Quantum Hall effect

Examples of Macroscopic Quantum Phenomena

- Superconductivity (Onnes, 1911)
- Superfluidity (Kapitsa, 1937)
- Quantum Hall effect (von Klitzing, 1980) < room temperature possible
- Bose-Einstein condensation (Cornell and Wieman, 1995)
- …

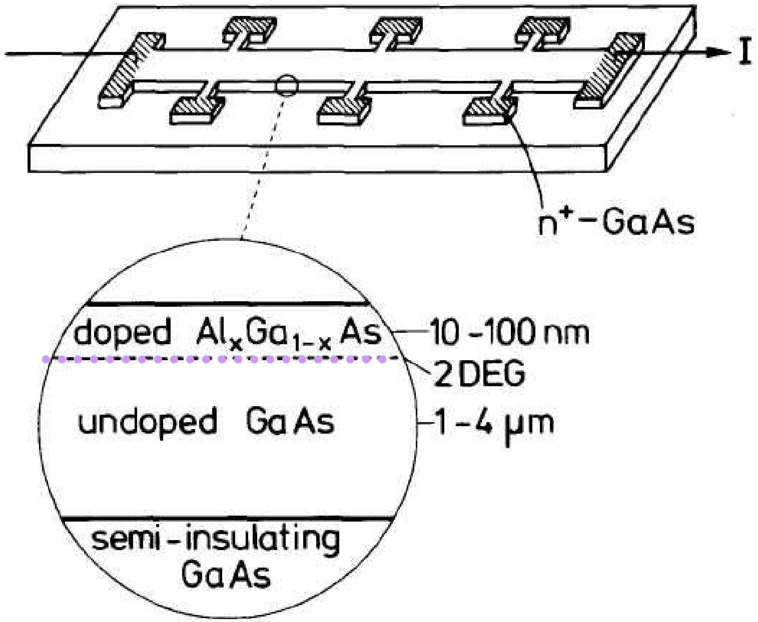
To observe IQHE, we need

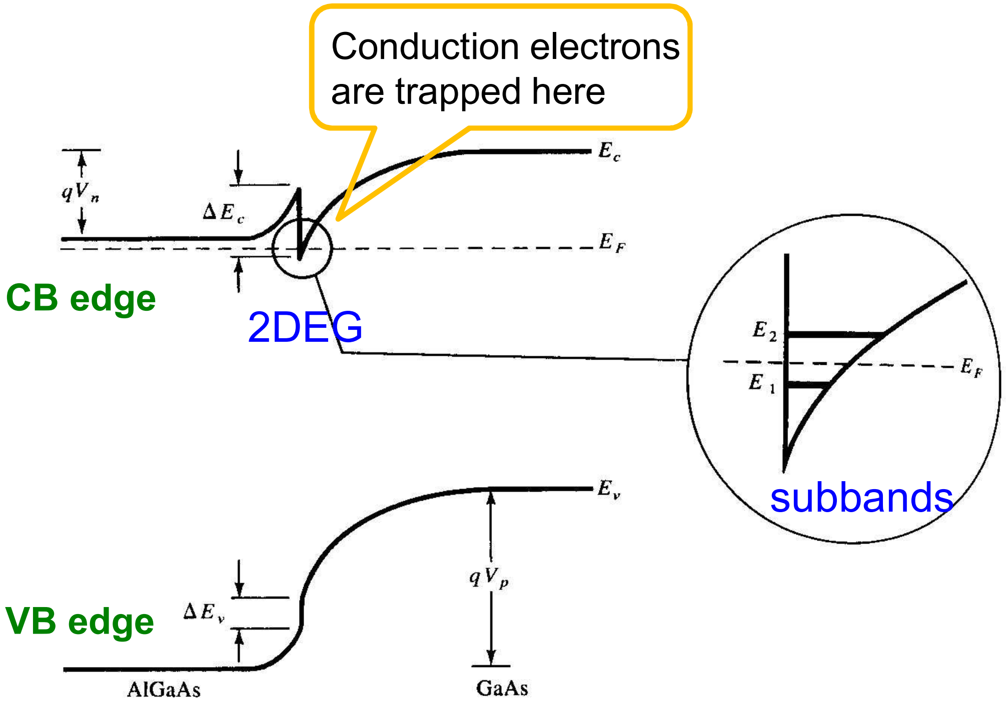
- Two-dimensional electron system
- Breaking time-reversal symmetry (with ***B*** or ***M***)
- Filled energy bands (insulator) with non-zero Chern numbers
  - Landau levels (IQHE)
  - Bands with magnetization (QAHE) ← next chap

(Low temp and high *B* field are usually required)

2-dimensional electron gas (2DEG)

GaAs/AlGaAs heterojunction





- At low T ($kT << E_2 - E_1$), the dynamics along the z-direction is frozen in the ground state → 2DEG

Other 2-dimensional electron systems

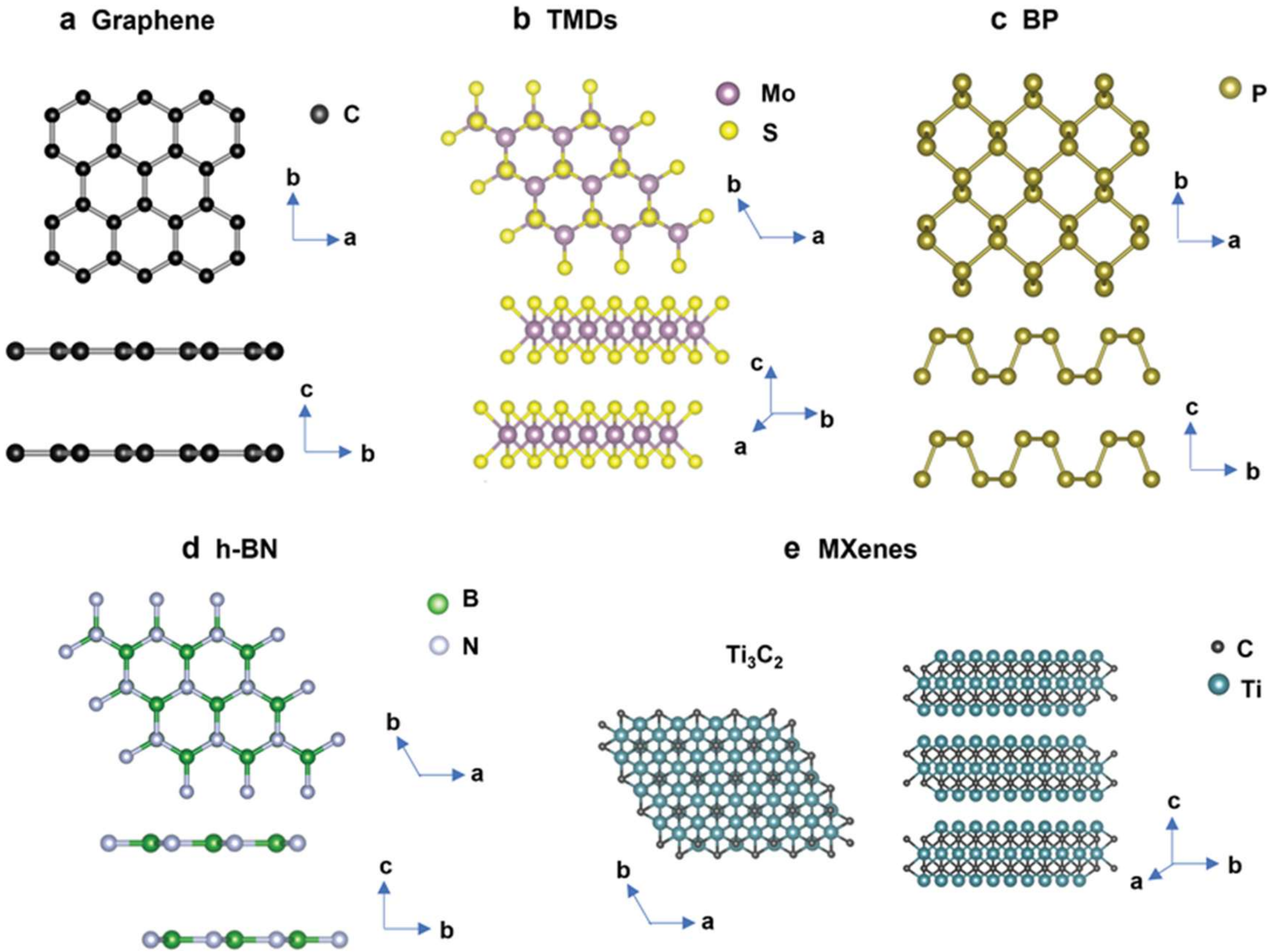
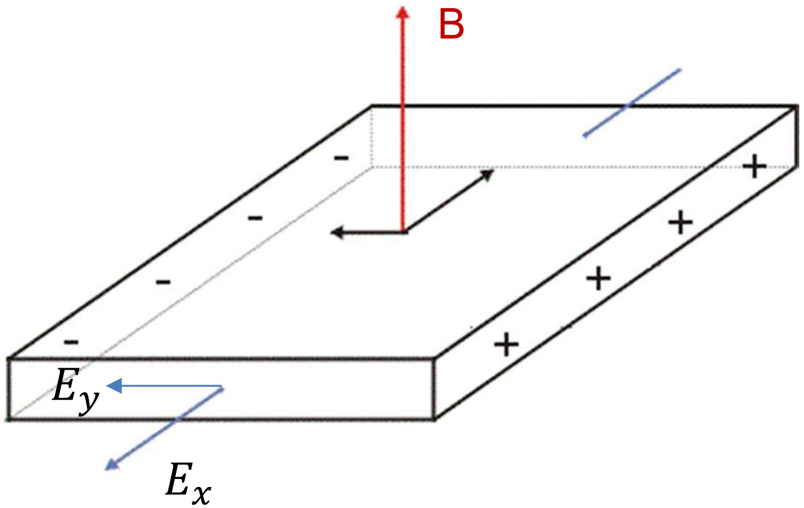


Fig from Chen et al, Small Methods 2023

Classical Hall effect (E. Hall, 1879)

$$m^*\frac{d\vec{v}}{dt} = -e\vec{E} - e\vec{v}\times\vec{B} - m^*\frac{\vec{v}}{\tau}$$

$$\vec{B} = B\hat{z};\ d\vec{v}/dt = \vec{0} \text{ at steady state}$$

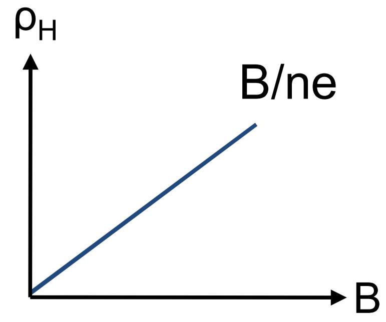


$$\Rightarrow \begin{pmatrix} m^*/\tau & eB \\ -eB & m^*/\tau \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -e\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

Current density $\vec{j} = -en\vec{v}$

$$\Rightarrow \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_0 & \frac{B}{ne} \\ -\frac{B}{ne} & \rho_0 \end{pmatrix}\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \rho_0\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}\begin{pmatrix} J_x \\ J_y \end{pmatrix}$$

Hall resistivity



(longitudinal) resistivity $\rho_0 = \frac{m^*}{ne^2\tau},\ \omega_c = \frac{eB}{m^*}$ Cyclotron frequency

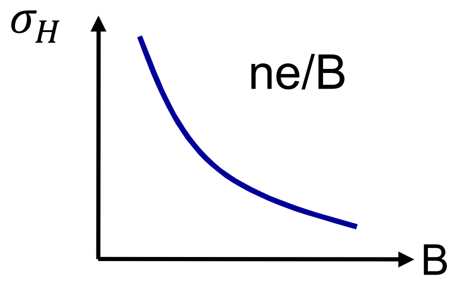
Alternatively,

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \frac{\sigma_0}{1+(\omega_c\tau)^2}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad \sigma_0 = \frac{ne^2\tau}{m^*}$$

if $\omega_c\tau \gg 1$, then

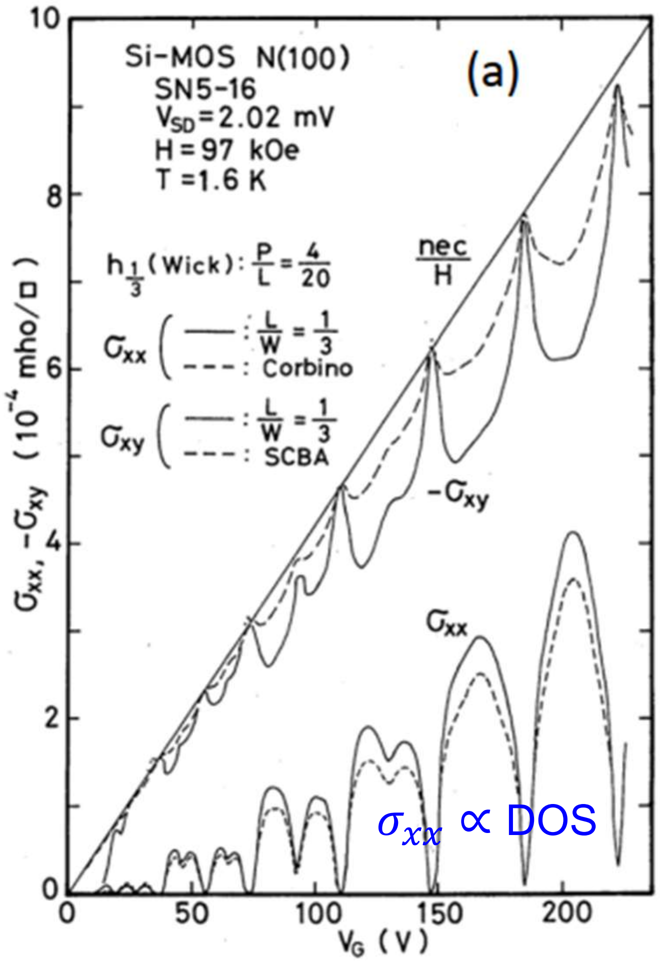
$$\boldsymbol{\sigma} \simeq \begin{pmatrix} \frac{\sigma_0}{(\omega_c\tau)^2} & -\frac{ne}{B} \\ \frac{ne}{B} & \frac{\sigma_0}{(\omega_c\tau)^2} \end{pmatrix}$$

Hall conductibity



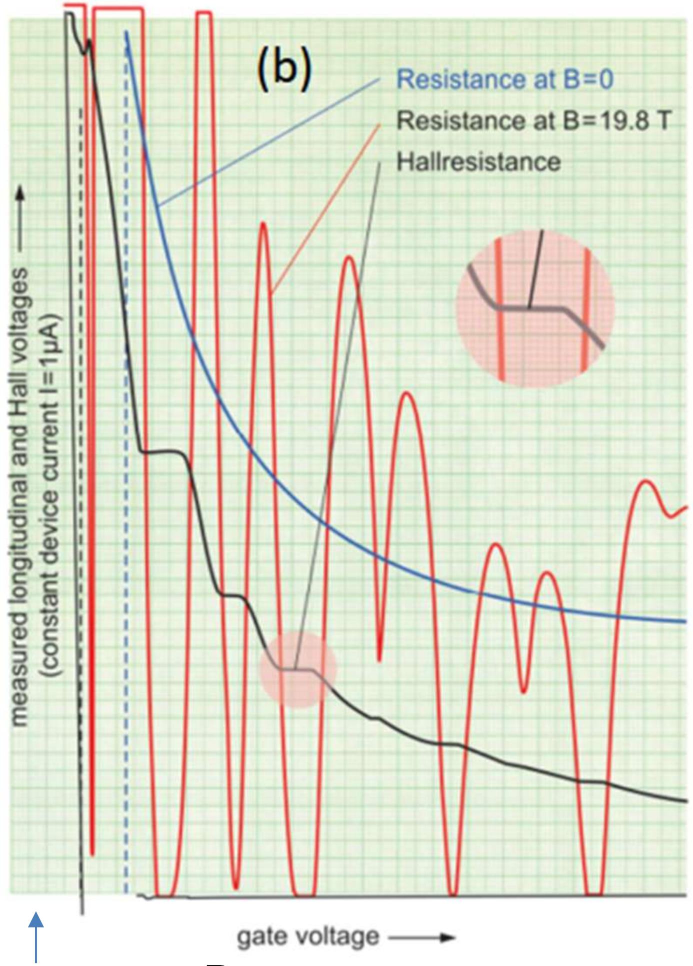
Note that for $\omega_c\tau \gg 1$, $\rho_L = 0$ and $\sigma_L = 0$ (QH insulator)

Hall conductivity (and resistivity) in a strong magnetic field



Bias potential $\sim n$

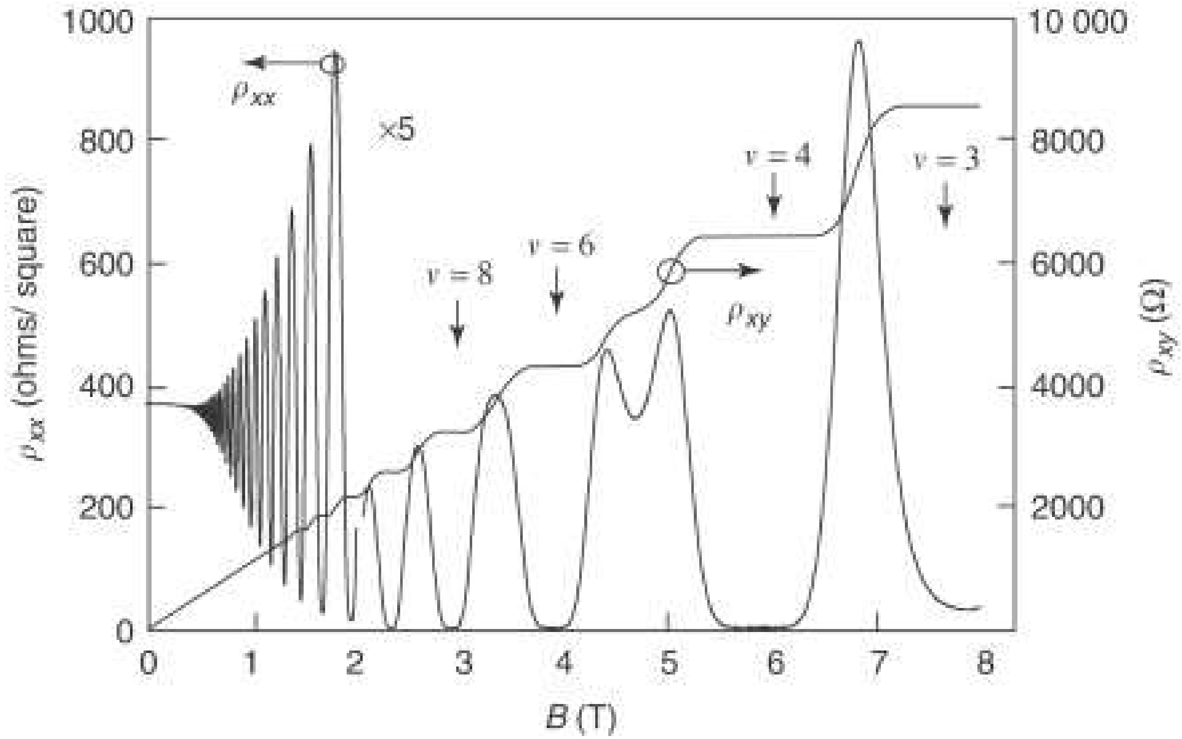
The first lab record that shows quantum Hall plateaus (von Klitzing 1979)



$$V_H \sim \rho_H \sim \frac{B}{ne}$$

Wakabayashi and Kawaji (1978)

[Integer] Quantum Hall effect (von Klitzing, 1980)




1985

Hall resistivity and Hall conductivity at plateaus

$$\rho_H = \frac{1}{n}\frac{h}{e^2}$$

$$\sigma_H = n\frac{e^2}{h}$$

h/e^2 = 25.81280745 k-ohm accurate to 10^{-9},
it became a defined value after 1990

fine structure constant, $\alpha \equiv e^2/4\pi\varepsilon_0\hbar c$.
[Note: After 2019, the values of e, h, and c are defined, and only ε_0 is uncertain.]

[Fractional] Quantum Hall effect (Tsui, Stormer, and Gossard, 1982)

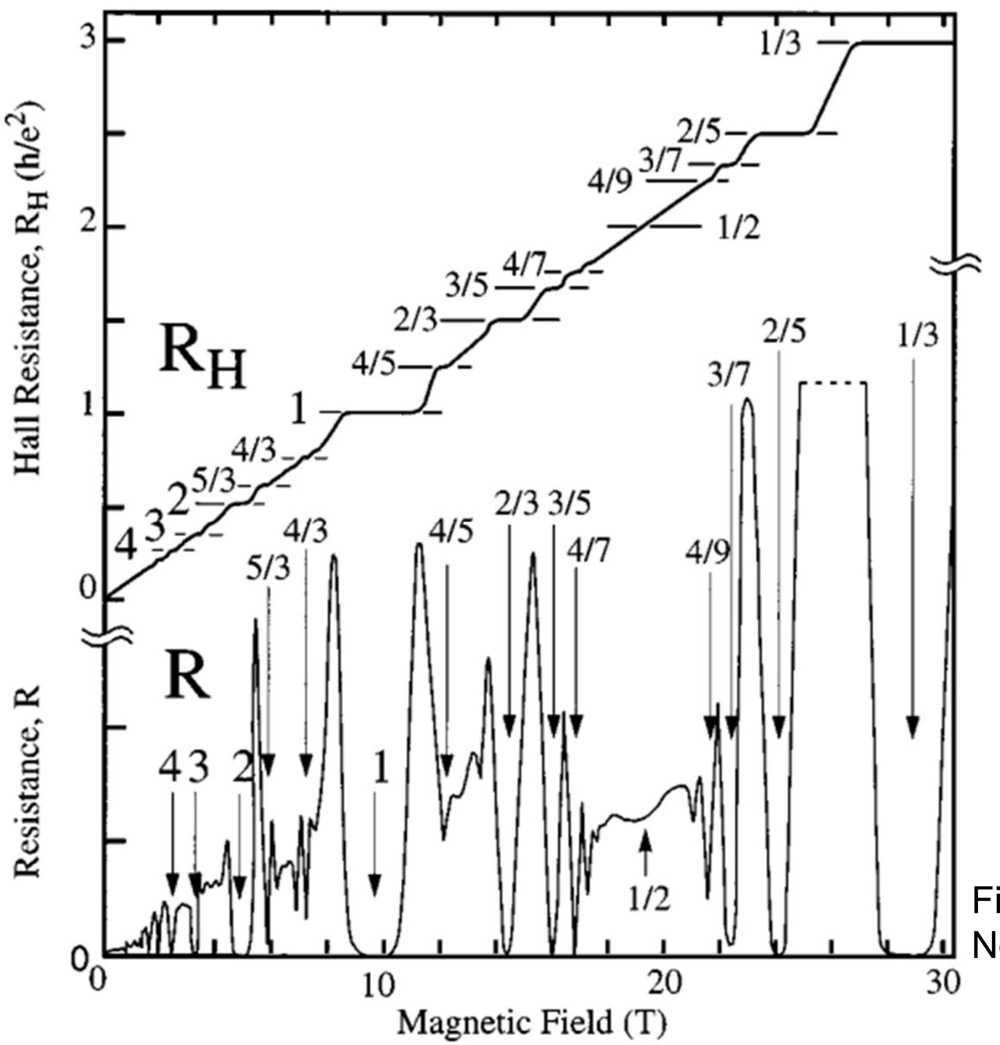
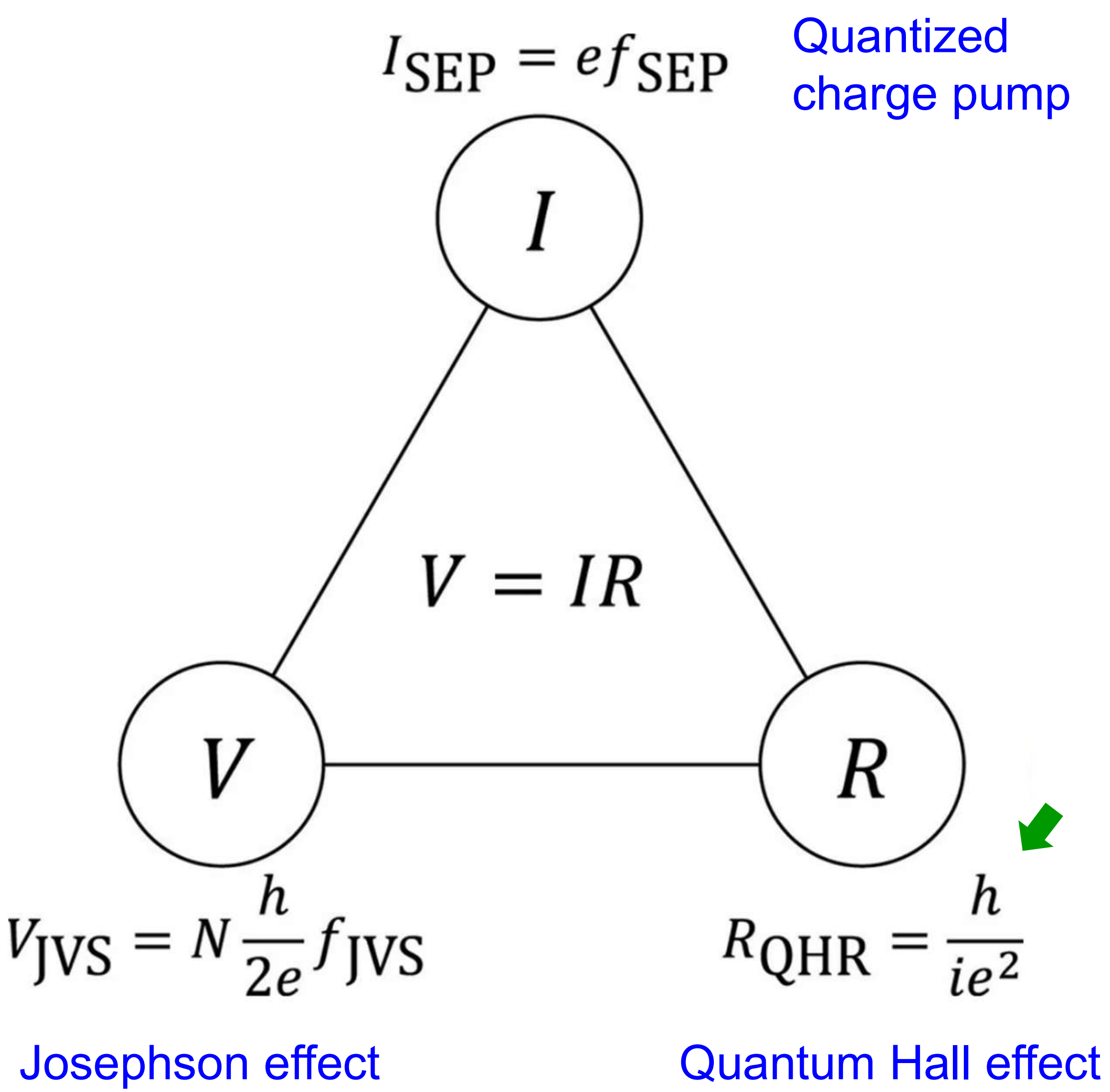


Fig from Stormer, Nobel Lecture

Fractional QHE has plateaus at fractional *n*.
Its theory is very different from the one below for IQHE.

Quantum metrology triangle



2DEG in a magnetic field

$$H = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m^*} - \boldsymbol{\mu} \cdot \mathbf{B}, \;\; \boldsymbol{\mu} = \frac{g^* q}{2m^*}\mathbf{S} \qquad q = -e$$

2 popular choices:

Linear gauge: $\mathbf{A}(\mathbf{r}) = B(0, x, 0)$,

Circular gauge: $\mathbf{A}(\mathbf{r}) = B(-\frac{y}{2}, \frac{x}{2}, 0)$.

1. Linear gauge

$$\Rightarrow \quad H = \frac{p_x^2}{2m} + \frac{(p_y + eBx)^2}{2m} + \mu_s B, \;\; \mu_s = \frac{ge\hbar}{2m} s, s = \pm\frac{1}{2}.$$

let $\phi(x, y) = \phi(x) e^{ik_y y}$

$$\Rightarrow \quad \left[\frac{p_x^2}{2m} + \frac{(\hbar k_y + eBx)^2}{2m} + \mu_s B\right]\phi(x) = \varepsilon\phi(x)$$

$$\left[\frac{p_x^2}{2m} + \frac{m}{2}\omega_c^2 (x - x_0)^2\right]\phi(x) = \bar{\varepsilon}_s \phi(x), x_0 \equiv -\frac{\hbar k_y}{eB},$$

$$\text{where } \bar{\varepsilon}_s \equiv \varepsilon - \mu_s B.$$

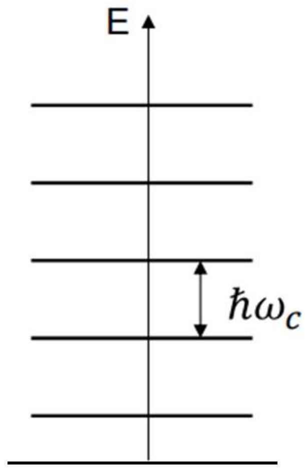
~ differential eq. for a simple harmonic oscillator located at x_0.

Landau levels (LLs) (see, e.g., Shankar QM)

$$\begin{cases} \varepsilon_{ns} = \left(n + \frac{1}{2}\right)\hbar\omega_c + \mu_s B, \\ \phi_n(x) = N_n H_n\left(\frac{x - x_0}{\lambda}\right) e^{-\frac{1}{2}\left(\frac{x-x_0}{\lambda}\right)^2} \end{cases}$$

Landau levels

Landau orbitals

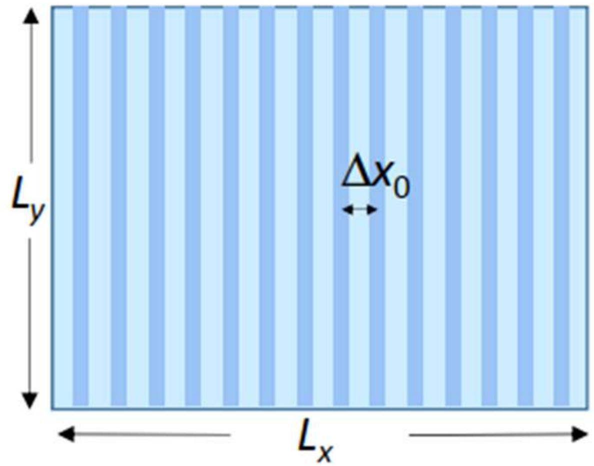


Magnetic length $\lambda = \sqrt{\hbar/m\omega_c} = \sqrt{\hbar/eB}$

$B = 1\,T, \lambda \simeq 256\,A$

$$\Delta k_y = 2\pi/L_y$$

$$\Delta x_0 = \frac{\hbar}{eB}\Delta k_y = \frac{\hbar}{eB}\frac{2\pi}{L_y}$$



Degeneracy of one LL

$$D = \frac{L_x}{\Delta x_0} = \frac{L_x L_y B}{h/e} = \frac{\Phi_B}{\phi_0}$$

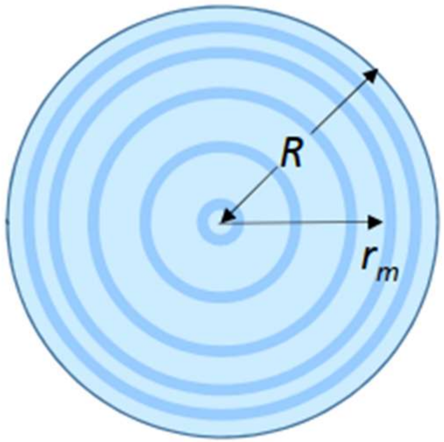
B=0.5 gauss, L^2=1 cm^2 → $D \simeq 10^6$

2. Circular gauge (see latex note for details)

$$H = \frac{\left(p_x - \frac{eB}{2}y\right)^2}{2m} + \frac{\left(p_y + \frac{eB}{2}x\right)^2}{2m} + \mu_s B$$

$$\psi(r,\phi) = u(r)e^{im\phi}$$

$$\Rightarrow \begin{cases} \varepsilon_{ns} = \left(n + \frac{1}{2}\right)\hbar\omega_c + \mu_s B \\ u_n(\tilde{r}) = N_n^{|m|}\tilde{r}^{|m|}e^{-\frac{\tilde{r}^2}{4}}L_n^{|m|}\left(\frac{\tilde{r}^2}{2}\right) \end{cases} \qquad \tilde{r} \equiv \frac{r}{\lambda}$$

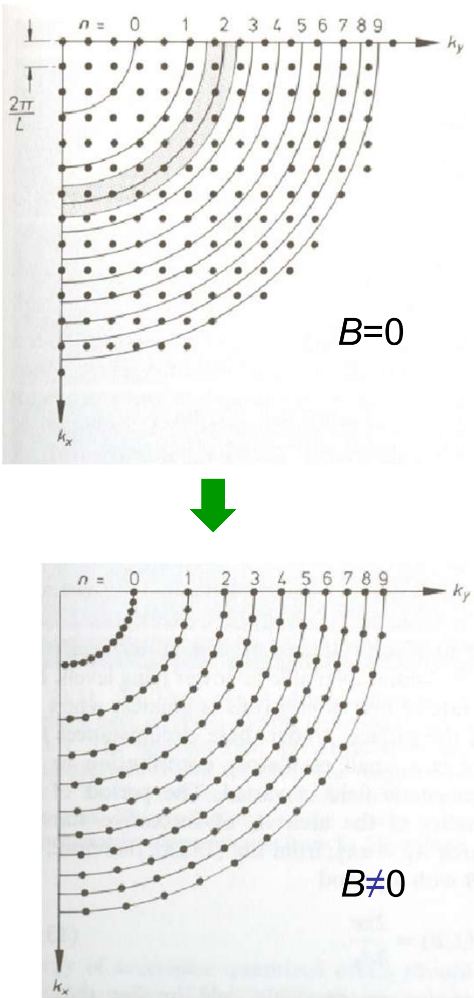


Degeneracy of one LL

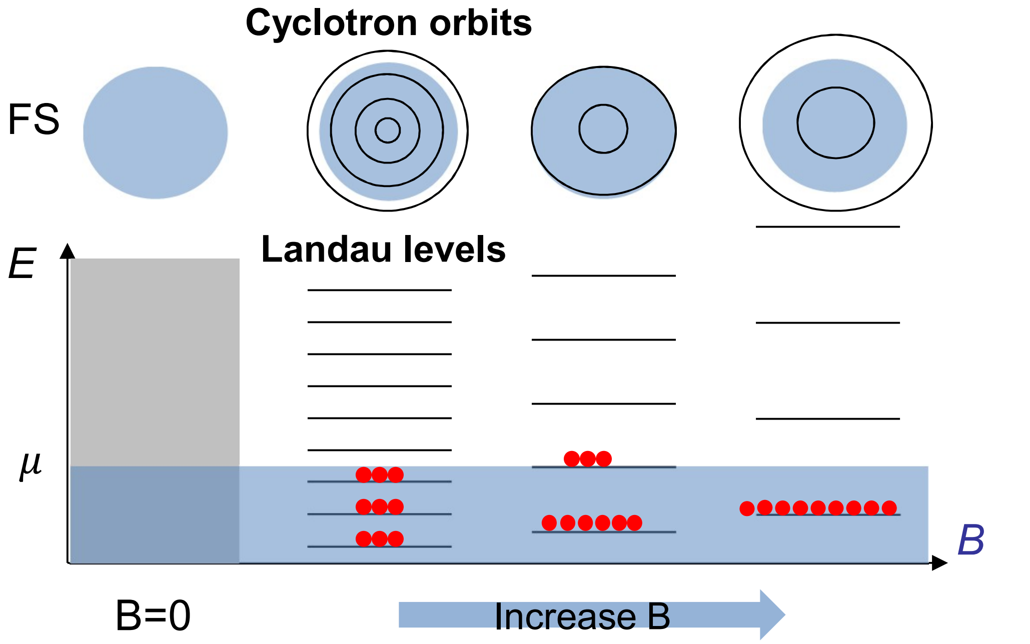
$$D = \frac{\pi R^2}{2\pi r \Delta r} = \frac{\pi R^2 B}{h/e} = \frac{\Phi_B}{\phi_0}$$

Same as before

Fermi sea (B=0) *vs* Landau levels (B≠0)

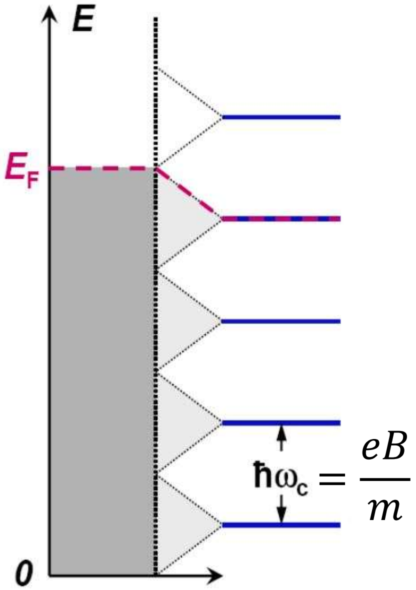


Cyclotron orbits in k-space,
with LL energies



larger level separation, and larger degeneracy (both ∝ B)

Landau levels and QHE:



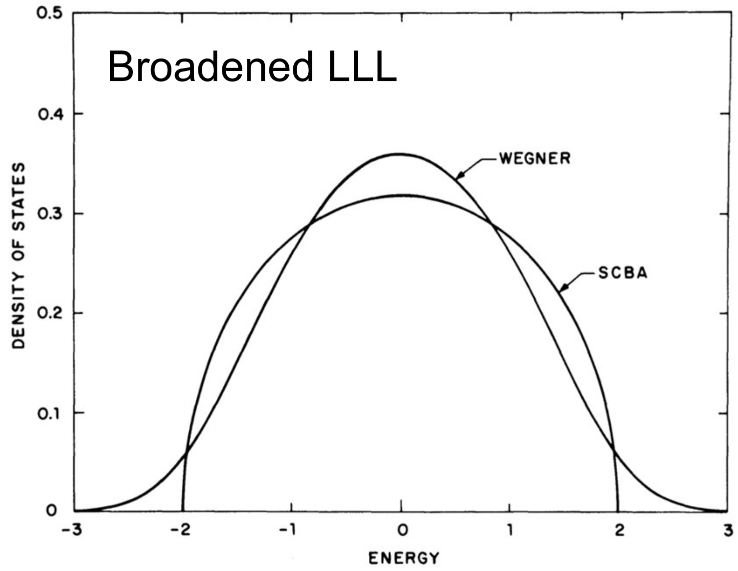


Figure 5.1. The exact (Wegner) and SCBA density of states in the lowest Landau level for a white-noise impurity potential.

Fig from Pruisken @ The quantum Hall effect

Cyclotron energy

$$\hbar\omega_c = 1.16H \times 10^{-8}\,\text{eV} \times \frac{m}{m^*}$$
$$= 1.34H \times 10^{-4}\,\text{K} \times \frac{m}{m^*}$$

(*H* in Gauss)

(for GaAs, m*=0.067m)

$B = 1\text{T}, \hbar\omega_c \simeq 1\ K$ (for GaAs)

- Landau levels have non-zero Chern numbers (TKNN paper)
- Hall conductance is quantized whenever the Fermi energy lies inside an energy gap (QH insulator)

LLs are broadened due to disorder

- Localized states vs Extended states in LLs

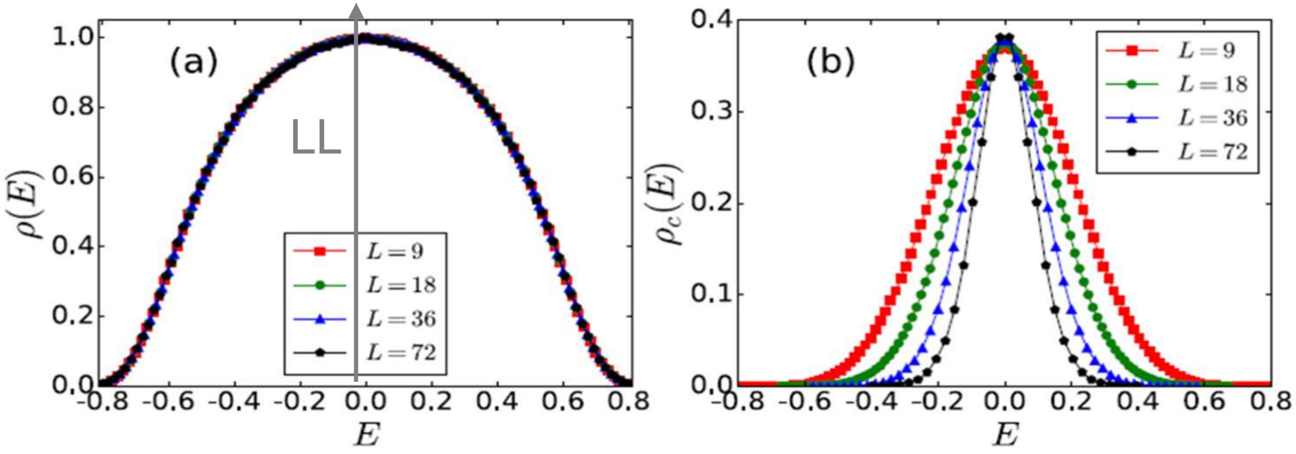


FIG. 5 (a) Density of states of a broadened LL. (b) Density of states of extended states in a LL. The distribution sharpens as the system gets larger.

Zhu et al, PRB 2019

According to numerical simulations, the range ΔE of the extended states in a LL shrinks as the sample area A gets larger (Fig. 5). Its dependence on size follows a scaling law (Huckestein, 1995),

$$\Delta E \simeq A^{-1/2\nu} \qquad \nu \simeq 7/3$$

展透模型 **Percolation model**, a semiclassical theory for smooth disorder potential

Energy landscape for a disordered LLL with a confining potential:

- Eigenstates are given by equipotential orbits.

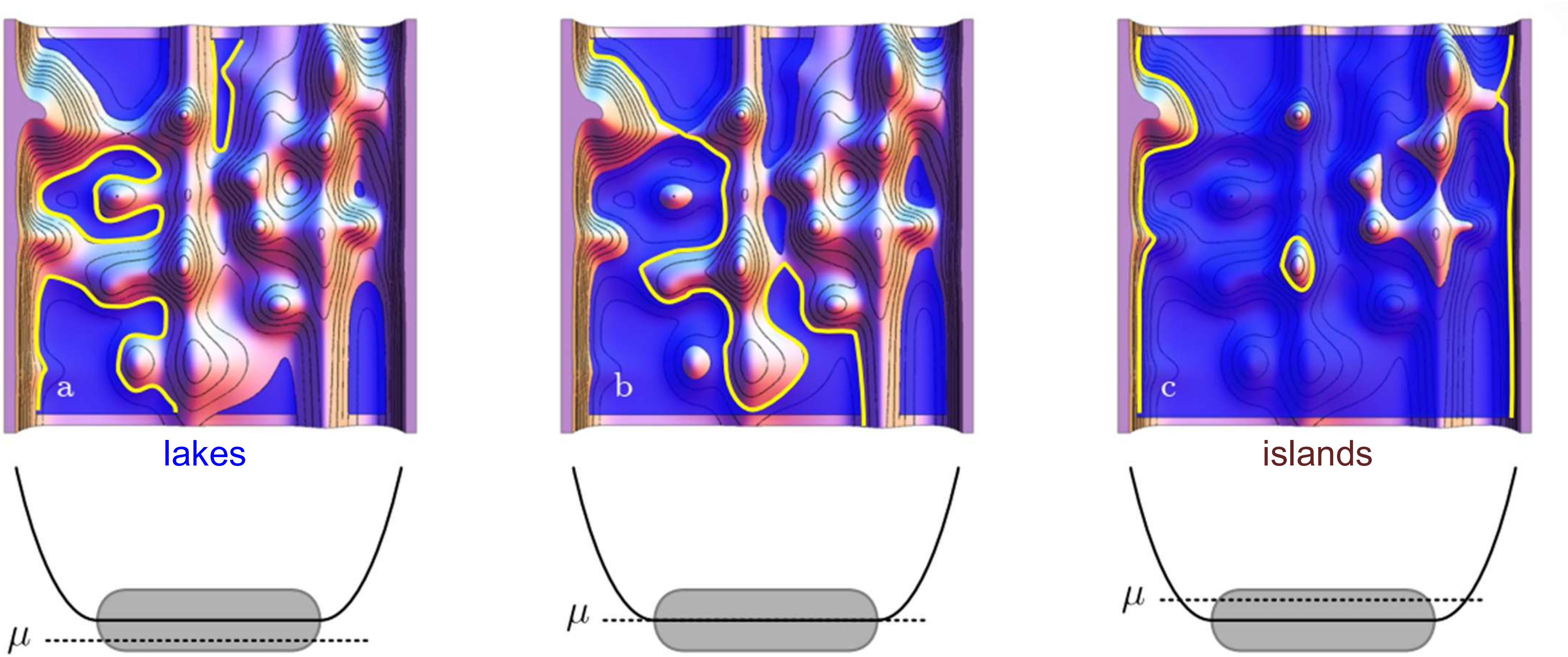
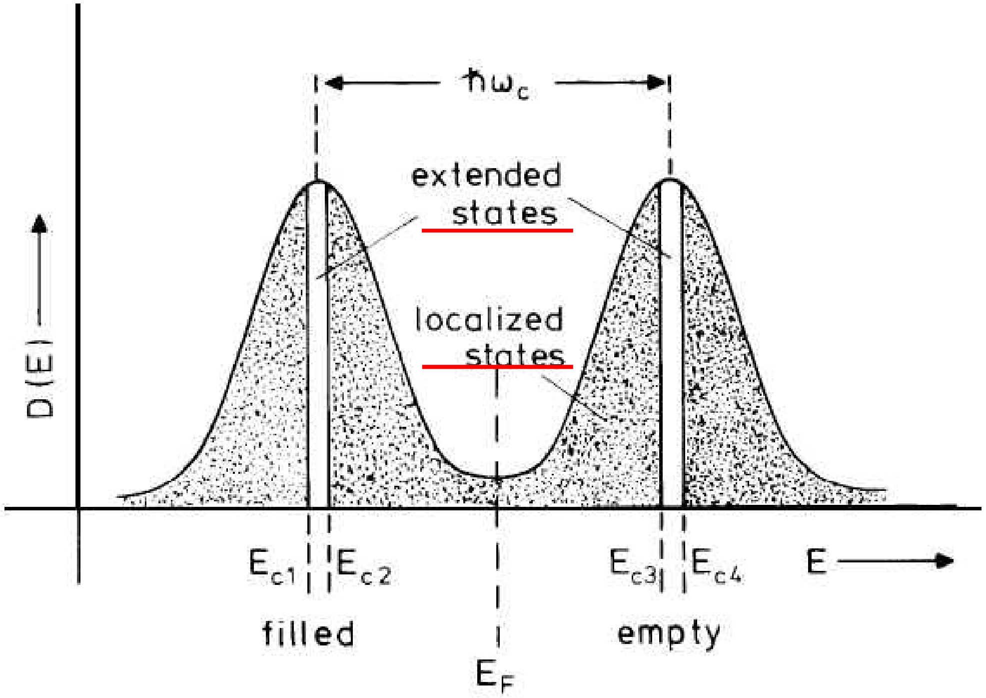


Fig from Hubert and Neupert - Topological Condensed Matter Physics

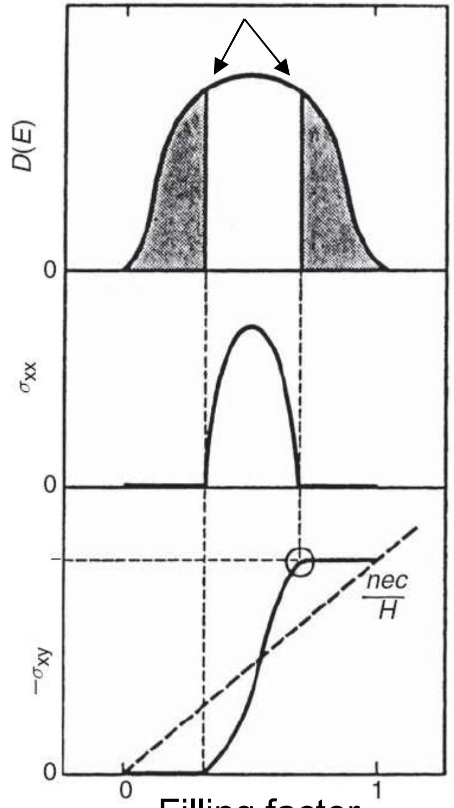
For experimental observation, see, e.g.,Hashimoto et al, PRL 101, 256802 (2008)

Quantum Hall plateaus and their transition

平台



Mobility edge



Aoki, CMST 2011

If there is no disorder, then there is no localized states, and no Hall plateau.

Explanation of the quantized Hall conductivity:

1. Laughlin's gauge argument (1981)

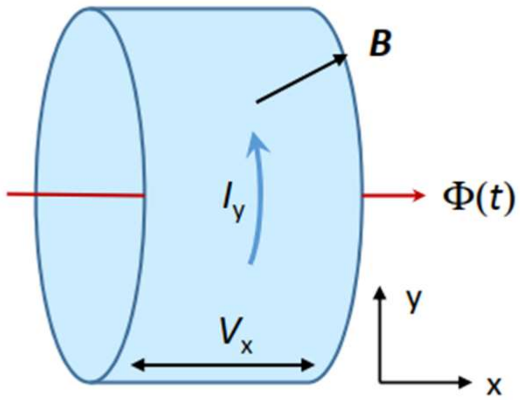
$$H = \sum_i \frac{1}{2m}\left[\vec{p}_i + e\vec{A}(\vec{r}_i) + e\vec{a}(\vec{r})\right]^2 + V(\vec{r}_i) + V_{ee}$$

• Simulate a longitudinal EMF by a fictitious time-dependent flux Φ

$$J_y = \frac{-e}{m}\frac{1}{L_x L_y}\sum_i\left[p_{iy} + eA_y(\vec{r}_i) + ea_y(\vec{r}_i)\right]$$

$$= -\frac{1}{L_x L_y}\frac{\partial H}{\partial a_y}$$

$$= -\frac{1}{L_x}\frac{\partial H}{\partial \Phi}, \quad \Phi = a_y L_y$$



solve $H_\Phi | \psi_\Phi > = E_\Phi | \psi_\Phi >$

$$\Rightarrow \quad <\psi_\Phi | \frac{\partial H_\Phi}{\partial \Phi} | \psi_\Phi > = \frac{\partial}{\partial \Phi} <\psi_\Phi | H_\Phi | \psi_\Phi > = \frac{\partial E_\Phi}{\partial \Phi}$$

Hellman-Feynman theorem

$$\therefore \quad J_y = -\frac{1}{L_x}\frac{\partial E_\Phi}{\partial \Phi}$$

• Aharonov-Bohm phase: $\exp\left(i\frac{e}{\hbar}\Phi\right)$. Due to gauge symmetry, the system needs to be invariant under $\Phi \rightarrow \Phi + \Phi_0$,

• E_F at localized states, , E_Φ is insensitive to Φ. No charge transfer whatever Φ is.

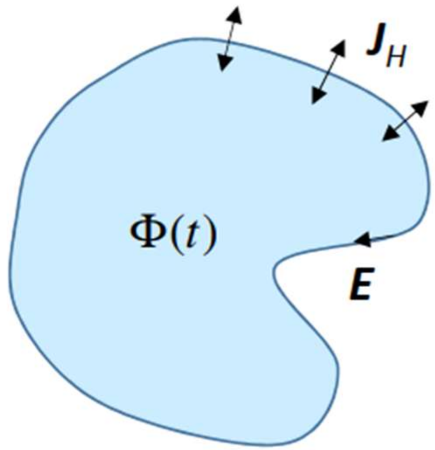
• E_F at extended states, only integer charges may transfer along x when Φ is changed by one Φ_0.

$$\Rightarrow \quad J_y = -\frac{N(-e)}{\Phi_0}\frac{V_x}{L_x} = N\frac{e^2}{h}E_x$$

2. Středa formula (Streda, 1982)

An increasing magnetic flux generates an EMF around the boundary

$$\frac{d\Phi(t)}{dt} = -\oint_C \mathbf{E}\cdot d\mathbf{r}$$
$$= \frac{1}{\sigma_H}\oint_C J_H dr = -\frac{1}{\sigma_H}\frac{dQ}{dt}$$
$$\text{or}\quad \delta\Phi = -\frac{1}{\sigma_H}\delta Q,$$



- B points out of paper
- C is counter-clockwise
- J_H flows out
- σ_H is the magnitude of Hall conductivity

For a uniform magnetic field, $d\Phi/dt = AdB/dt$, also the charge density $\rho_e = Q/A$, where A is the area bounded by C. Hence, we have

$$\sigma_H = -\frac{\partial \rho_e}{\partial B}$$

Due to the gauge symmetry, only integer number of electrons can cross the boundary when $\Delta\Phi = \phi_0$. Therefore,

$$\sigma_H = -\frac{\partial \rho_e}{\partial B} = -\frac{N(-e)/A}{\Delta\Phi/A} = N\frac{e^2}{h}$$

3. Theory of linear response, an outline

Both the current response and the density response can be calculated with the theory of linear response.

$$H = \frac{1}{2m}(\mathbf{p}+e\mathbf{A})^2 + V_L$$
$$= H_0 + \frac{e}{m}\mathbf{A}\cdot\mathbf{p} + O(A^2)$$

$$|\psi_m\rangle = |m\rangle + \sum_{\ell\neq m}\frac{|\ell\rangle\langle\ell|H'|m\rangle}{\varepsilon_m-\varepsilon_\ell}$$

➡ $$\langle\mathbf{J}\rangle = \sum_m f_m\langle\psi_m|\mathbf{J}|\psi_m\rangle$$
$$= \langle\mathbf{J}\rangle_0 + \delta\langle\mathbf{J}\rangle,$$

➡ $$\sigma_{\alpha\neq\beta}(0) = \frac{e^2}{im^2}\frac{1}{V_0}\sum_{\ell m(\ell\neq m)} f_\ell \frac{p^\alpha_{\ell m}p^\beta_{m\ell} - p^\beta_{\ell m}p^\alpha_{m\ell}}{\omega^2_{\ell m}}$$

➡ $$\sigma^{DC}_{xy} = \frac{e^2}{\hbar V_0}\sum_{nk} f_{n\mathbf{k}}\frac{1}{i}\left(\left\langle\frac{\partial u_{n\mathbf{k}}}{\partial k_x}\middle|\frac{\partial u_{n\mathbf{k}}}{\partial k_y}\right\rangle - \left\langle\frac{\partial u_{n\mathbf{k}}}{\partial k_y}\middle|\frac{\partial u_{n\mathbf{k}}}{\partial k_x}\right\rangle\right)$$

Same as the result from the anomalous velocity in Chap 3

$$\frac{1}{2\pi}\int_{\text{filled BZ}} d^2k\ F^z_{n\mathbf{k}}$$ is an integer, but where is the BZ here?

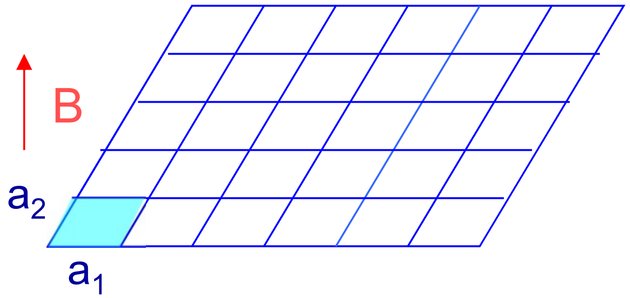
Lattice electron in a magnetic field:

magnetic translation symmetry

Consider a uniform B field,

$$\left\{\frac{1}{2m}[\boldsymbol{p}+e\mathbf{A}(\boldsymbol{r})]^2+V_L(\boldsymbol{r})\right\}\psi(\boldsymbol{r})=E\psi(\boldsymbol{r})$$

$$\rightarrow \left\{\frac{1}{2m}[\boldsymbol{p}+e\mathbf{A}(\boldsymbol{r}+\boldsymbol{a})]^2+V_L(\boldsymbol{r})\right\}\psi(\boldsymbol{r}+\boldsymbol{a})=E\psi(\boldsymbol{r}+\boldsymbol{a})$$



write

$$\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{a})=\boldsymbol{A}(\boldsymbol{r})+\nabla f(\boldsymbol{r}),$$

where $\nabla f(\boldsymbol{r})=\mathbf{A}(\boldsymbol{r}+\boldsymbol{a})-\mathbf{A}(\boldsymbol{r})\equiv\Delta\mathbf{A}(\boldsymbol{a})$.

$f=\Delta\mathbf{A}\cdot\boldsymbol{r}$

Indep of r

The extra vector potential ∇f can be removed by a gauge transformation,

$$\left\{\frac{1}{2m}[\boldsymbol{p}+e\mathbf{A}(\boldsymbol{r})]^2+V_L(\boldsymbol{r})\right\}e^{i(e/\hbar)f}\psi(\boldsymbol{r}+\boldsymbol{a})$$

$$=Ee^{i(e/\hbar)f}\psi(\boldsymbol{r}+\boldsymbol{a}).$$

- Magnetic translation operator

$$T_{\boldsymbol{a}}\psi(\boldsymbol{r})=e^{i(e/\hbar)\Delta\mathbf{A}\cdot\boldsymbol{r}}\psi(\boldsymbol{r}+\boldsymbol{a})$$

$$[H,T_{\boldsymbol{a}}]=0$$

$$T_{\boldsymbol{a}_2}T_{\boldsymbol{a}_1}=T_{\boldsymbol{a}_1}T_{\boldsymbol{a}_2}\exp\left(i\frac{e}{\hbar}\oint\mathbf{A}\cdot d\boldsymbol{r}\right)$$

Commute if this is 1

Xiao et al, RMP 2010

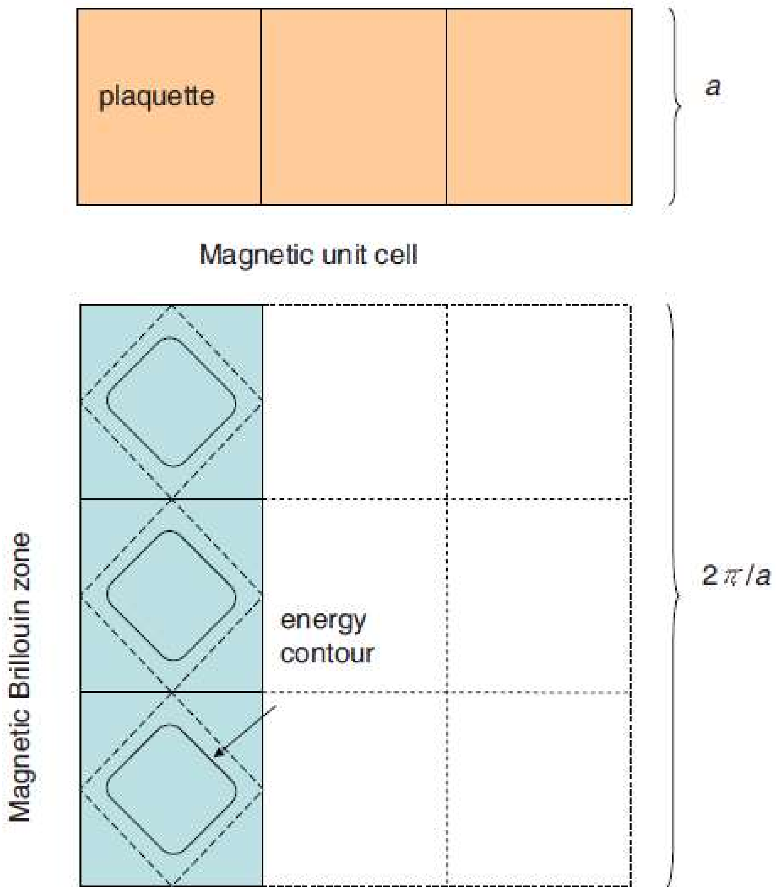
Simultaneous eigenstates:

magnetic Bloch states

$$\begin{cases} H\psi_{nk} = E_{nk}\psi_{nk}, \\ T_{qa_1}\psi_{nk} = e^{ik\cdot qa_1}\psi_{nk}, \\ T_{a_2}\psi_{nk} = e^{ik\cdot a_2}\psi_{nk}. \end{cases}$$

- If $\Phi=(p/q)\Phi_0$ per plaquette, then Magnetic Brillouin zone = BZ/q.

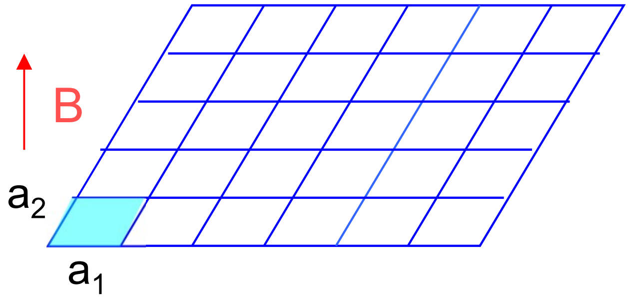
e.g., p/q=1/3



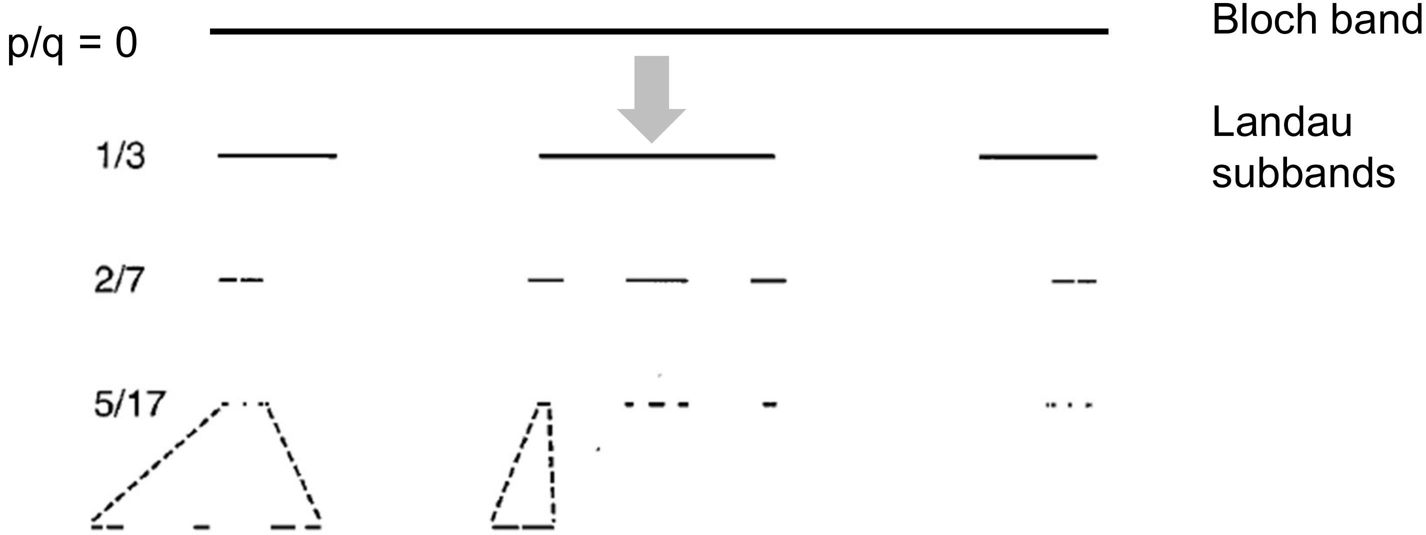
Lattice electrons in a magnetic field

$$\left\{\frac{1}{2m}[\boldsymbol{p}+e\mathbf{A}(\boldsymbol{r})]^2+V_L(\boldsymbol{r})\right\}\psi(\boldsymbol{r})=E\psi(\boldsymbol{r})$$

Commensurability between 2 scales: a, λ



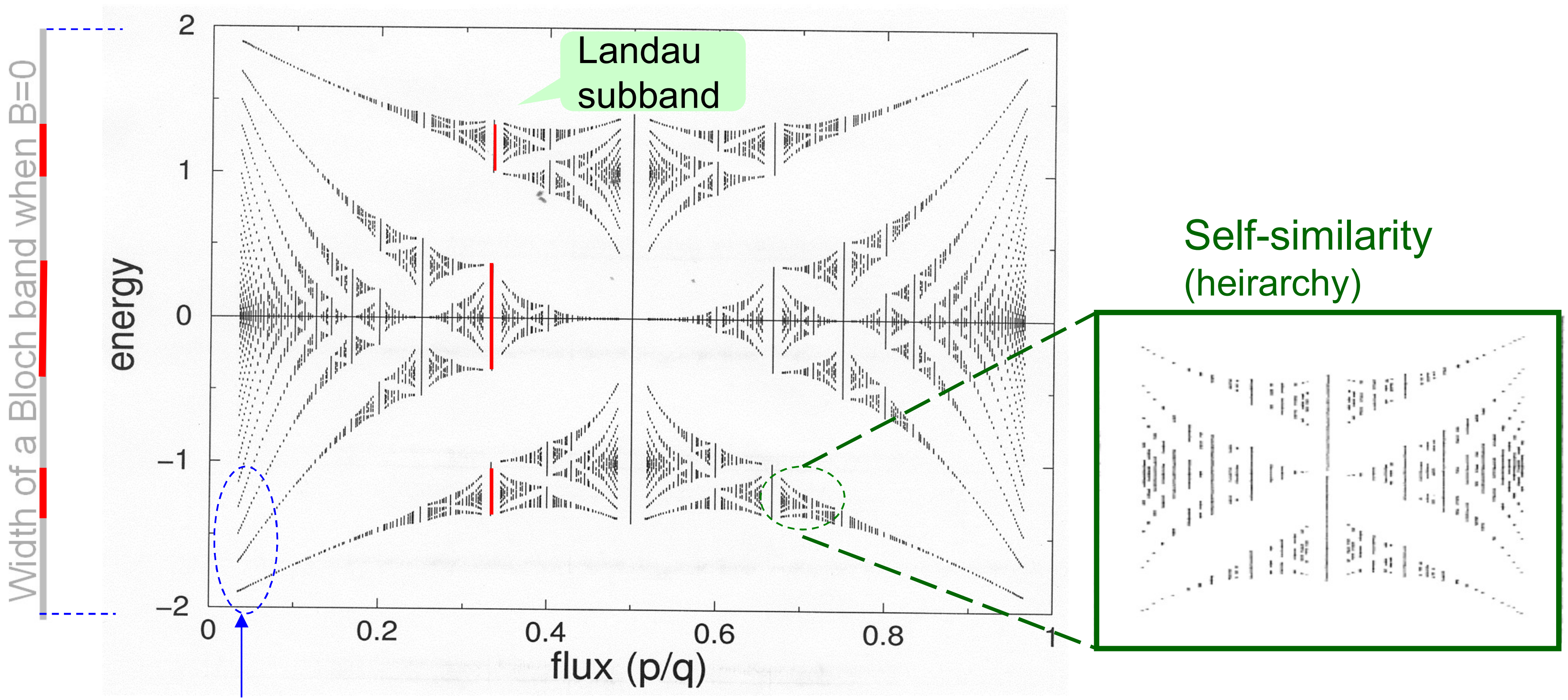
Splitting of Bloch band $\phi=\frac{p}{q}\phi_0,\quad \phi_0=\frac{h}{e}$ (flux quantum)



Xiao, Rev Mod Phys 2010

Hofstadter's butterfly (Hofstadter, PRB 1976)

• A fractal spectrum with self-similarity structure



B → 0 near band button, evenly-spaced LLs

• The total band width for an irrational q is of measure zero (as in a Cantor set).

Quantum Hall Effect in 2D systems

- Si MOSFET (von Klitzing et al, 1980)
- GaAs heterojunction (Stormer, 1982)
- Graphene (Novoselov, Science 2007)
- Polar oxide heterostructures (Tsukazaki et al, Science 2007)
- Twisted bilayer graphene (Lee et al, PRL 2011)
- TMD: WSe_2 (Movva et al, PRL 2017)
- InSe (Bandurin et al, Nat Nanotech 2017)
- Tellurene (Qiu et al, Nat Nanotech 2020)
- …

Transition Metal Dichalcogenide

過渡金屬硫化物

Room-Temperature Quantum Hall Effect in Graphene

2007

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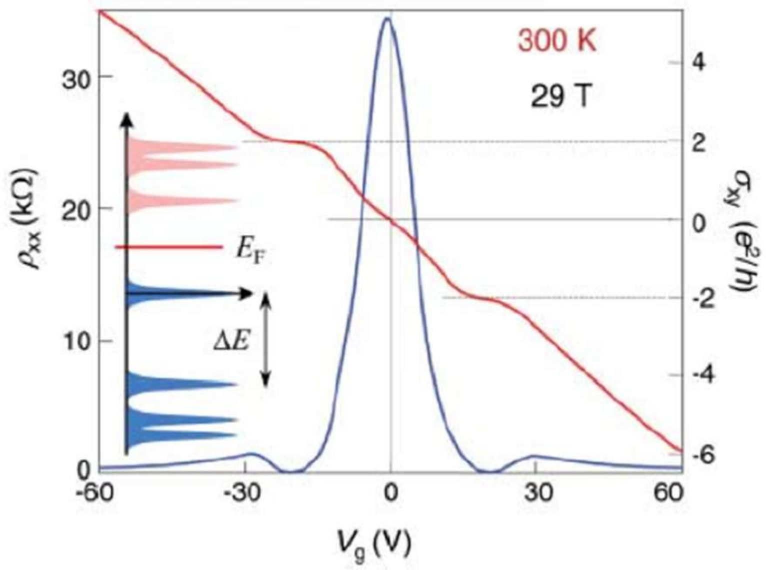
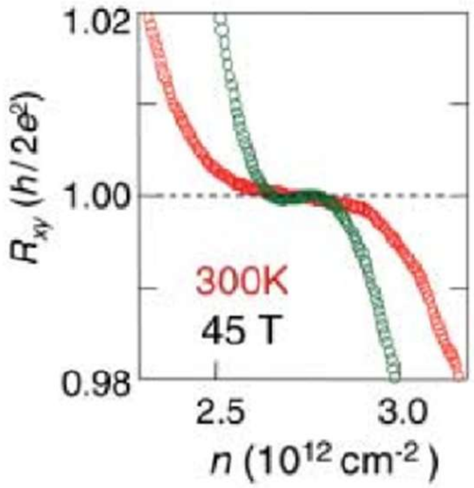




In graphene, due to the Dirac cone in energy spectrum, the LLs are not uniformly spaced, but locate at $\varepsilon_n = v_F\sqrt{2e\hbar B}\sqrt{n}$, where $v_F \simeq 10^6$ m/s. The energy separation between the lowest and the first LLs is about 60 K at 1 Tesla. Thus, one can observe the IQHE *at room temperature* when $B = 45$ T (Novoselov *et al.*, 2007).

Edge state in quantum Hall insulator

3 levels of understanding

1. Classical

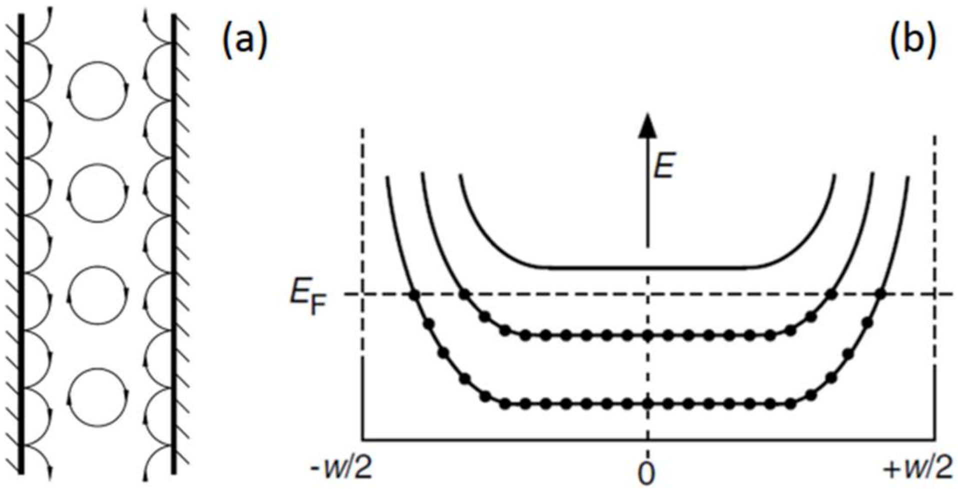
   Skipping orbit (chiral)

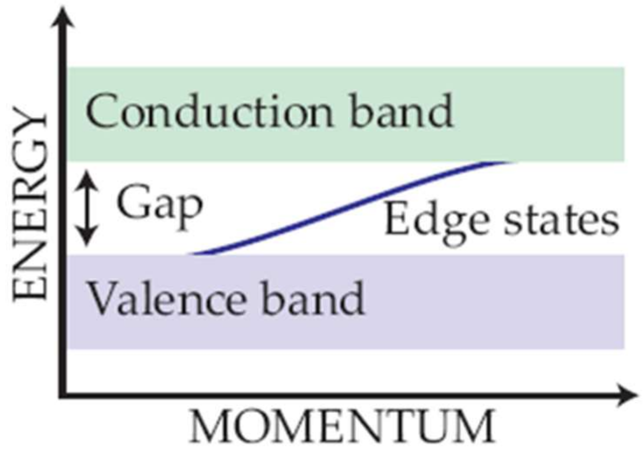
2. Semiclassical

   Bending of LLs near boundary

3. Quantum

   Energy levels of edge state appear within an energy gap





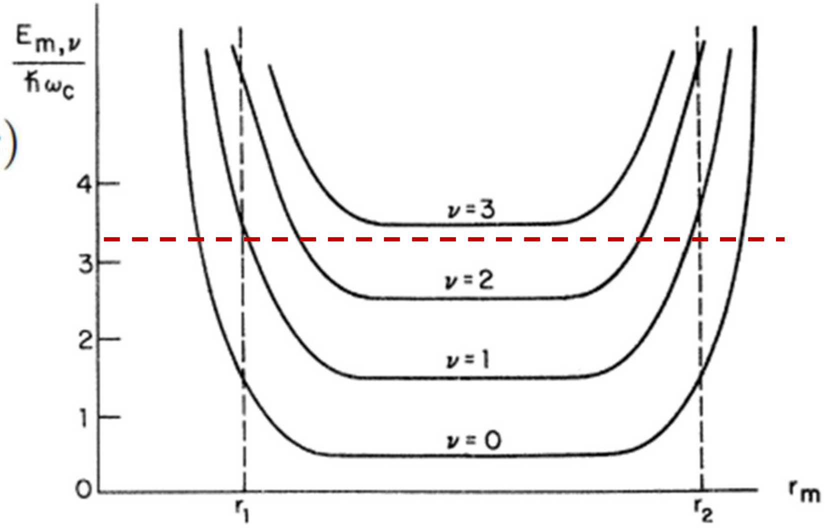
Semiclassical approach

- Soft wall boundary

$$\left[\frac{p_x^2}{2m} + \frac{m}{2}\omega_c^2(x-x_0)^2 + eU(x)\right]\phi(x) = \varepsilon\phi(x)$$

If the potential varies smoothly with respect to the cyclotron radius, then U(x) can be replaced with $U(x_0)$.

$$\varepsilon_{nx_0} = \left(n+\frac{1}{2}\right)\hbar\omega_c + eU(x_0)$$



- Hard wall boundary

$$U(x) = \begin{cases} \infty & \text{for } |x| < w/2, \\ 0 & \text{for } |x| > w/2. \end{cases}$$

Bohr-Sommerfeld quantization:

$$S = \oint_C dx\, p_x = (n+\gamma)h,\ \ n \in Z$$

- For an orbit in the bulk

$$S = 2\int_{x_0-R}^{x_0+R} dx\sqrt{2m\varepsilon - m\omega_c^2(x-x_0)^2}$$
$$= 2\pi\frac{\varepsilon}{\omega_c} = \left(n+\frac{1}{2}\right)h. \quad \Rightarrow \quad \varepsilon_n = \left(n+\frac{1}{2}\right)\hbar\omega_c$$

- For an orbit near the edge (see latex note for details)

$$S = 2\int_{x_0-R}^{0} dx\sqrt{2m\varepsilon - m^2\omega_c^2(x-x_0)^2}$$
$$= \frac{\varepsilon}{\omega_c}\left[2\theta - \sin(2\theta)\right] = \left(n+\frac{3}{4}\right)h, \qquad \cos\theta = x_0/R$$

$$\Rightarrow \quad \varepsilon_{nx_0} = \frac{2\pi}{2\theta - \sin(2\theta)}\left(n+\frac{3}{4}\right)\hbar\omega_c$$

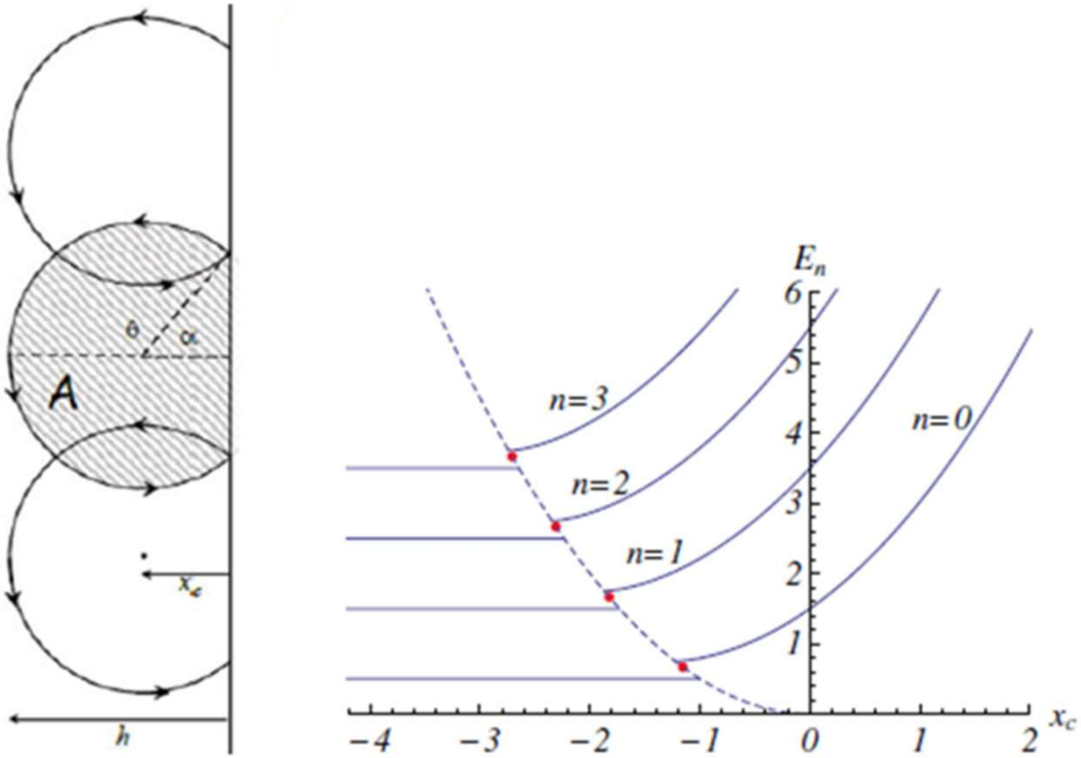
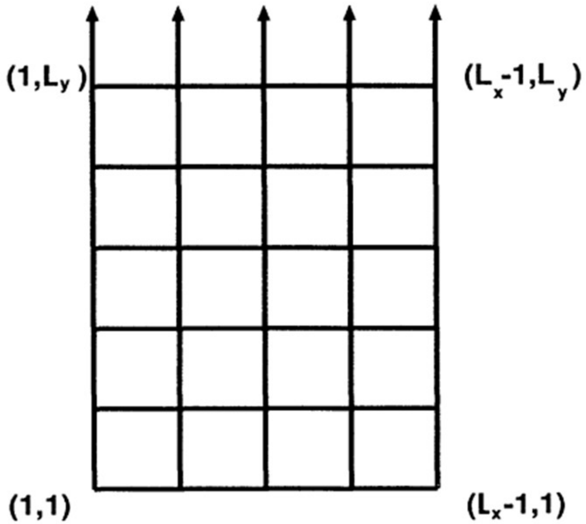


Fig from Montambaux, The Euro Phys J, 2011

Quantum approach (with lattice)



$\phi = \frac{2}{5}\phi_0$
per plaquette

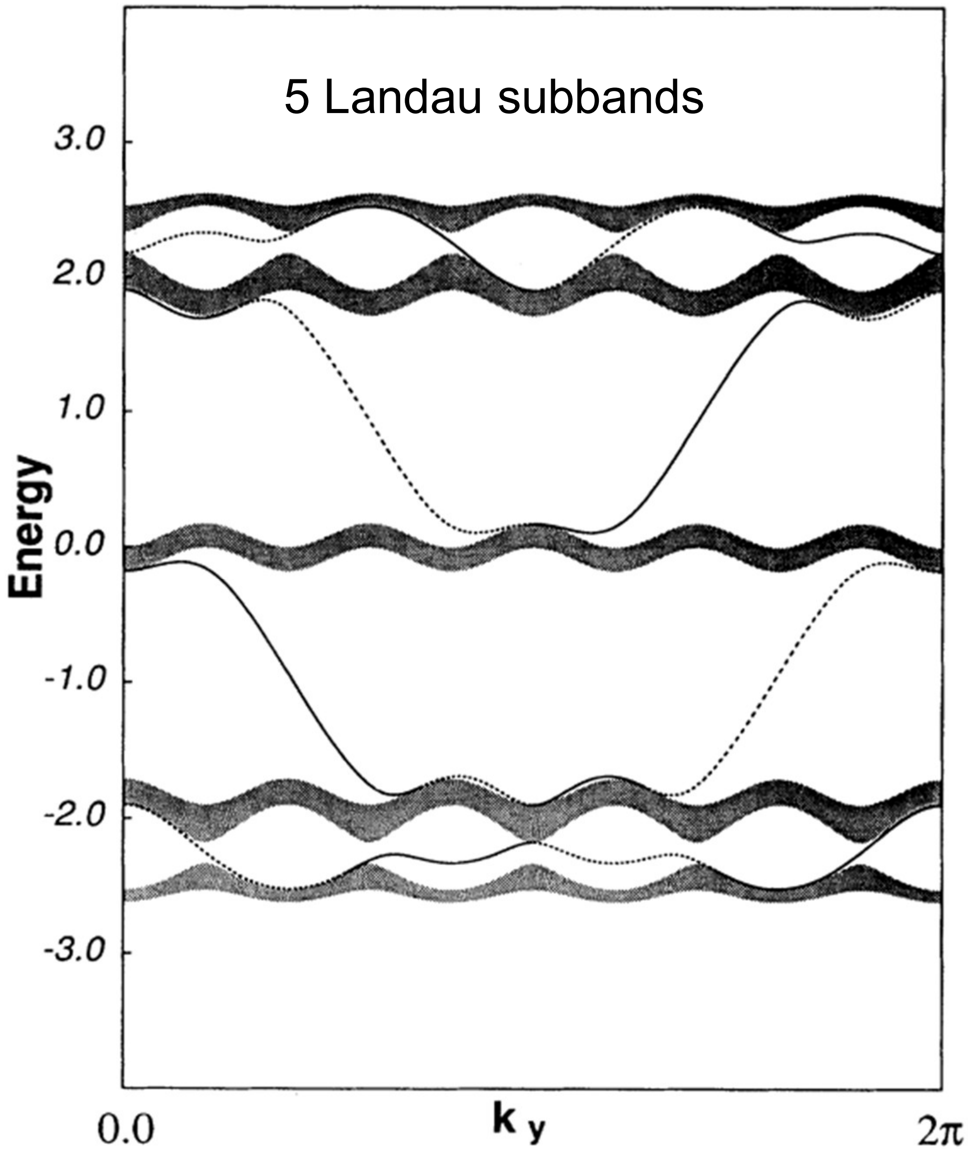


Fig from Hatsugai, PRB 1993