} d_{\mathbf{k}} + h.c. \\
&+ \Delta \sum_{\mathbf{k}} \left(c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} - d^{\dagger}_{\mathbf{k}} d_{\mathbf{k}}\right) \\
&= \sum_{\mathbf{k}} \left(c^{\dagger}_{\mathbf{k}}, d^{\dagger}_{\mathbf{k}}\right) \mathsf{H}(\mathbf{k}) \begin{pmatrix} c_{\mathbf{k}} \\ d_{\mathbf{k}} \end{pmatrix}.
\end{aligned}$$

$$\mathsf{H}(\mathbf{k}) = \begin{pmatrix} 2t_2\sum_i \cos\mathbf{k}\cdot\mathbf{a}_i + \Delta & t_1\sum_i e^{i\mathbf{k}\cdot\boldsymbol{\delta}_i} \\ t_1\sum_i e^{-i\mathbf{k}\cdot\boldsymbol{\delta}_i} & 2t_2\sum_i \cos\mathbf{k}\cdot\mathbf{a}_i - \Delta \end{pmatrix}$$
$$= h_0(\mathbf{k}) + \mathbf{h}(\mathbf{k})\cdot\boldsymbol{\sigma},$$

$$h_0(\mathbf{k}) = 2t_2\sum_i \cos\mathbf{k}\cdot\mathbf{a}_i,$$
$$\mathbf{h}(\mathbf{k}) = \left(t_1\sum_i \cos\mathbf{k}\cdot\boldsymbol{\delta}_i, -t_1\sum_i \sin\mathbf{k}\cdot\boldsymbol{\delta}_i, \Delta\right)$$



$$\varepsilon_\pm(\mathbf{k}) = h_0(\mathbf{k}) \pm |\mathbf{h}(\mathbf{k})|$$

$$|\mathbf{h}| = \sqrt{3t_1^2 + 2t_1^2(\cos\mathbf{k}\cdot\mathbf{a}_1 + \cos\mathbf{k}\cdot\mathbf{a}_2 + \cos\mathbf{k}\cdot\mathbf{a}_3) + \Delta^2}$$
$$= \sqrt{3t_1^2 + 2t_1^2 f(\mathbf{k}) + \Delta^2}, \qquad t_1 \simeq 2.7\ eV$$

$$f(\mathbf{k}) \equiv \cos\mathbf{k}\cdot\mathbf{a}_1 + \cos\mathbf{k}\cdot\mathbf{a}_2 + \cos\mathbf{k}\cdot\mathbf{a}_3$$
$$= \cos\sqrt{3}ak_x + 2\cos\frac{\sqrt{3}}{2}ak_x\cos\frac{3}{2}ak_y.$$

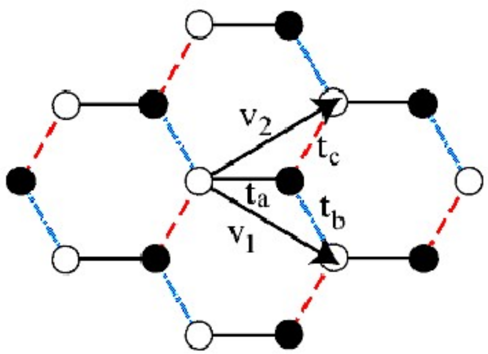
Stability of the nodal point in graphene

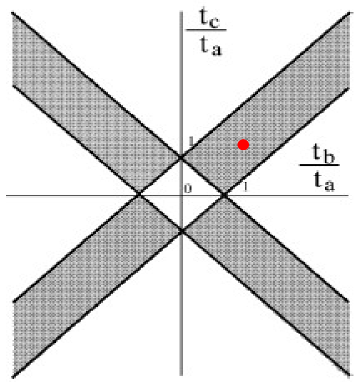
- TRS
  $$H(\vec{k})^* = H(-\vec{k})$$
  $$\rightarrow h_x(\vec{k}), h_y(\vec{k}), h_z(\vec{k}) = \text{even, odd, even}$$

- SIS (for graphene, $\pi=\sigma_x$)
  $$\sigma_x H(\vec{k})\sigma_x = H(-\vec{k})$$
  $$\rightarrow h_x(\vec{k}), h_y(\vec{k}), h_z(\vec{k}) = \text{even, odd, odd}$$

- TRS+SIS $\rightarrow$ no σ_z term ← Co-dimension is 2
  (point degeneracy in 2D BZ)

- Global stability: point degeneracy is further protected by C_3 symmetry
  (Ch7, Bernevig)



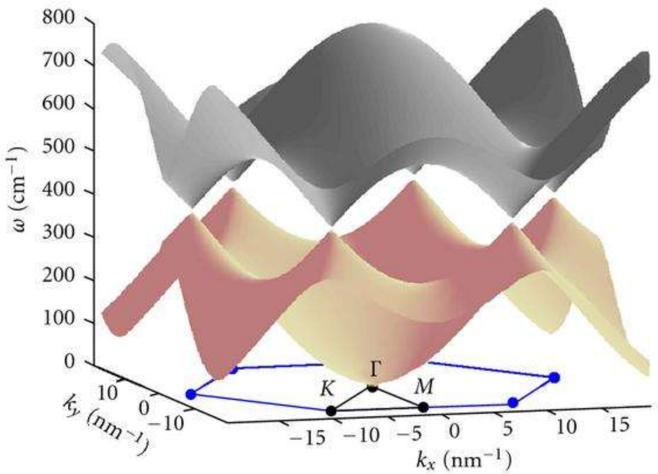


Hasegawa et al, PRB 2006

Low-energy effective Hamiltonian near Dirac point

$$\mathbf{K}_1 = \frac{2\pi}{a}\left(\frac{2}{3\sqrt{3}}, 0\right),$$
$$\mathbf{K}_2 = \frac{2\pi}{a}\left(\frac{1}{3\sqrt{3}}, \frac{1}{3}\right)$$

Alternative:
$\mathbf{K}_2 = -\mathbf{K}_1$



$$\mathbf{k} = \mathbf{K}_i + \mathbf{k}_i, \quad |\mathbf{k}_i| \ll |\mathbf{K}_i|; \; i = 1, 2.$$

$$H_{12}(\mathbf{k}) = t_1\left(e^{i\mathbf{K}_i\cdot\boldsymbol{\delta}_1} e^{i\mathbf{k}_i\cdot\boldsymbol{\delta}_1} + e^{i\mathbf{K}_i\cdot\boldsymbol{\delta}_2} e^{i\mathbf{k}_i\cdot\boldsymbol{\delta}_2} + e^{i\mathbf{K}_i\cdot\boldsymbol{\delta}_3} e^{i\mathbf{k}_i\cdot\boldsymbol{\delta}_3}\right)$$

(rewrite $k_{1,2}$ simply as k)

$$H_{12}(\mathbf{k}) \simeq it_1 a\mathbf{k}\cdot\left(\pm\frac{3}{2}i, -\frac{3}{2}\right)$$
$$= t_1 a\left(\mp\frac{3}{2}k_x - \frac{3}{2}ik_y\right)$$

$$\mathrm{H}(\mathbf{k}) \simeq \begin{pmatrix} \Delta & \frac{3}{2}t_1 a(\mp k_x - ik_y) \\ \frac{3}{2}t_1 a(\mp k_x + ik_y) & -\Delta \end{pmatrix}$$
$$= \hbar v_F(\tau k_x\sigma_x + k_y\sigma_y) + \Delta\sigma_z,$$

Valley $\tau = \pm$

$$v_F = \frac{\frac{3}{2}at_1}{\hbar} \simeq \frac{c}{300}$$

In Chap 3, we learned that

Time reversal symmetry: $\vec{F}_n\left(\vec{k}\right) = -\vec{F}_n(-\vec{k})$

Space inversion symmetry: $\vec{F}_n\left(\vec{k}\right) = +\vec{F}_n(-\vec{k})$

- With both symmetries, (for non-degenerate state) the Berry curvature is zero. So we need to break either TRS or SIS.
- If we break only SIS, then

$$\vec{F}_n\,(K_2) = -\vec{F}_n(K_1) \quad \text{valley Hall effect}$$

- If we break only TRS, then

$$\vec{F}_n\,(K_2) = +\vec{F}_n(K_1) \quad \text{Haldane model}$$

e.g. with TRS

$$H \longrightarrow H' = \hbar v_F \vec{\sigma}^* \cdot \left(-\vec{k} - \vec{K}_2\right) + \Delta\sigma_z$$
$$= -\hbar v_F \vec{\sigma}^* \cdot \left(\vec{k} - \vec{K}_1\right) + \Delta\sigma_z$$

$$\left(h_x, h_y, h_z\right) \to (-h_x, h_y, h_z)$$

So the Berry curvature at opposite valleys have opposite signs.

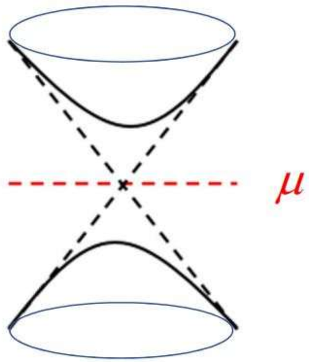
- Effective Hamiltonian near degenerate point

$$H_0 = \hbar v_F\left(\mp k_x\sigma_x + k_y\sigma_y\right) + \Delta\sigma_z \text{ at } \pm K, \qquad \hbar v_F \equiv \frac{3}{2}t_1 a$$

- Berry curvature

$$F_{z\tau}^{\pm}(\mathbf{k}) = \pm\tau\frac{1}{2}\frac{\hbar^2 v_F^2\Delta}{h^3}$$

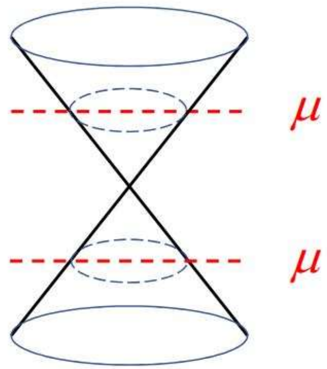
$$\int_{all} d^2k\, F_\tau^{\pm} = \pm\tau\pi \qquad \text{(Drop sub z)}$$



➡ $$\sigma_H = \frac{e^2}{h}\frac{1}{2\pi}\int d^2k F_\tau^- = -\frac{\tau}{2}\frac{e^2}{h}$$

Total Hall conductivity is zero due to the cancellation between 2 valleys.

- Gapless limit (at +***K***):



➡ Berry phase (surrounding +***K***): $\gamma_C = \pm\pi$

Berry phase remains the same as long as C encircles the origin

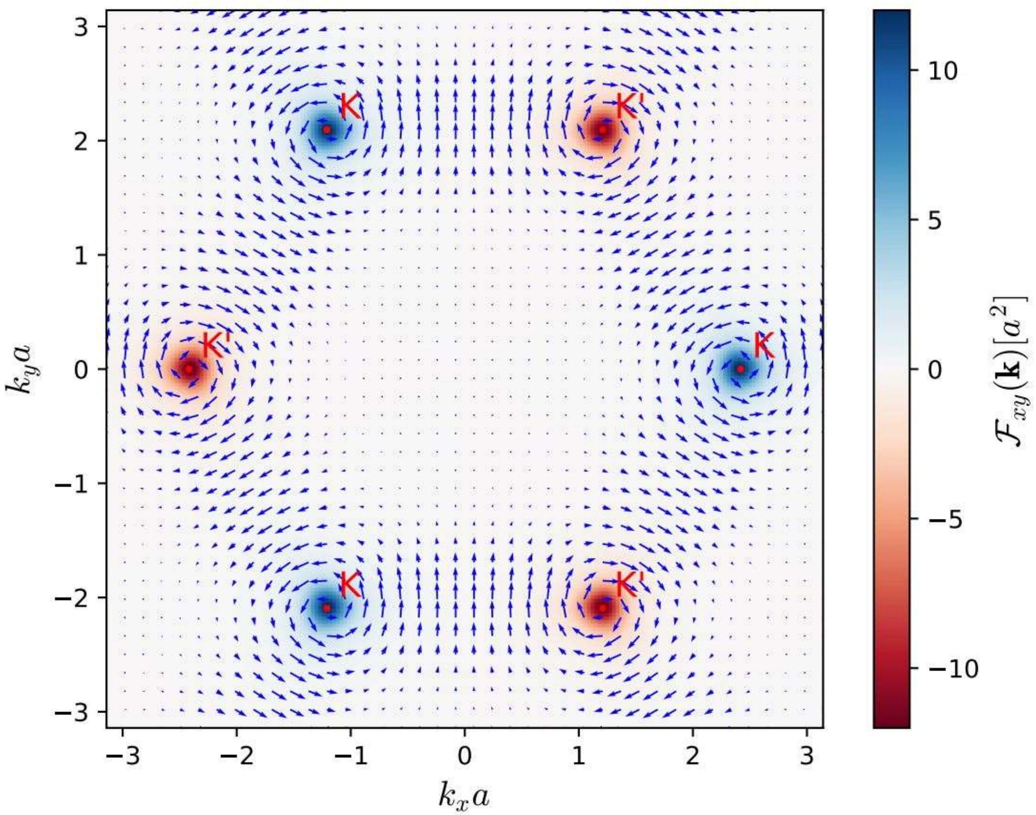
➡ $$F^{\pm}(\vec{k}) = \pm\pi\delta^2(\vec{k})$$

Berry curvature over the whole BZ $F_z^{\pm}(\mathbf{k}) = \mp\frac{1}{2h^3}\mathbf{h}\cdot\frac{\partial\mathbf{h}}{\partial k_x}\times\frac{\partial\mathbf{h}}{\partial k_y}.$

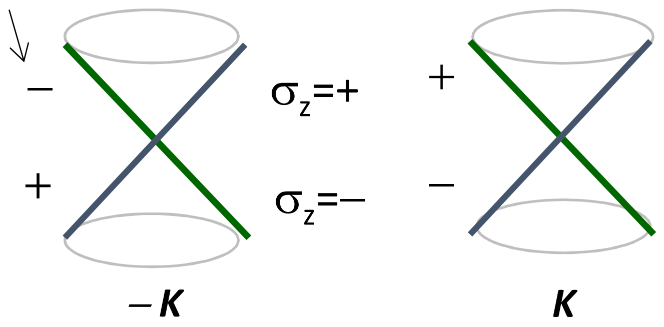
$$\Rightarrow F_z^{\pm}(\mathbf{k}) = \mp\frac{\sqrt{3}/2}{2h^3}a^2t_1^2\Delta\left(\sin\mathbf{k}\cdot\mathbf{a}_1+\sin\mathbf{k}\cdot\mathbf{a}_2+\sin\mathbf{k}\cdot\mathbf{a}_3\right)$$

Berry connection and Berry curvature

Fig from Atteia's thesis, 2018



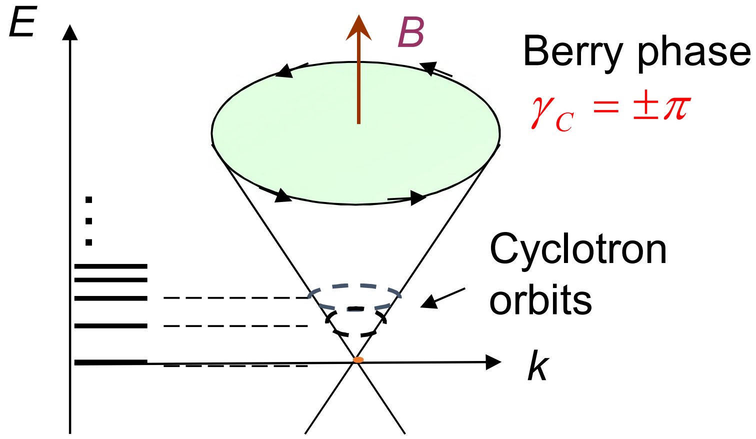
Sign of F



Cayssol and Fuchs, 2021

Cf: $F_{xy}^{\pm}(\boldsymbol{k}) = \pm a^2\frac{\sqrt{3}t^2M}{|E(\boldsymbol{k})|^3}\sin\left(\boldsymbol{k}\,.\,\frac{\boldsymbol{\delta}_2-\boldsymbol{\delta}_3}{2}\right)\sin\left(\boldsymbol{k}\,.\,\frac{\boldsymbol{\delta}_3-\boldsymbol{\delta}_1}{2}\right)\sin\left(\boldsymbol{k}\,.\,\frac{\boldsymbol{\delta}_1-\boldsymbol{\delta}_2}{2}\right)$

Graphene in magnetic field



Bohr-Sommerfeld quantization (Kittel)

$$\underbrace{\frac{1}{2}\oint_C \left(\vec{k} \times d\vec{k}\right) \cdot \hat{z}}_{\text{Orbital area}} = 2\pi\left(n + \frac{1}{2} - \frac{\gamma_C}{2\pi}\right)\frac{eB}{\hbar}$$

Correction from Berry phase

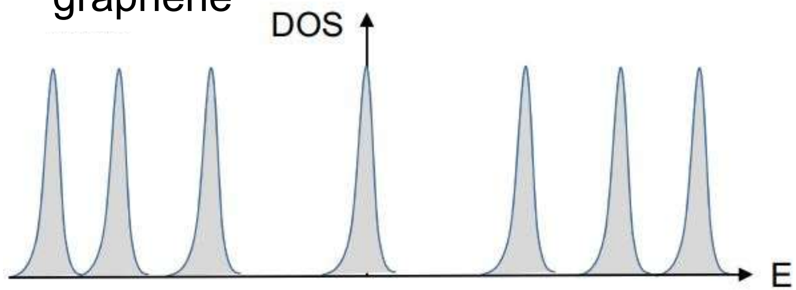
$$\Rightarrow \quad \pi k^2 = 2\pi\left(n + \cancel{\frac{1}{2}} - \cancel{\frac{\gamma_C}{2\pi}}\right)\frac{eB}{\hbar}$$

$$E(k) = \hbar v_F k$$

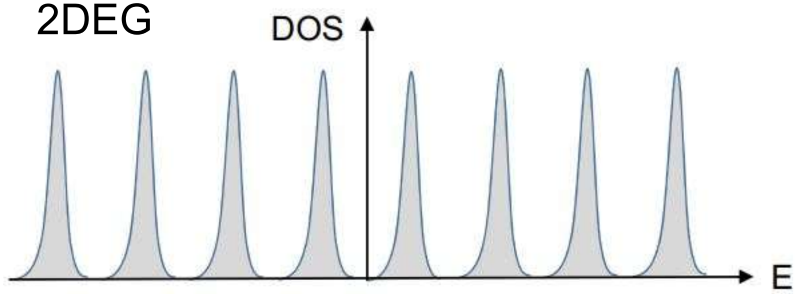
$$\Rightarrow \quad E_n = v_F\sqrt{2eB\hbar n}$$

Landau levels in graphene

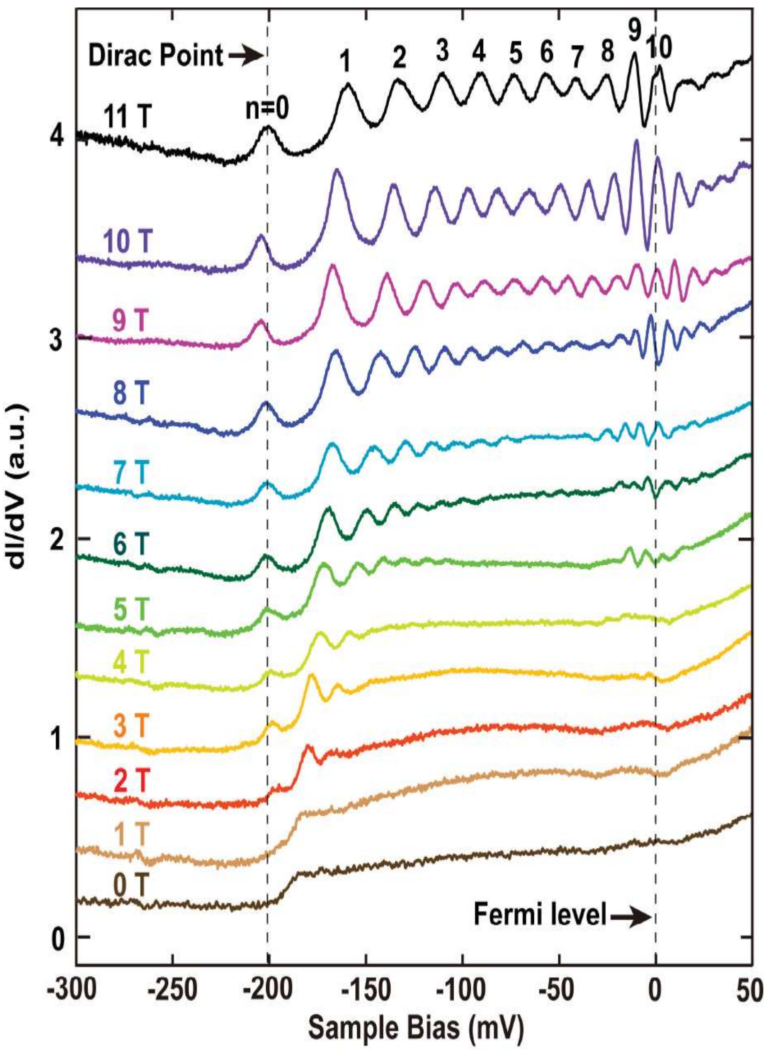
graphene



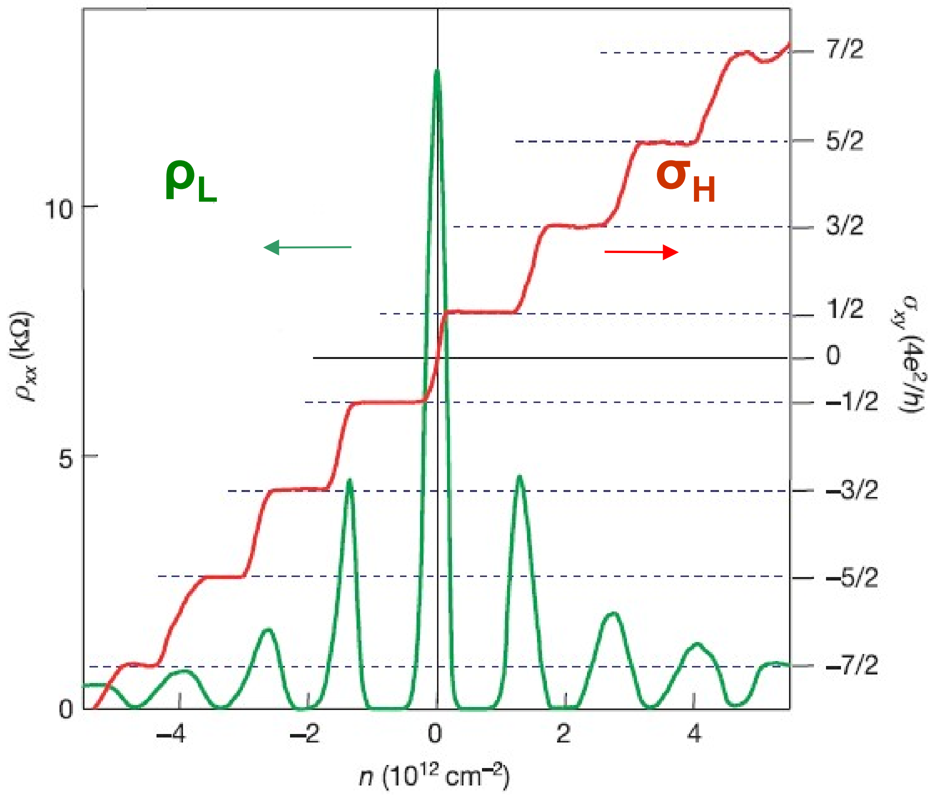
2DEG



STM experiment
(Cheng et al, PRL 2010)



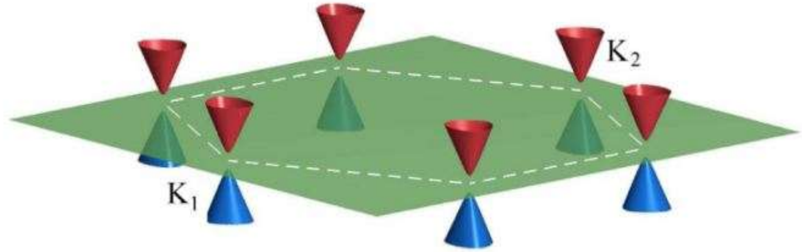
Quantum Hall effect in graphene

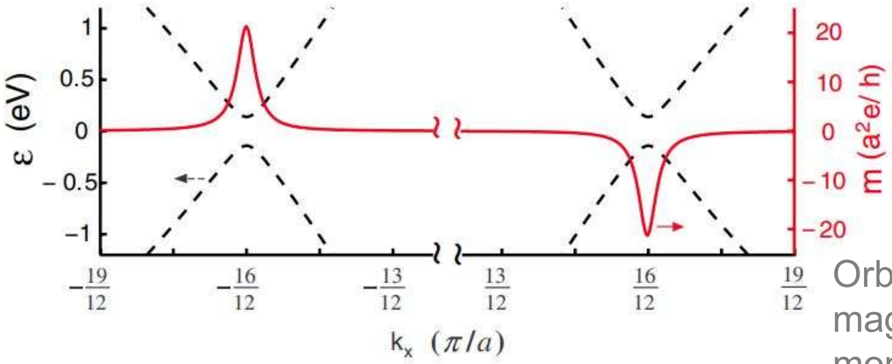


Novoselov et al, Nature 2005

Valley Hall effect (Xiao et al, PRL 2007)

B = 0



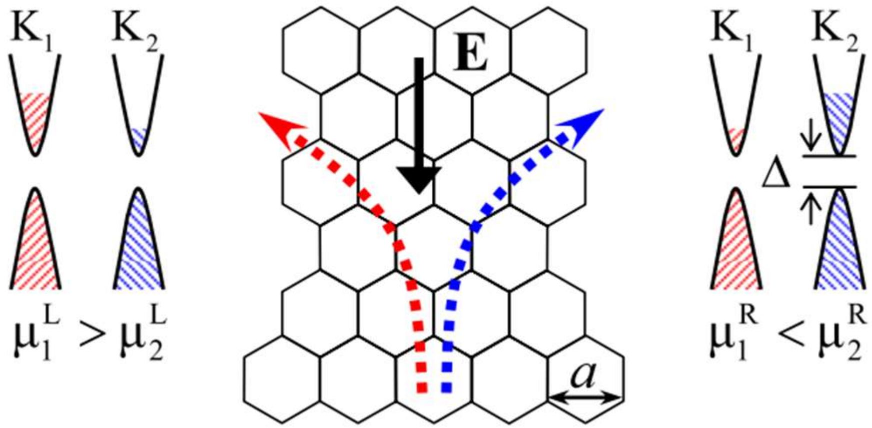


$\sim \Omega$

Orbital magnetic moment

Opposite anomalous velocities
from two valleys

$$\mathbf{v}_n(\mathbf{k}) = \frac{1}{\hbar}\frac{\partial \varepsilon_{n\mathbf{k}}}{\partial \mathbf{k}} + \boxed{\frac{e}{\hbar}\mathbf{E}\times\mathbf{F}_n(\mathbf{k})}$$



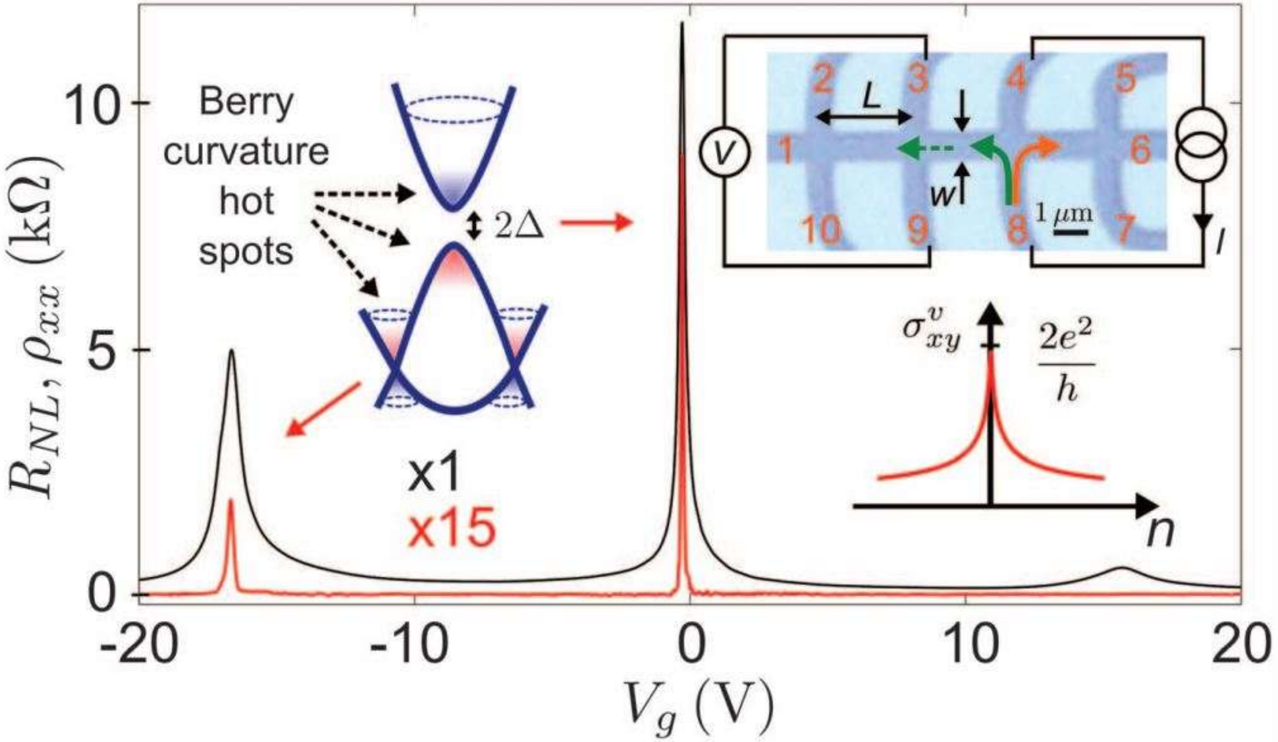
VALLEYTRONICS

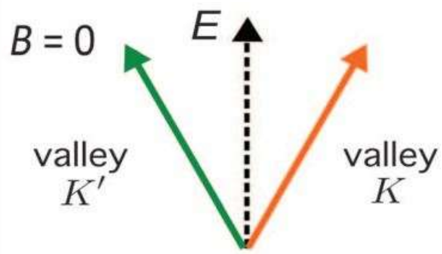
Detecting topological currents in graphene superlattices

Science, 2014

R. V. Gorbachev,[1,2*] J. C. W. Song,[3,4*] G. L. Yu,[1] A. V. Kretinin,[2] F. Withers,[2] Y. Cao,[1] A. Mishchenko,[1] I. V. Grigorieva,[2] K. S. Novoselov,[2] L. S. Levitov,[3*] A. K. Geim[1,2]†

Graphene on hBN (breaking inversion symm → 2Δ = 30 meV)



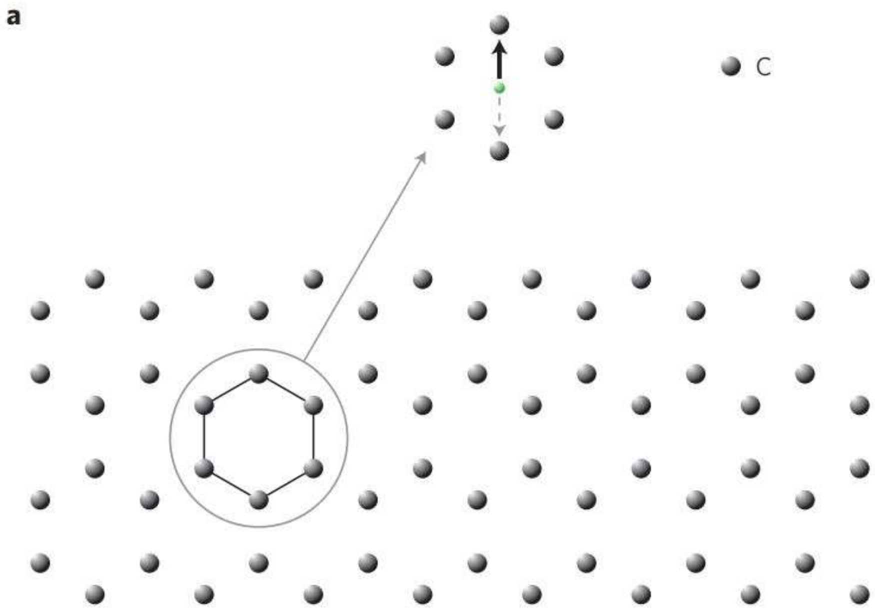


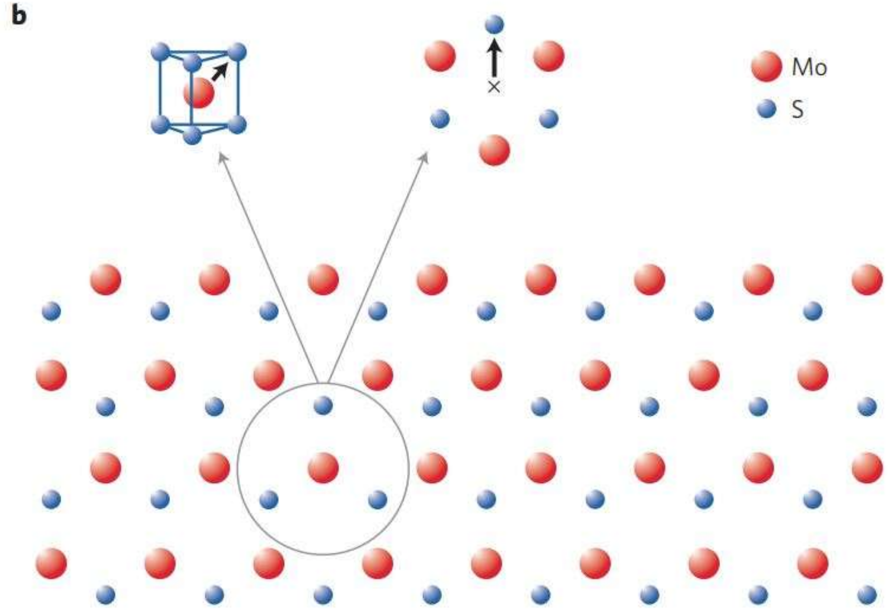
矽烯

2D material: graphene, silicene, hBN, TMD …e6c

Graphene (with SIS)

monolayer MoS_2 (without SIS)

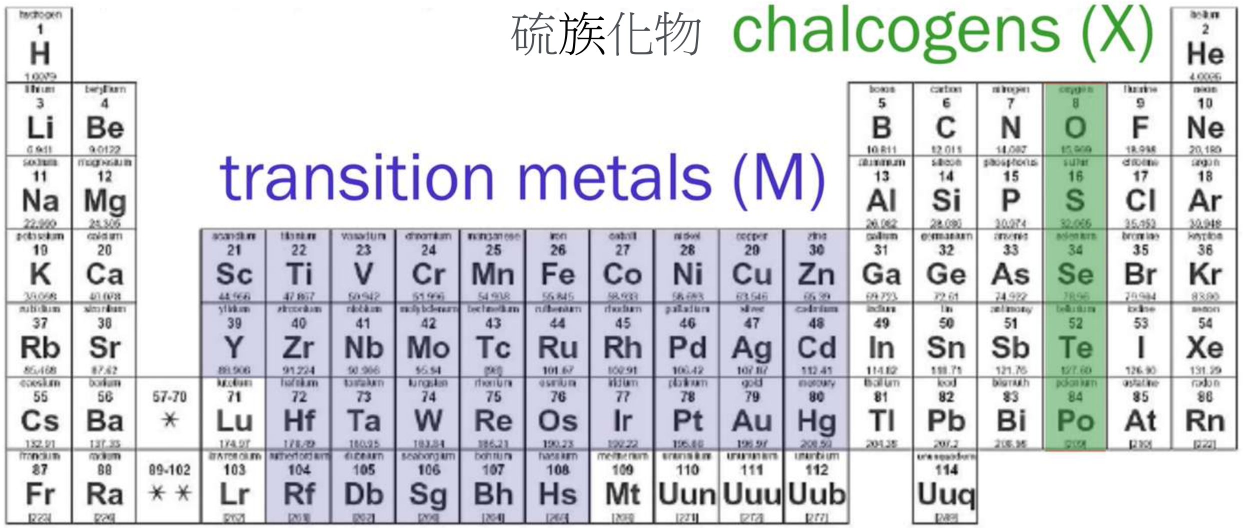




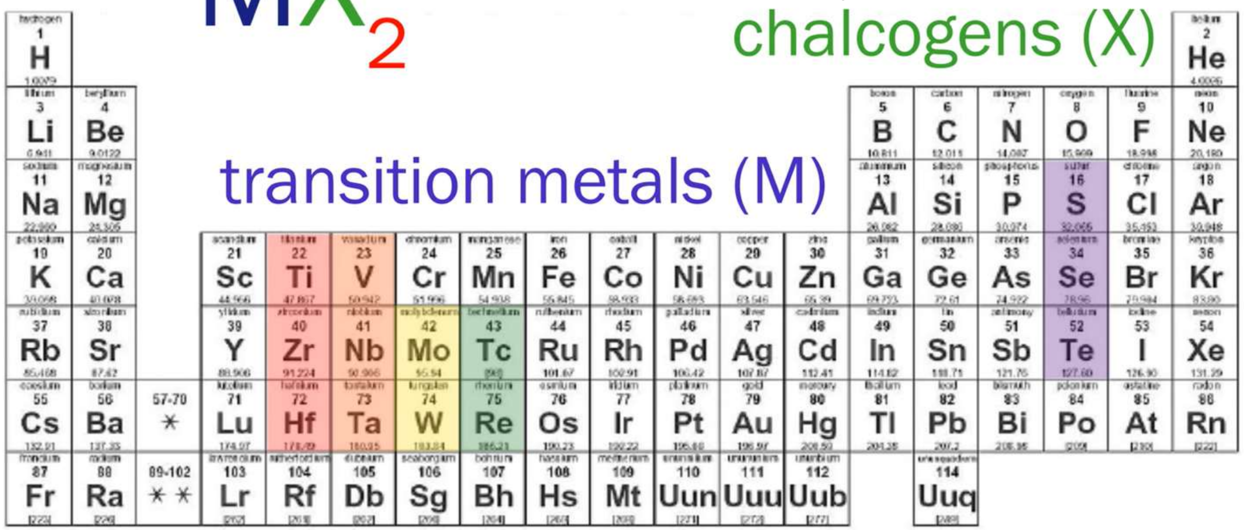
Graphene on hBN breaks SIS,
opens a gap ~ 30 mV

An energy gap ~ 1.9 eV
(easier for optical and electrical control)

2 Transition Metal Dichalcogenide (TMD) 過渡金屬二硫屬化物



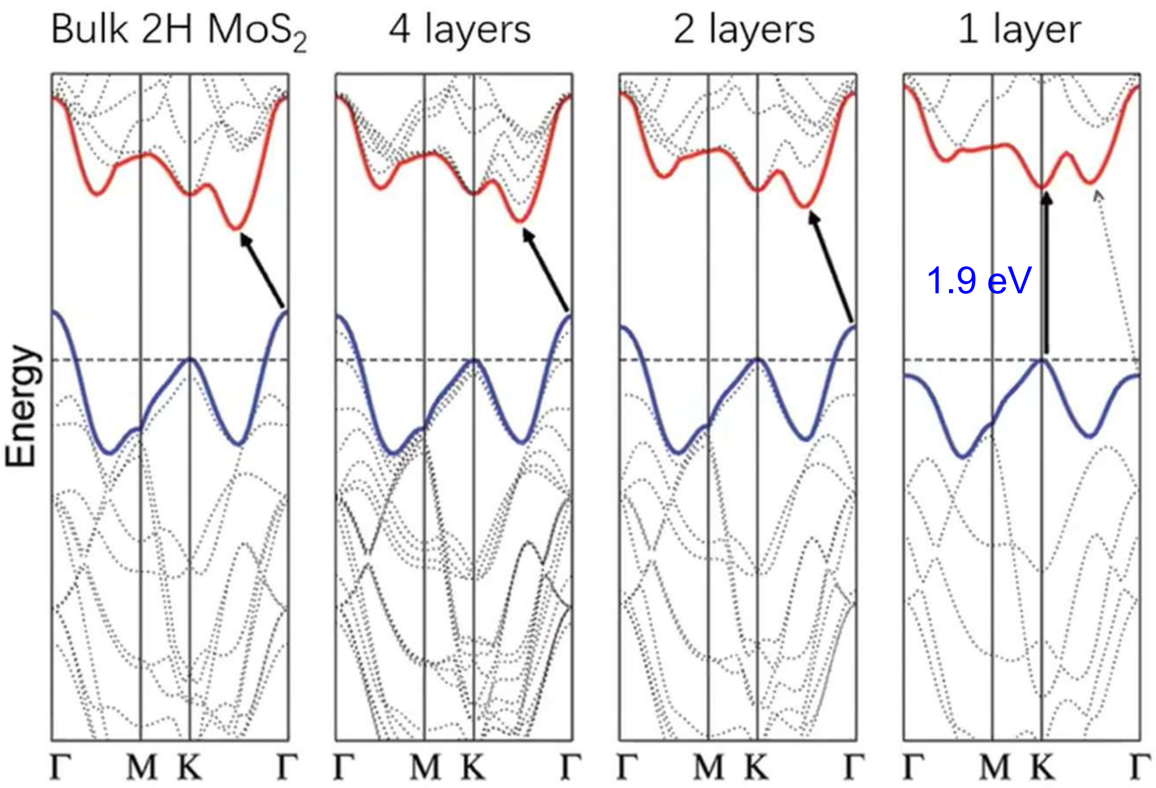
MX_2 stable as a monolayer



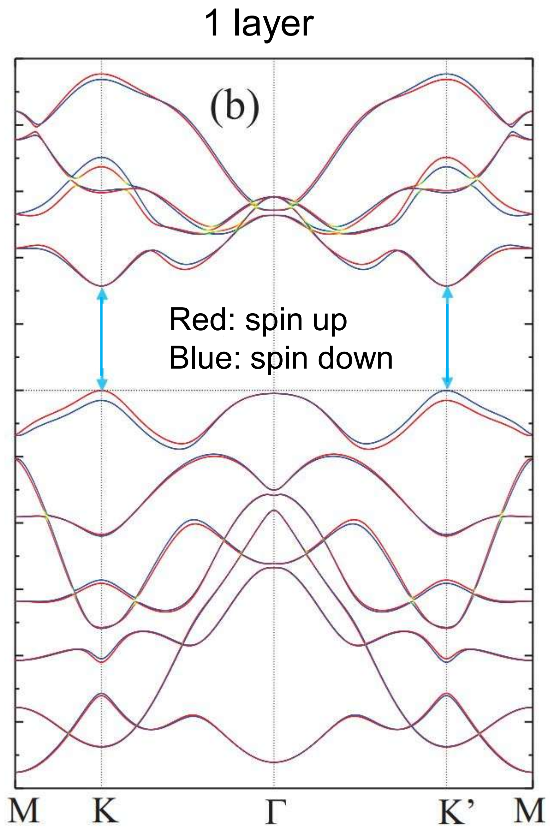
Transition Metal Dichalcogenide: MoS_2

- Bulk material indirect band gap (with SIS)

  Single layer direct band gap (without SIS) → Massive Dirac fermion
- Large SO coupling, spin-valley locking (due to Kramer degeneracy)
- Room temperature mobility 3-4 order of magnitude lower than in graphene
- Strong Coulomb interaction, excitons … etc.



Splendiani et al, Nano Lett 2010



Feng et al, PRB 2012

Valley Hall effect in a MoS_2 monolayer

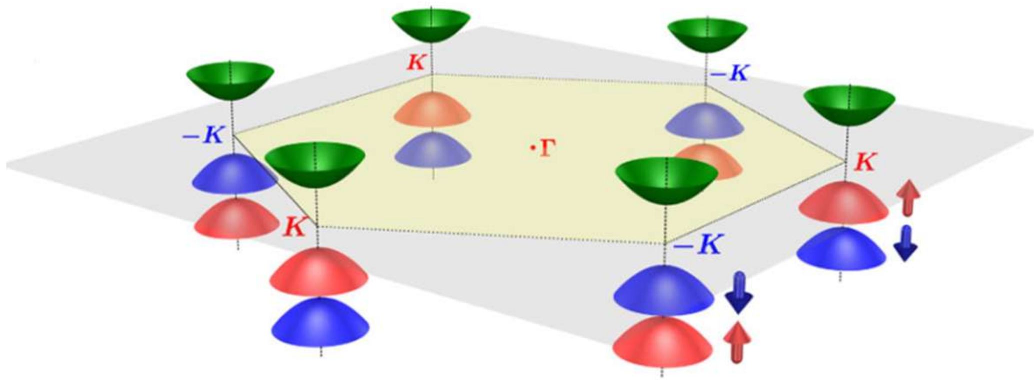


Fig from Xiao et al, PRL 2012

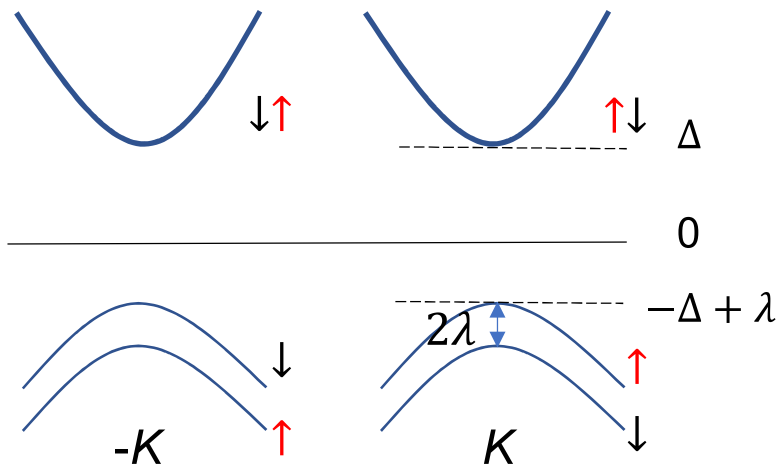
- 2-band model near massive Dirac points:

$$H = \alpha(\tau k_x \sigma_x + k_y \sigma_y) + \Delta' \sigma_z + \frac{\lambda}{2}$$

$$\Delta' \equiv \Delta - \lambda/2$$

$$\varepsilon_k^{c/v} = \pm\sqrt{\alpha^2 k^2 + \Delta'^2} + \frac{\lambda}{2}$$

- Berry curvature



$$F_c^\tau(\mathbf{k}) = \frac{\tau}{2} \frac{\alpha^2 \Delta'}{\left[\alpha^2 k^2 + \Delta'^2\right]^{3/2}}$$

Opposite valleys have opposite BCs

- For partially-filled valence band,

$$\sigma_H^\tau = \frac{1}{2\pi} \int d^2 k F_c^\tau(\mathbf{k})$$

$$= \frac{\tau}{2}\left[1 - \frac{\Delta'}{\sqrt{\alpha^2 k_F^2 + \Delta'^2}}\right]$$

VALLEYTRONICS Science, 2014

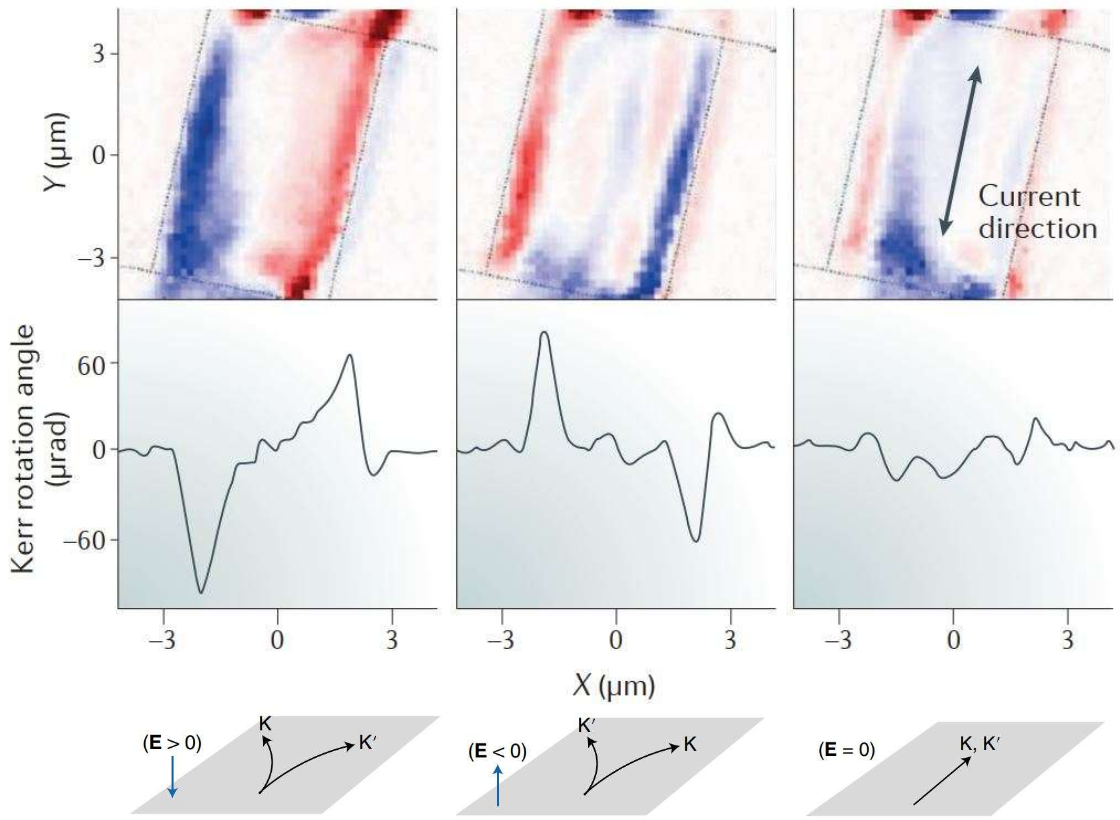
The valley Hall effect in MoS_2 transistors

K. F. Mak,[1,2]* K. L. McGill,[2] J. Park,[1,3] P. L. McEuen[1,2]*

Lee et al, Nature Nano Tech, 2016

Bilayer MoS_2 with SIS (broken by E field)

Local Kerr signal



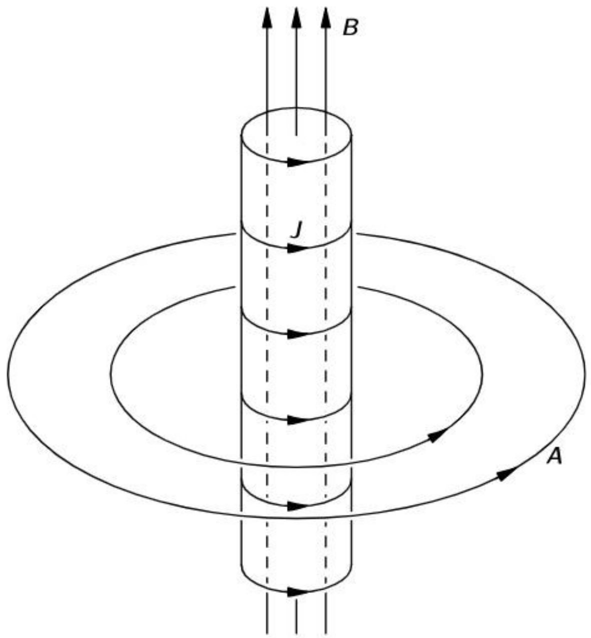
- Graphene
  - symmetries
  - Effective Hamiltonian
- Transition metal dichalcogenide
  - Berry curvature
  - Optical transitions
- Haldane model
  - Haldane flux
  - Berry curvature

First, electron hopping in a magnetic field

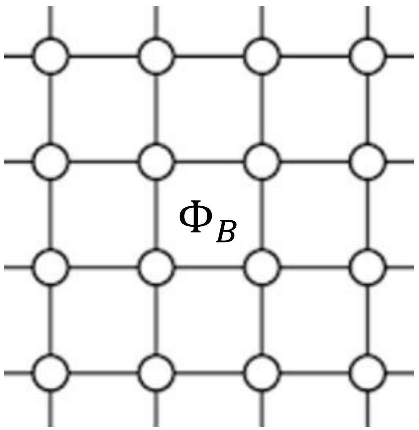
- Aharonov-Bohm (AB) phase

For a closed loop C,

$$\psi' = e^{i\frac{e}{\hbar}\oint_C d\vec{r}\cdot\vec{A}}\psi = e^{2\pi i\frac{\Phi_B}{\Phi_0}}, \;\; \Phi_0 \equiv h/e$$



- AB phase and electron hopping



3 Haldane's graphene model, PRL 1989

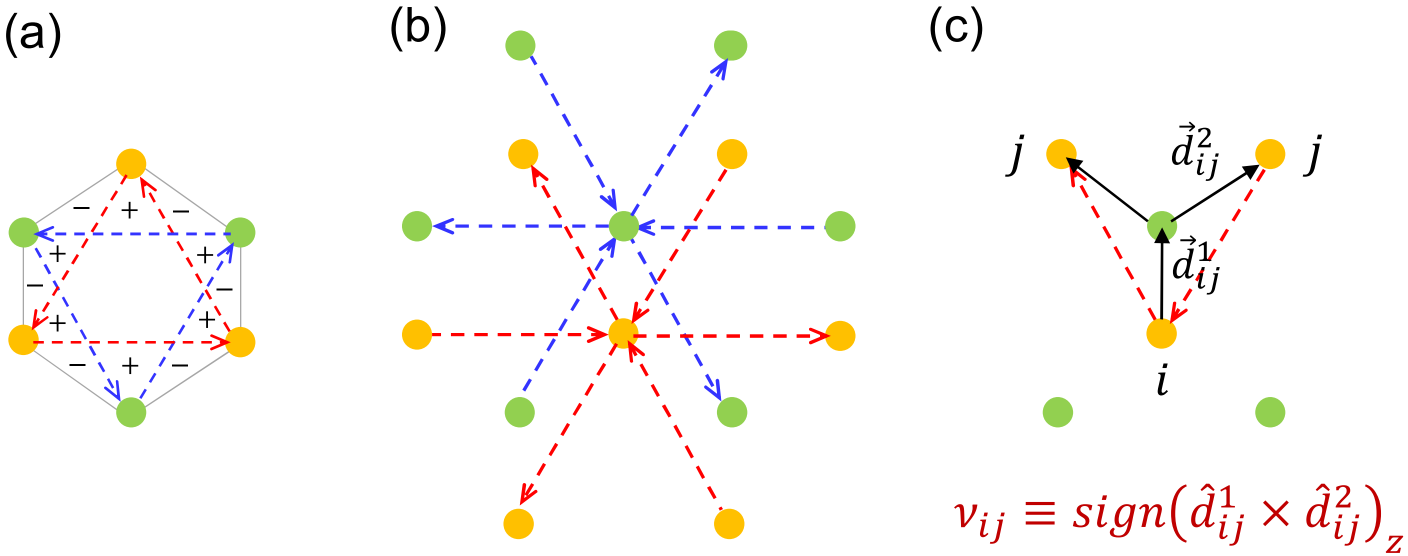
(no net magnetic field, but TRS is broken)

$$H = -t_1 \sum_{<i,j>} c_i^+ c_j + \Delta \sum_i \xi_i c_i^+ c_i + t_2 \sum_{<<i,j>>} e^{i\nu_{ij}\phi} c_i^+ c_j \quad \text{NNN}$$

^ break SIS ^ break TRS

As a result of Haldane flux

$$\begin{aligned}
\mathbf{R}_A &\to \mathbf{R}_A + \mathbf{a}_i \text{ get } e^{-i\phi}, \quad i = 1,2,3 \\
\mathbf{R}_A &\to \mathbf{R}_A - \mathbf{a}_i \text{ get } e^{+i\phi}, \\
\mathbf{R}_B &\to \mathbf{R}_A + \mathbf{a}_i \text{ get } e^{+i\phi}, \\
\mathbf{R}_B &\to \mathbf{R}_A - \mathbf{a}_i \text{ get } e^{-i\phi}.
\end{aligned}$$



$$\nu_{ij} \equiv sign\left(\hat{d}_{ij}^{1} \times \hat{d}_{ij}^{2}\right)_z$$

NNN coupling

$$
\begin{aligned}
\hat{H}_{NNN} &= t_2 \sum_{\mathbf{R}} \sum_{i=1}^{3} \left(c^{\dagger}_{\mathbf{R}+\mathbf{a}_i} c_{\mathbf{R}} e^{-i\phi} + h.c. \right) \\
&+ t_2 \sum_{\mathbf{R}} \sum_{i=1}^{3} \left({}^{\dagger}_{\mathbf{R}+\mathbf{a}_i} d_{\mathbf{R}} e^{+i\phi} + h.c. \right) . \\
&= 2t_2 \sum_{\mathbf{k}} \sum_{i} \left[\cos(\mathbf{k}\cdot\mathbf{a}_i + \phi) c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} + \cos(\mathbf{k}\cdot\mathbf{a}_i - \phi) d^{\dagger}_{\mathbf{k}} d_{\mathbf{k}} \right] \\
&= \sum_{\mathbf{k}} \left(c^{\dagger}_{\mathbf{k}}, d^{\dagger}_{\mathbf{k}} \right) \mathsf{H}_{NNN}(\mathbf{k}) \begin{pmatrix} c_{\mathbf{k}} \\ d_{\mathbf{k}} \end{pmatrix}
\end{aligned}
$$

$$
\mathsf{H}_{NNN}(\mathbf{k}) = 2t_2 \begin{pmatrix} \sum_i \cos(\mathbf{k}\cdot\mathbf{a}_i + \phi) & 0 \\ 0 & \sum_i \cos(\mathbf{k}\cdot\mathbf{a}_i - \phi) \end{pmatrix}
$$

$$
\mathsf{H}(\mathbf{k}) = \frac{h_{11} + h_{22}}{2} + \mathbf{h}\cdot\boldsymbol{\sigma}
$$

$$
\mathbf{h} = \left(t_1 \sum_i \cos \mathbf{k}\cdot\boldsymbol{\delta}_i, -t_1 \sum_i \sin \mathbf{k}\cdot\boldsymbol{\delta}_i, \frac{h_{11} - h_{22}}{2} + \Delta \right)
$$

$$
= -2t_2 \sum_i \sin \mathbf{k}\cdot\mathbf{a}_i \sin\phi + \Delta
$$

Near Dirac point

$$h_3(\mathbf{k}) = \pm 3\sqrt{3} t_2 \sin\phi + \Delta + O(k^2).$$

let $h_3 = m_\tau(\phi, \Delta)$ a ϕ-dependent effective mass

$$\mathrm{H}_\tau(\mathbf{k}) = \frac{h_{11} + h_{22}}{2} + \begin{pmatrix} m_\tau & \hbar v_F(\tau k_x - i k_y) \\ \hbar v_F(\tau k_x + i k_y) & -m_\tau \end{pmatrix}$$

$$F^{+}_{z\tau}(\mathbf{k}) = \frac{\tau}{2} \frac{\hbar^2 v_F^2 m_\tau}{(\hbar^2 v_F^2 k^2 + m_\tau^2)^{3/2}}$$

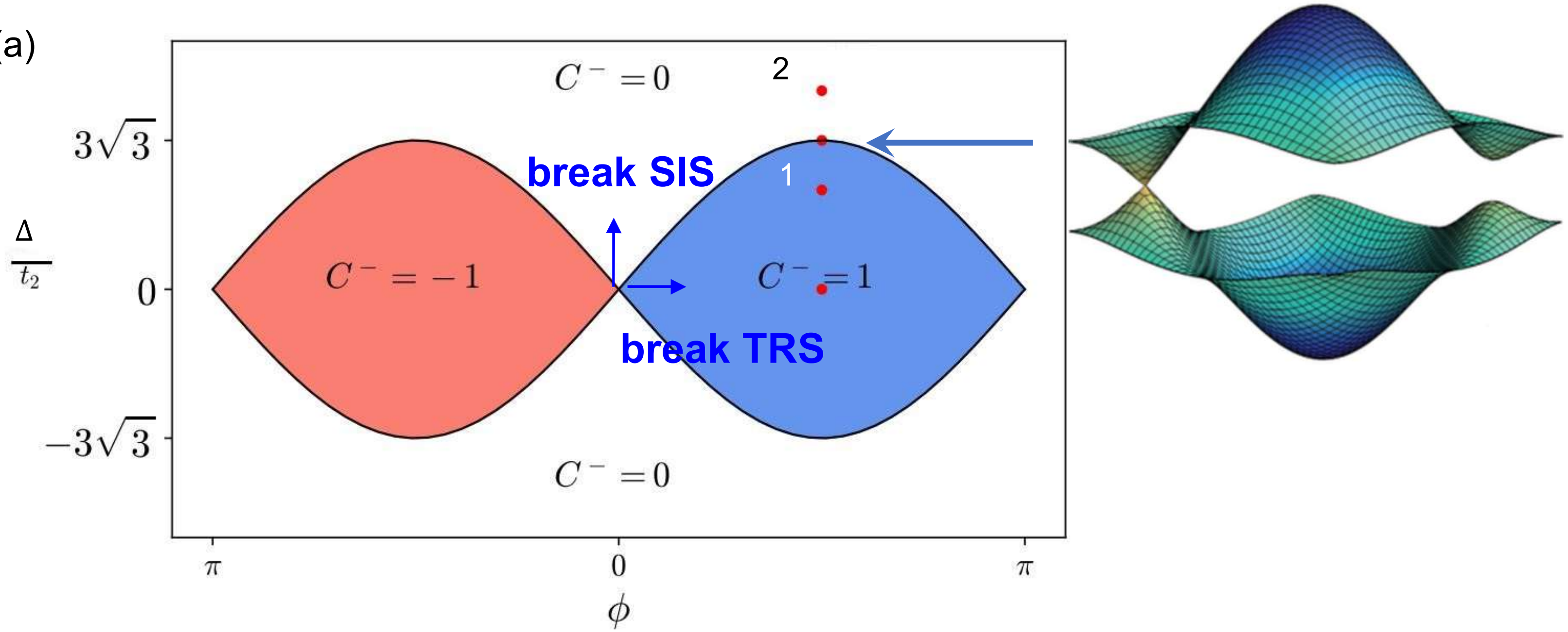
The Dirac gap is closed when

$$m_\pm(\phi, \Delta) = \pm 3\sqrt{3} t_2 \sin\phi + \Delta = 0$$

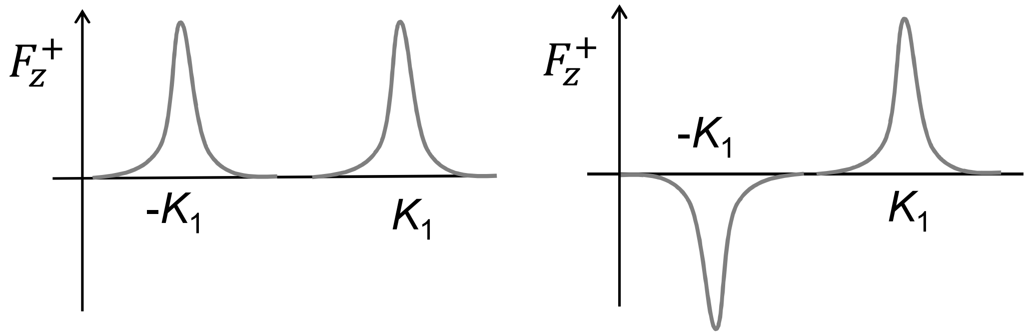
Phase diagram Zero-field quantum Hall effect phases ($\nu = \pm 1$, where $\sigma^{xy} = \nu e^2/h$) occur if $|M/t_2| < 3\sqrt{3}|\sin\phi|$.

Fig from Atteia's thesis, 2018

(a)

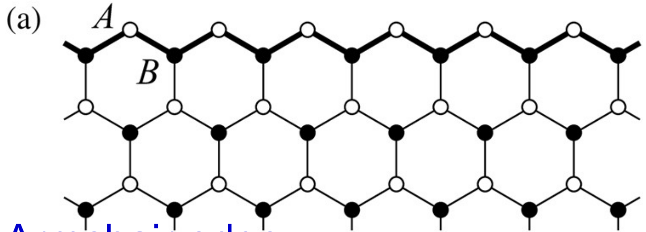


(b) Berry curvatures of point 1 (left) and 2 (right)

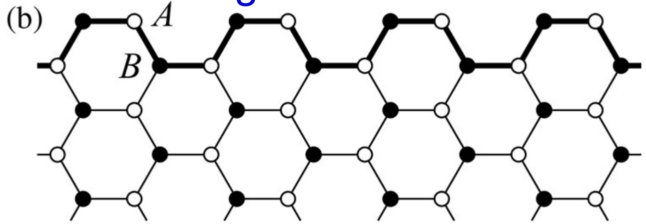


Edge state in different graphene models (for zigzag edge)

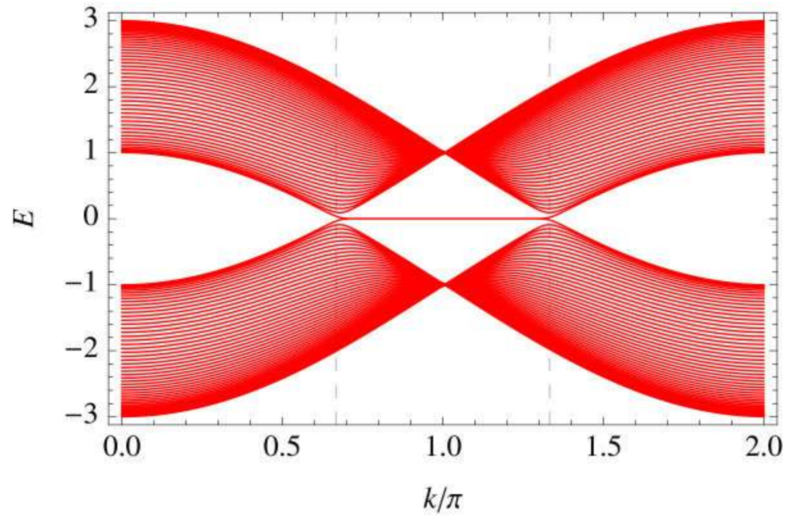
Zigzag edge

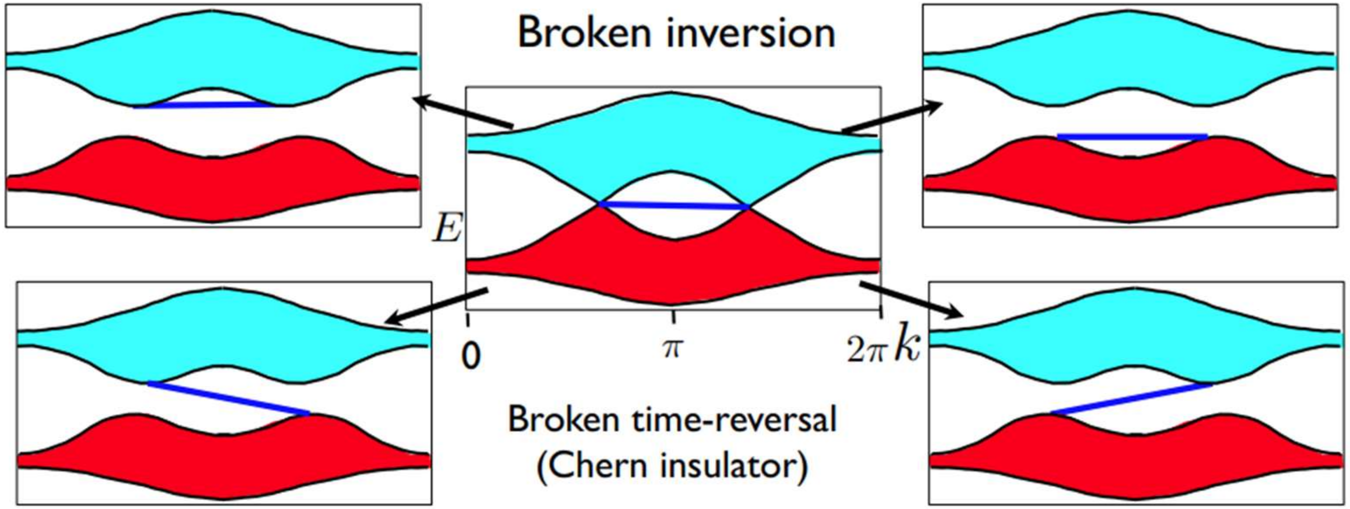


Armchair edge



Edge states for zigzag edge (Fujita, 1996)





Haldane, Nobel lecture