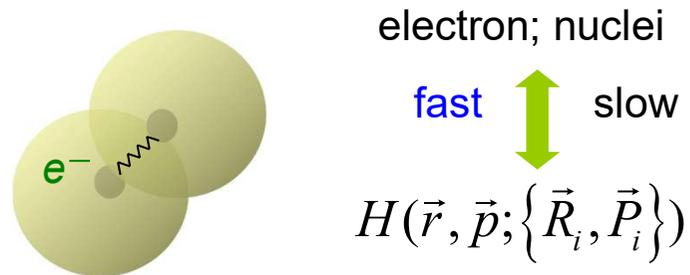


Review of Berry phase

- A. Non-degenerate energy level
- B. Geometric analogy
- C. Dynamics of the parameters
- D. Degenerate energy levels

System with fast and slow variables

Example: a vibrating H_2^+ molecule,



Instead of solving the full time-dependent Schroedinger eq.,
one can use the [Born-Oppenheimer approximation](#):

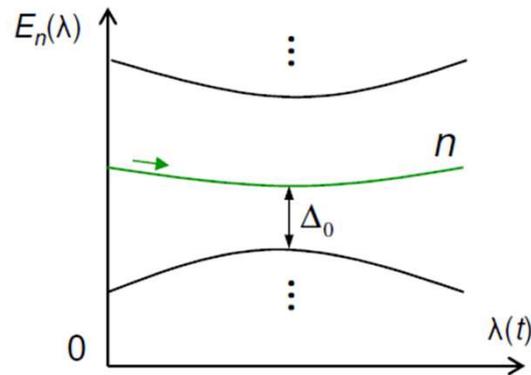
- “Slow variables \mathbf{R}_i are treated as *parameters* $\lambda(t)$
(the kinetic energies from \mathbf{P}_i are neglected)
- Solve time-**in**dependent Schroedinger eq.

$$H(\vec{r}, \vec{p}; \vec{\lambda}) \left| n, \vec{\lambda} \right\rangle = \varepsilon_{n, \vec{\lambda}} \left| n, \vec{\lambda} \right\rangle$$

“**snapshot**” state (single-valued in λ)

Adiabatic evolution of a quantum system

- Energy spectrum



If the characteristic frequency of motion $\Omega_0 \ll \Delta_0/\hbar$, then there is *no* inter-level transition. (Quantum adiabatic theorem)

$$H_{\lambda}|n, \lambda\rangle = \varepsilon_{n\lambda}|n, \lambda\rangle$$

- After time t

$$|\Psi_{n\lambda}(t)\rangle = e^{-\frac{i}{\hbar} \int_0^t dt' \varepsilon_{n\lambda}(t')} |n, \lambda(t)\rangle$$

(accumulated)
dynamical phase

- Phases of the snapshot states at different λ 's are *independent* and can be assigned arbitrarily

$$|n, \vec{\lambda}'\rangle = e^{i\chi_n(\vec{\lambda})} |n, \vec{\lambda}\rangle$$

Do we need to worry about this phase?

No need! • Fock, Z. Phys 1928

• Schiff, *Quantum Mechanics* (3rd ed.) p.290

Pf : Consider the n -th level,

$$H|n, \vec{\lambda}\rangle = \varepsilon_{n, \vec{\lambda}} |n, \vec{\lambda}\rangle$$

snapshot state

Allow a λ -dependent phase growing out of evolution

$$|\Psi_{n\vec{\lambda}}(t)\rangle = e^{i\gamma_n(\vec{\lambda})} e^{-i\int_0^t dt' \varepsilon_n(t')} |n, \vec{\lambda}\rangle$$

$$H|\Psi_{n\vec{\lambda}}(t)\rangle = i\hbar \frac{\partial}{\partial t} |\Psi_{n\vec{\lambda}}(t)\rangle$$

$$\rightarrow \dot{\gamma}_n = i \langle n, \vec{\lambda} | \frac{\partial}{\partial \vec{\lambda}} | n, \vec{\lambda} \rangle \cdot \dot{\vec{\lambda}} = \vec{A}_n \cdot \dot{\vec{\lambda}} \neq 0$$

$$\vec{A}_n(\vec{\lambda}) \equiv i \langle n, \vec{\lambda} | \frac{\partial}{\partial \vec{\lambda}} | n, \vec{\lambda} \rangle$$

Redefine the phase of snapshot states,

$$|n, \vec{\lambda}\rangle' = e^{i\phi_n(\vec{\lambda})} |n, \vec{\lambda}\rangle \quad (f_n \text{ is single-valued})$$

$$\rightarrow \mathbf{A}_n'(\vec{\lambda}) = \mathbf{A}_n(\vec{\lambda}) - \frac{\partial \phi_n}{\partial \vec{\lambda}}$$

Choose a $\phi(\vec{\lambda})$ such that, $\mathbf{A}_n'(\vec{\lambda})=0$, hence removing this extra phase.

However, there is one problem:

$\nabla_{\vec{\lambda}} \phi = \vec{A}(\vec{\lambda})$ does not always have a well-defined (global) solution!

Two possible cases:

1. $\nabla \times \vec{A} = 0$ everywhere

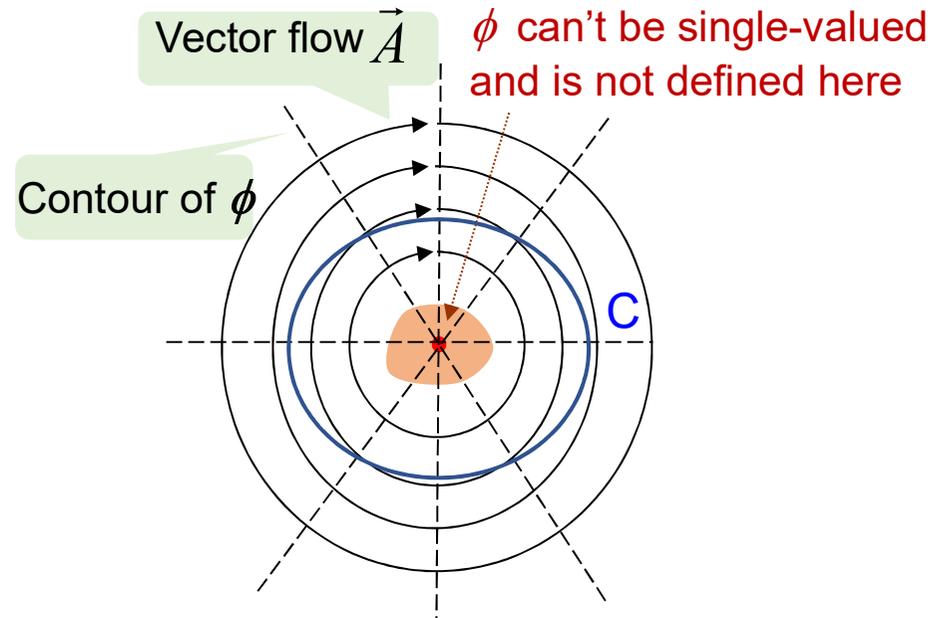
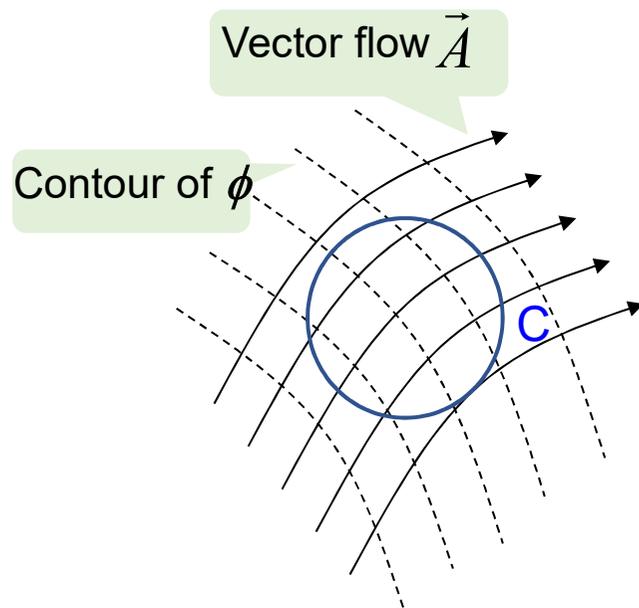
2. $\nabla \times \vec{A} \neq 0$ somewhere

or $\oint_C \vec{A} \cdot d\vec{\lambda} = 0$

or $\oint_C \vec{A} \cdot d\vec{\lambda} \neq 0$

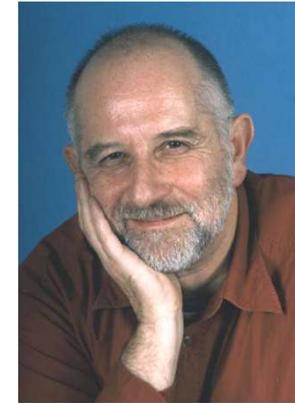
e.g., electrostatic field

e.g., vector potential



M. Berry, 1984 :

The parameter-dependent phase is NOT always removable!



For periodic motion with $\lambda(T)=\lambda(0)$, we have, in general

$$\left| \psi_{\vec{\lambda}(T)} \right\rangle = e^{i\gamma_C} e^{-i \int_0^T dt' \varepsilon(t')} \left| \psi_{\vec{\lambda}(0)} \right\rangle \quad \text{Index } n \text{ neglected}$$

- Berry phase (aka geometric phase)

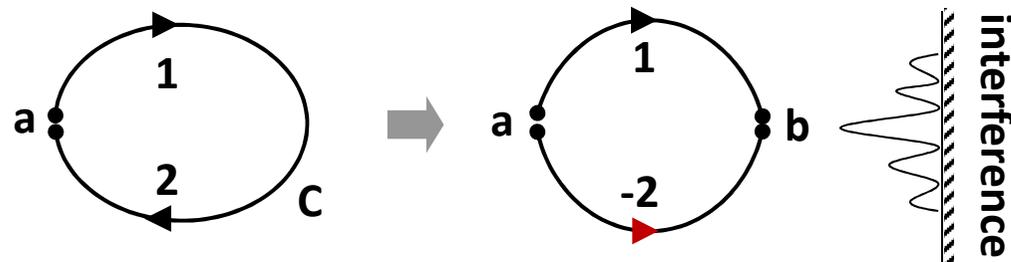
$$\gamma_C = \oint_C \langle \vec{\lambda} | i \frac{\partial}{\partial \vec{\lambda}} | \vec{\lambda} \rangle \cdot d\vec{\lambda} \neq 0$$

Depends on the geometry of the path C, independent of the rate $\dot{\gamma}$

- Berry phase is path-dependent

$$\text{if } \oint_C = \int_1 + \int_2 \neq 0, \quad \text{then } \int_1 - \int_2 \left(= \int_1 + \int_2 \right) \neq 0$$

Phase difference



Some terminology

- **Berry connection** (aka Berry potential)

$$\vec{A}(\vec{\lambda}) \equiv i \langle \vec{\lambda} | \nabla_{\lambda} | \vec{\lambda} \rangle$$

- **Stokes theorem** (3-dim here, can be higher)

$$\gamma_C = \oint_C \vec{A} \cdot d\vec{\lambda} = \int_S \nabla_{\vec{\lambda}} \times \vec{A} \cdot d\vec{a}$$

- **Berry curvature** (aka Berry field)

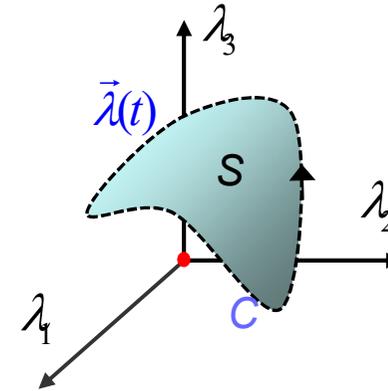
$$\vec{F}(\vec{\lambda}) \equiv \nabla_{\lambda} \times \vec{A}(\vec{\lambda}) = i \langle \nabla_{\lambda} \psi_{\vec{\lambda}} | \times | \nabla_{\lambda} \psi_{\vec{\lambda}} \rangle$$

- **Gauge transformation**

- $|\psi_{\vec{\lambda}}\rangle \rightarrow e^{i\chi(\vec{\lambda})} |\psi_{\vec{\lambda}}\rangle$
- $\vec{A}(\vec{\lambda}) \rightarrow \vec{A}(\vec{\lambda}) - \nabla_{\lambda} \chi$
- $\vec{F}(\vec{\lambda}) \rightarrow \vec{F}(\vec{\lambda})$
- $\gamma_C \rightarrow \gamma_C$

Redefine the phases of the snapshot states (χ is single-valued)

Berry curvature and Berry phase are not changed under the G.T.



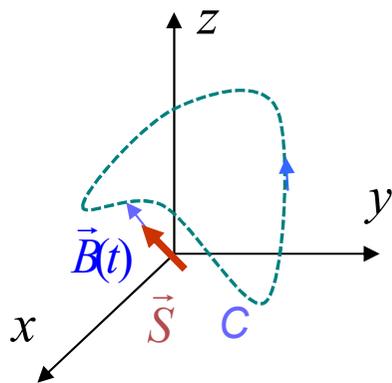
For a small loop,

$$\gamma_C = \int_S \vec{F} \cdot d\vec{a} \simeq \vec{F} \cdot d\vec{a}$$

A canonical example (we'll cite this result several times later)

A **spin-1/2 particle** in a *slowly changing B field*

• Real space



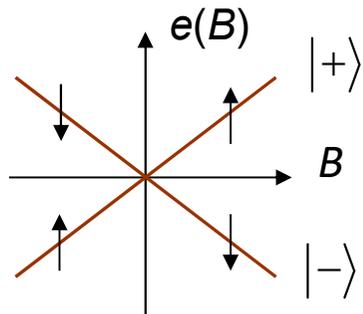
$$H_{\vec{\lambda}=\vec{B}} = \mu_B \vec{B} \cdot \vec{\sigma}$$

- Eigenvalues and eigenstates

$$\epsilon_{\pm} = \pm \mu_B B$$

$$|\hat{n}, +\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix}, \quad |\hat{n}, -\rangle = \begin{pmatrix} -e^{-i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}.$$

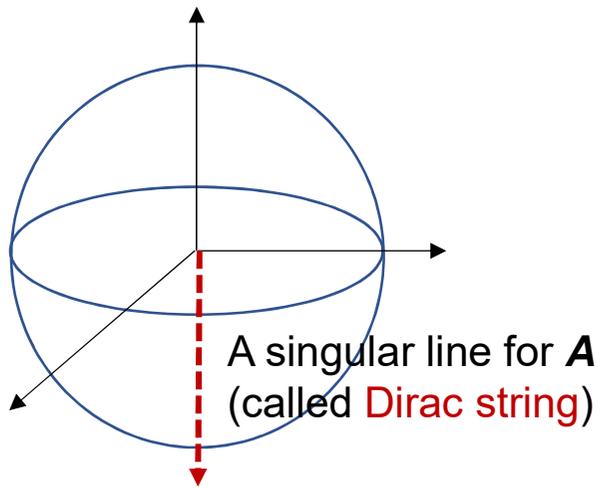
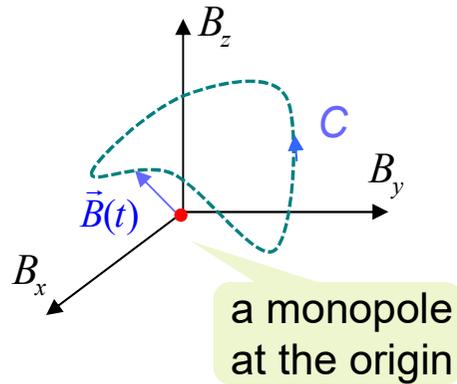
Level crossing at $B=0$



- Different choices of phases (gauge choices)

$|\hat{n}, \pm\rangle' = e^{\mp i\phi} |\hat{n}, \pm\rangle$ are also single-valued. You can check that $|\hat{n}, \pm\rangle$ have ϕ -ambiguity at $\theta = \pi$ (but not at $\theta = 0$), while $|\hat{n}, \pm\rangle'$ have ϕ -ambiguity at $\theta = 0$ (but not at $\theta = \pi$).

- Parameter space



- **Berry connection**

$$\frac{\partial}{\partial \mathbf{B}} = \frac{\partial}{\partial B} \hat{e}_r + \frac{1}{B} \frac{\partial}{\partial \theta} \hat{e}_\theta + \frac{1}{B \sin \theta} \frac{\partial}{\partial \phi} \hat{e}_\phi$$

$$\begin{aligned} \mathbf{A}_+(\mathbf{B}) &= i \langle \mathbf{B}, + | \frac{\partial}{\partial \mathbf{B}} | \mathbf{B}, + \rangle \\ &= -\frac{1}{2B} \frac{1 - \cos \theta}{\sin \theta} \hat{e}_\phi. \end{aligned}$$

~ **vector potential** of a monopole

Similarly,
$$\mathbf{A}_-(\mathbf{B}) = \frac{1}{2B} \frac{1 - \cos \theta}{\sin \theta} \hat{e}_\phi$$

Both $\mathbf{A}_\pm(\mathbf{B})$ are singular along $\theta = \pi$.
(relates to the ϕ -ambiguity)

- **Berry curvature**

→
$$\mathbf{F}_\pm(\mathbf{B}) = \nabla_{\mathbf{B}} \times \mathbf{A}_\pm(\mathbf{B}) = \mp \frac{1}{2} \frac{\hat{B}}{B^2}$$

~ **magnetic field** of a monopole

Point of level crossing is the source of Berry curvature

- Berry phase

→ $\gamma_{\pm}(C) = \mp \frac{1}{2} \Omega(C)$
 spin \times solid angle

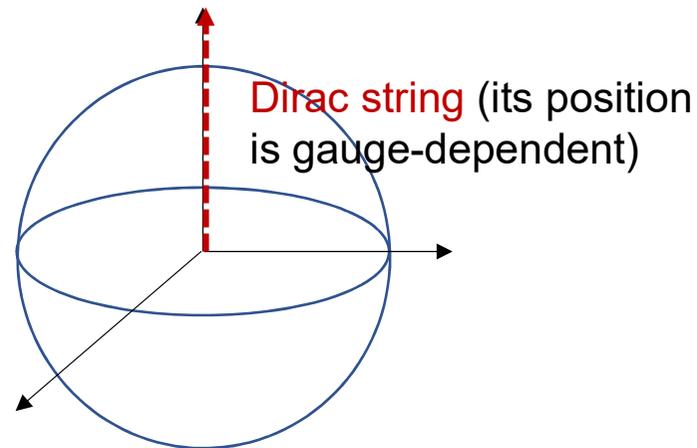
- Berry index
(topological charge)

$$\frac{1}{2\pi} \int_{S_B^2} d^2\mathbf{a} \cdot \mathbf{F}_{\pm}(\mathbf{B}) = \mp 1$$

- Gauge transformation

$$\begin{aligned} |\hat{n}, \pm\rangle' &= e^{\mp i\phi} |\hat{n}, \pm\rangle \\ \mathbf{A}'_{\pm}(\mathbf{B}) &= \mathbf{A}_{\pm}(\mathbf{B}) \pm \frac{\partial\phi}{\partial\mathbf{B}} \\ &= \mathbf{A}_{\pm}(\mathbf{B}) \pm \frac{1}{B \sin\theta} \hat{e}_{\phi} \\ &= \pm \frac{1}{2B} \frac{1 + \cos\theta}{\sin\theta} \hat{e}_{\phi} \end{aligned}$$

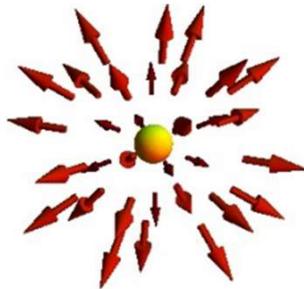
Both $\mathbf{A}'_{\pm}(\mathbf{B})$ are singular along $\theta = 0$.



Analogy with electromagnetism

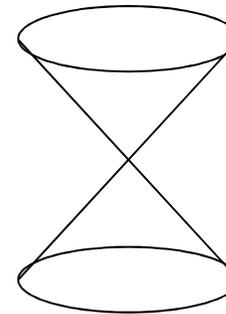
Electromagnetism	Quantum anholonomy
vector potential $\mathbf{A}(\mathbf{r})$	Berry connection $\mathbf{A}(\boldsymbol{\lambda})$
magnetic field $\mathbf{B}(\mathbf{r})$	Berry curvature $\mathbf{F}(\boldsymbol{\lambda})$
magnetic monopole	degenerate point
magnetic charge	Berry index ← (aka monopole charge, topological charge, etc)
magnetic flux $\Phi(C)$	Berry phase $\gamma(C)$

Magnetic monopole
in **real space**



Magnetic flux (monopole charge)
is quantized

“Magnetic monopole”
in **momentum space**

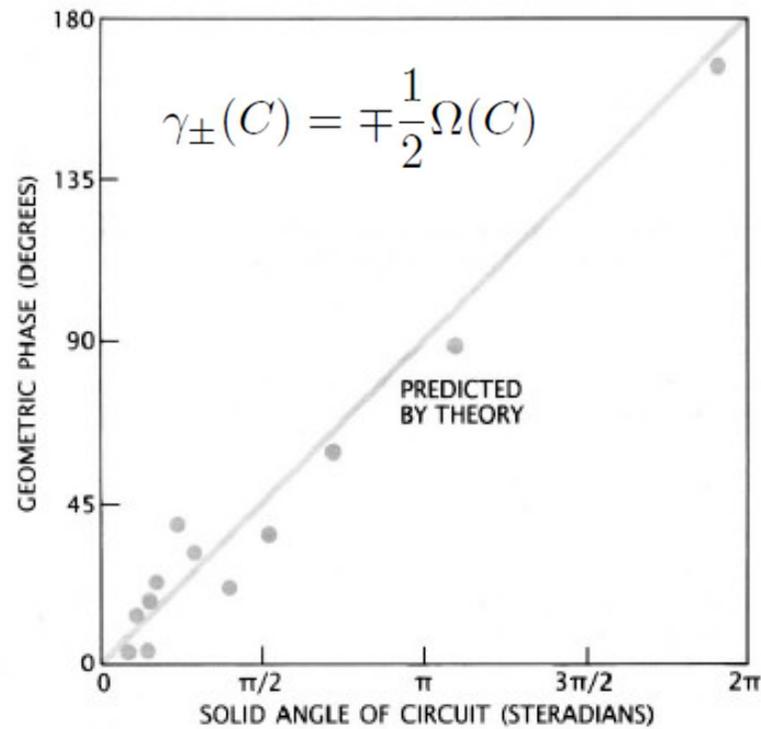
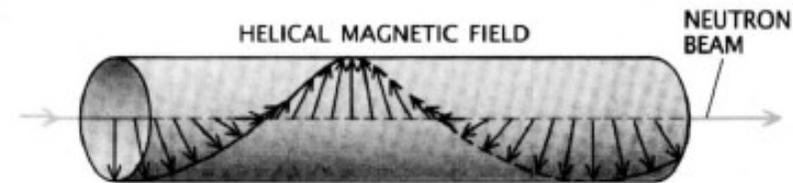


Point degeneracy in
Weyl semimetal

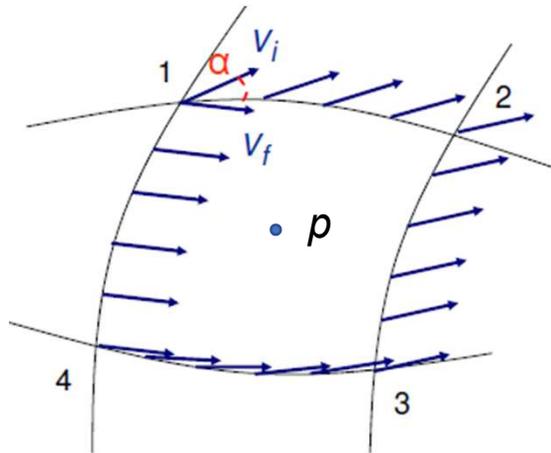
Berry index (topological charge)
is quantized

Experiments : Bitter and Dubbers , PRL 1987

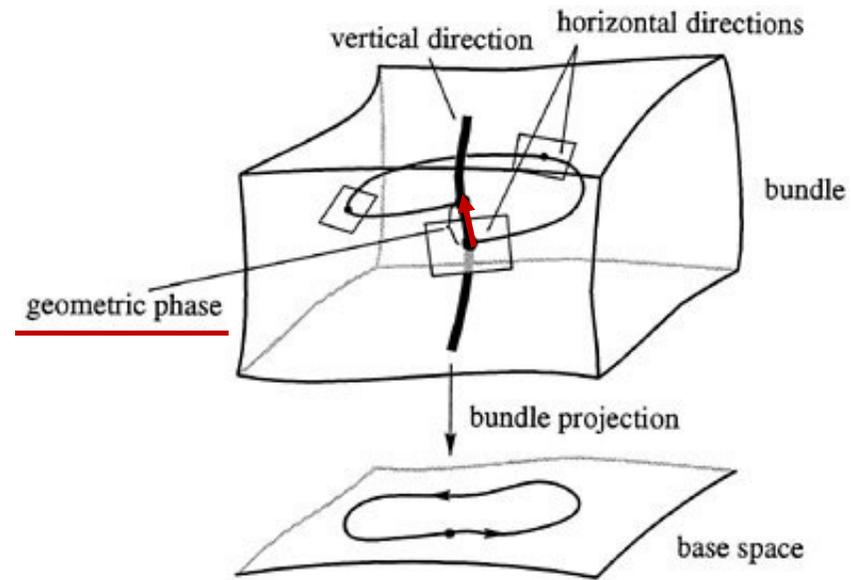
Neutrons fly through a helical magnetic field



Berry phase ~ Anholonomy angle



Fiber bundle = λ -space x U(1) phase



$$G \equiv \lim_{A \rightarrow 0} \frac{\alpha_A}{A} \quad \text{Gaussian curvature}$$



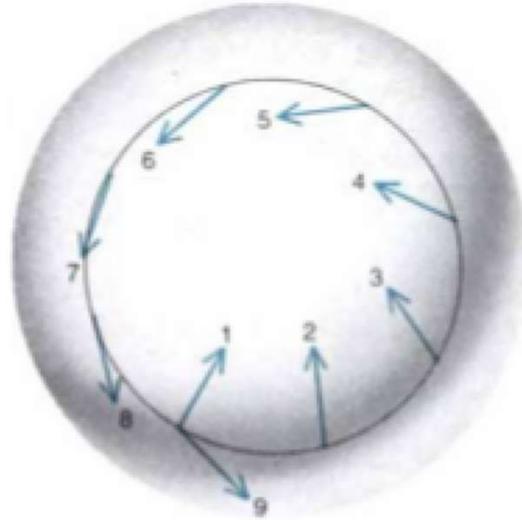
$$\vec{F} \cdot \hat{n} \equiv \lim_{A \rightarrow 0} \frac{\gamma_C}{A} \quad \text{Berry curvature}$$

For a small s, $\gamma_C = \int_S \vec{F} \cdot d\vec{a} \simeq \vec{F} \cdot d\vec{a}$

Fig. from *Fiber bundles and quantum theory*, by Bernstein and Phillips, Sci. Am. 1981

Revisiting parallel transport (PT)

- PT along a general curve

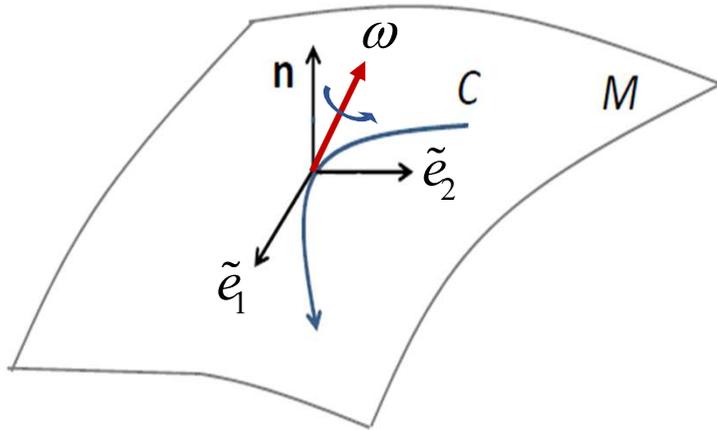


$$\alpha_A = 2\pi(1 - \cos\theta)$$

Parallel transport:

v does not twist around the local vertical axis
(normal vector **n**) as we move along a curve **C**.

A moving frame on a curved surface



Parallel transport condition of
a moving triad $(\mathbf{n}, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2)$:

No rotation around \mathbf{n} ,

$$\boldsymbol{\omega} \cdot \mathbf{n} = 0$$

Angular velocity

$$\dot{\tilde{\mathbf{e}}}_1 = \boldsymbol{\omega} \times \tilde{\mathbf{e}}_1$$

$$\begin{aligned} \boldsymbol{\omega} \cdot \mathbf{n} &= \boldsymbol{\omega} \cdot \tilde{\mathbf{e}}_1 \times \tilde{\mathbf{e}}_2 \\ &= \boldsymbol{\omega} \times \tilde{\mathbf{e}}_1 \cdot \tilde{\mathbf{e}}_2 = \dot{\tilde{\mathbf{e}}}_1 \cdot \tilde{\mathbf{e}}_2 = 0 \end{aligned}$$

PT condition

Define complex vector

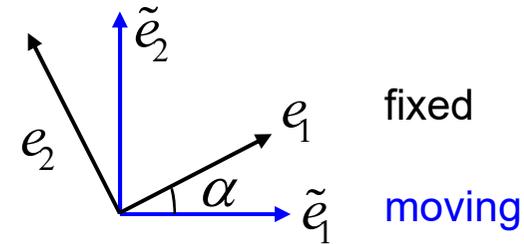
$$\psi = \frac{1}{\sqrt{2}} (\tilde{\mathbf{e}}_1 + i\tilde{\mathbf{e}}_2)$$

$$\boldsymbol{\omega} \cdot \mathbf{n} = 0 \implies \text{Im}(\psi^* \cdot \dot{\psi}) = 0, \text{ or } \boxed{i\psi^* \cdot \dot{\psi} = 0.}$$

Alternative form of
the PT condition

PT frame vs fixed frame:

- fixed triad $(\mathbf{n}, \mathbf{e}_1, \mathbf{e}_2)$
- moving triad $(\mathbf{n}, \tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2)$



define $\phi = \frac{1}{\sqrt{2}} (\mathbf{e}_1 + i\mathbf{e}_2)$

$$\psi = \frac{1}{\sqrt{2}} (\tilde{\mathbf{e}}_1 + i\tilde{\mathbf{e}}_2)$$

then $\psi(\mathbf{r}) = \phi(\mathbf{r})e^{i\alpha(\mathbf{r})}$

→ $\psi^* \cdot d\psi = \phi^* \cdot d\phi + i d\alpha$

→ $\alpha(C) = i \oint_C \phi^* \cdot \frac{d\phi}{d\mathbf{r}} \cdot d\mathbf{r}$

Fixed states,
single-valued

PT condition

versus

$$i\psi^* \cdot \dot{\psi} = 0$$

PT states, not
single-valued

Analogy: $\gamma(C) = i \oint_C \langle \phi_{\vec{\lambda}} | \nabla_{\vec{\lambda}} \phi_{\vec{\lambda}} \rangle \cdot d\vec{\lambda}$

Snapshot states



$$i \langle \psi_{\vec{\lambda}} | \nabla_{\vec{\lambda}} \psi_{\vec{\lambda}} \rangle = 0$$

PT states

Analogy

	geometry	quantum state
fixed basis	$\phi(x)$	$ \phi; \boldsymbol{\lambda}\rangle$
PT basis	$\psi(x)$	$ \psi; \boldsymbol{\lambda}\rangle$
connection	$\phi^* \cdot \dot{\phi}$	$\langle \phi \dot{\phi} \rangle$
PT condition	$\psi^* \cdot \dot{\psi} = 0$	$\langle \psi \dot{\psi} \rangle = 0$
holonomy	anholonomy angle	Berry phase
curvature	Gaussian curvature	Berry curvature
<u>topological number</u>	<u>Euler characteristic</u>	<u>Chern number</u>

$$\chi = \frac{1}{2\pi} \int_S da G$$

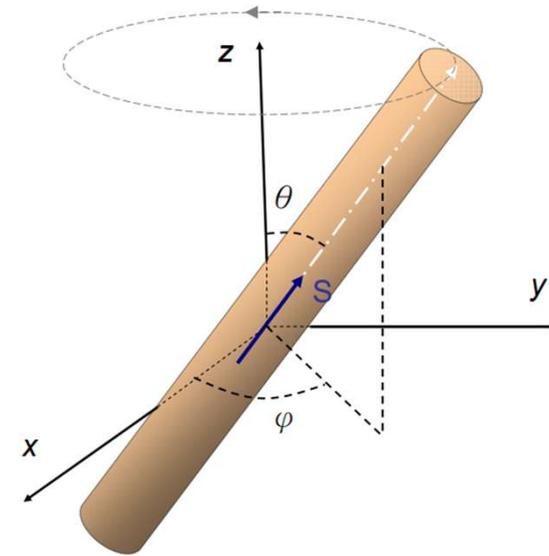
$$C = \frac{1}{2\pi} \int_M d\vec{a} \cdot \vec{F}$$

Dynamics of parameters

e.g., a rotating solenoid

$$H = \frac{L^2}{2I} + \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}$$

$$L^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$$



Full quantum state

$$|\Psi\rangle = \sum_{s=\pm} \overset{\text{Slow}}{\psi_s(\mathbf{B})} \overset{\text{fast}}{|\mathbf{B}, s\rangle}$$

$$H|\Psi\rangle = E|\Psi\rangle$$

$$\rightarrow \left[\frac{\hbar^2}{2I \sin^2 \theta} \left(\frac{1}{i} \frac{d}{d\phi} - \mathbf{A}_s \right)^2 + \varepsilon_s \right] \psi_s = E_s \psi_s$$

$$\mathbf{A}_s = i \langle \mathbf{B}, s | \frac{d}{d\phi} | \mathbf{B}, s \rangle$$

C. Degenerate energy levels

- Non-degenerate level
Wave function is a scalar

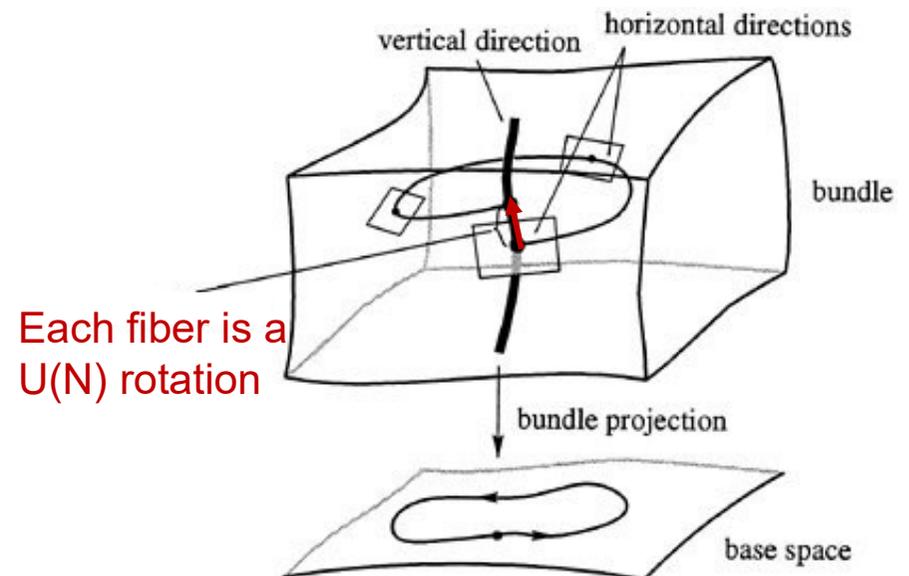
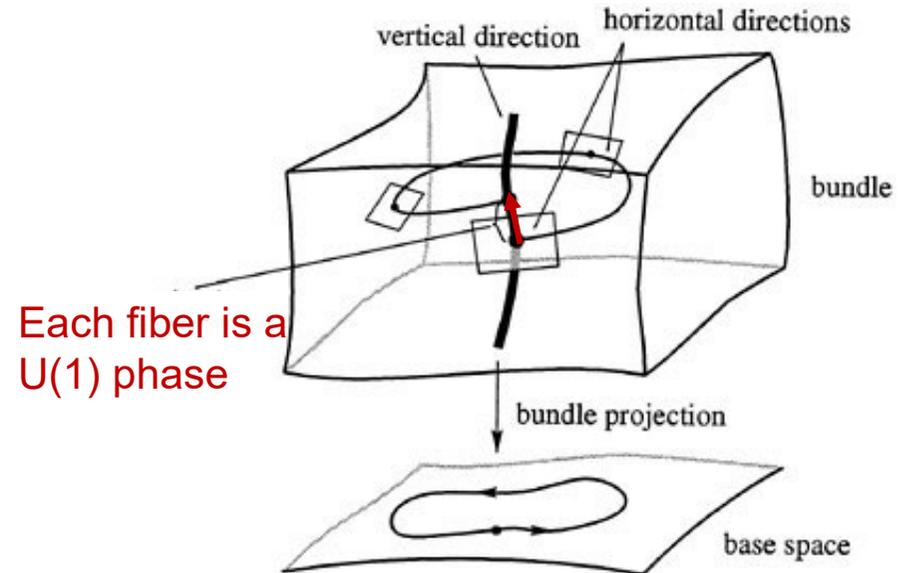
$$\left| \psi_{\vec{\lambda}(T)} \right\rangle = e^{i\gamma_C} e^{-i \int_0^T dt' \varepsilon(t')} \left| \psi_{\vec{\lambda}(0)} \right\rangle$$

Initial state and final state differ by a U(1) phase

- Degenerate levels (N-fold degeneracy)
Wave function is a N-component spinor

Initial state and final state differ by a U(N) rotation.

After diagonalization, you get N U(1) phases



For example, 2-fold degeneracy



$$\left\{ \begin{aligned} |\Psi_{n,1}(t)\rangle &= e^{-\frac{i}{\hbar} \int_0^t dt' \varepsilon_{n\lambda}(t')} \\ &\quad \times (|n, 1, \boldsymbol{\lambda}(t)\rangle \Gamma_{11}(t) + |n, 2, \boldsymbol{\lambda}(t)\rangle \Gamma_{21}(t)), \\ |\Psi_{n,2}(t)\rangle &= e^{-\frac{i}{\hbar} \int_0^t dt' \varepsilon_{n\lambda}(t')} \\ &\quad \times (|n, 1, \boldsymbol{\lambda}(t)\rangle \Gamma_{12}(t) + |n, 2, \boldsymbol{\lambda}(t)\rangle \Gamma_{22}(t)). \end{aligned} \right.$$

or $|\Psi_{n\beta}(t)\rangle = e^{-\frac{i}{\hbar} \int_0^t dt' \varepsilon_{n\lambda}(t')} \sum_{\alpha} |n\alpha\boldsymbol{\lambda}(t)\rangle \Gamma_{\alpha\beta}(t).$

Dynamical phase
 α
Berry rotation matrix

$$\langle \Psi_{n\alpha} | \Psi_{n\beta} \rangle = \delta_{\alpha\beta}$$

→ $\Gamma^\dagger \Gamma = \Gamma \Gamma^\dagger = 1$ Unitary rotation, U(2) matrix

$$H|\Psi_{n\beta}(t)\rangle = i\hbar \frac{\partial}{\partial t} |\Psi_{n\beta}(t)\rangle$$

$$\begin{aligned} \rightarrow \frac{d\Gamma_{\alpha\beta}}{dt} &= - \sum_{\gamma} \langle n\alpha\lambda | \frac{\partial}{\partial t} | n\gamma\lambda \rangle \Gamma_{\gamma\beta} \\ &= i \sum_{\gamma} \dot{\lambda}(t) \cdot \mathbf{A}_{\alpha\gamma}^{(n)}(\lambda) \Gamma_{\gamma\beta}, \end{aligned}$$

where

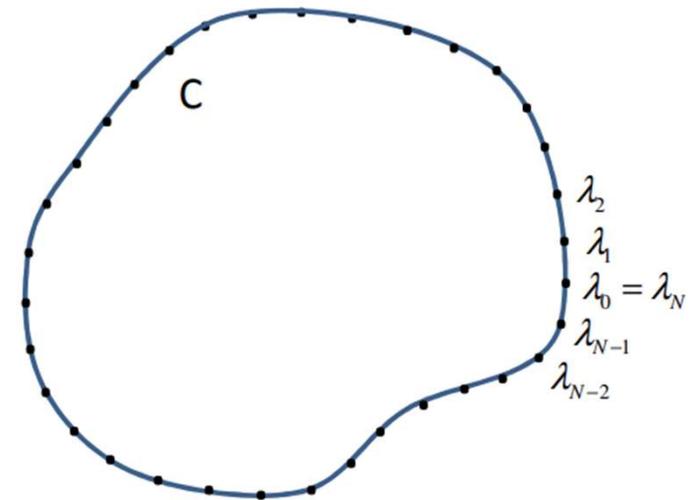
$$\mathbf{A}_{\alpha\beta}^{(n)}(\lambda) \equiv i \langle n\alpha\lambda | \frac{\partial}{\partial \lambda} | n\beta\lambda \rangle$$

Berry connection (2x2 matrix)

Different **A** matrices do not commute with each other

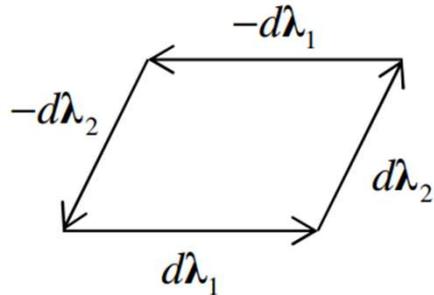
$$\begin{aligned} \Gamma(t + dt) &= \Gamma(t) + idt \dot{\lambda}(t) \cdot \vec{A}(t) \Gamma(t) \\ &\simeq e^{idt \dot{\lambda}(t) \cdot \vec{A}(t)} \Gamma(t) \end{aligned}$$

$$\begin{aligned} \rightarrow \Gamma(t) &= \dots e^{id\lambda \cdot \vec{A}(\lambda_1)} e^{id\lambda \cdot \vec{A}(\lambda_0)} \Gamma(0) \\ &\equiv \underline{P}e^{i \int_{\lambda_0}^{\lambda(t)} d\lambda \cdot \vec{A}(\lambda)}, \quad \Gamma(0) = 1, \\ &\text{path-ordering operator.} \end{aligned}$$



Aka **Wilson loop**

Berry curvature (Berry rotation per unit area)



$$i\vec{F}_{\square} \cdot \hat{\mathbf{n}} \equiv \frac{\Gamma_{\square} - 1}{d^2 a}$$

$$\begin{aligned} & \Gamma_{\square}(\boldsymbol{\lambda}) \\ &= \Gamma(\boldsymbol{\lambda}, \boldsymbol{\lambda} + d\boldsymbol{\lambda}_2) \Gamma(\boldsymbol{\lambda} + d\boldsymbol{\lambda}_2, \boldsymbol{\lambda} + d\boldsymbol{\lambda}_1 + d\boldsymbol{\lambda}_2) \\ & \times \Gamma(\boldsymbol{\lambda} + d\boldsymbol{\lambda}_1 + d\boldsymbol{\lambda}_2, \boldsymbol{\lambda} + d\boldsymbol{\lambda}_1) \Gamma(\boldsymbol{\lambda} + d\boldsymbol{\lambda}_1, \boldsymbol{\lambda}) \end{aligned}$$

➔ $F_{k\ell} = \partial_k A_{\ell} - \partial_{\ell} A_k - i[A_k, A_{\ell}]$. 2x2 matrix for each k, ℓ

non-commutative: **Non-Abelian** Berry curvature

A 3x3 antisymmetric matrix (with indices k, ℓ) is equivalent to a vector (see latex note for details)

Alternative form: $F_{k\ell} d\lambda_{1k} d\lambda_{2\ell} = \vec{F} \cdot d^2 \mathbf{a}$,

where $\vec{F} = \nabla_{\boldsymbol{\lambda}} \times \vec{A} - i\vec{A} \times \vec{A}$. 2x2 matrix for each vector component

Gauge transformation

$$|\alpha\boldsymbol{\lambda}'\rangle = \sum_{\gamma} |\gamma\boldsymbol{\lambda}\rangle U_{\gamma\alpha}(\boldsymbol{\lambda}) \quad \leftrightarrow \quad \langle\alpha\boldsymbol{\lambda}'| = \sum_{\gamma} U_{\alpha\gamma}^{\dagger}(\boldsymbol{\lambda}) \langle\gamma\boldsymbol{\lambda}|.$$

$$\begin{aligned} \mathbf{A}'_{\alpha\beta}(\boldsymbol{\lambda}) &= i\langle\alpha\boldsymbol{\lambda}'| \frac{\partial}{\partial\boldsymbol{\lambda}} |\beta\boldsymbol{\lambda}'\rangle & \text{or} & \quad \mathbf{A}'_k = \mathbf{U}^{\dagger} \mathbf{A}_k \mathbf{U} + i\mathbf{U}^{\dagger} \frac{\partial}{\partial\lambda_k} \mathbf{U} \\ &= U_{\alpha\gamma}^{\dagger} \mathbf{A}_{\gamma\delta} U_{\delta\beta} + iU_{\alpha\gamma}^{\dagger} \frac{\partial}{\partial\boldsymbol{\lambda}} U_{\gamma\beta} \end{aligned}$$

$$\begin{aligned} e^{id\boldsymbol{\lambda}\cdot\vec{\mathbf{A}}'} &\simeq 1 + id\boldsymbol{\lambda}\cdot\vec{\mathbf{A}}' \\ &= 1 + id\boldsymbol{\lambda}\cdot\mathbf{U}^{\dagger}\vec{\mathbf{A}}\mathbf{U} - d\boldsymbol{\lambda}\cdot\mathbf{U}^{\dagger}\frac{\partial}{\partial\boldsymbol{\lambda}}\mathbf{U} \\ &\simeq \mathbf{U}^{\dagger} \left(1 + id\boldsymbol{\lambda}\cdot\vec{\mathbf{A}}\right) \left(\mathbf{U} - d\boldsymbol{\lambda}\cdot\frac{\partial}{\partial\boldsymbol{\lambda}}\mathbf{U}\right) \\ &\simeq \mathbf{U}^{\dagger}(\boldsymbol{\lambda}) e^{id\boldsymbol{\lambda}\cdot\vec{\mathbf{A}}} \mathbf{U}(\boldsymbol{\lambda} - d\boldsymbol{\lambda}). \end{aligned}$$

$$\Rightarrow \Gamma'[C] = \mathbf{U}^{\dagger}(\boldsymbol{\lambda}_0) \Gamma[C] \mathbf{U}(\boldsymbol{\lambda}_0) \quad \text{Gauge covariant}$$

Similarly,

$$\begin{aligned} F_{k\ell} \rightarrow F'_{k\ell} &= \partial_k A'_{\ell} - \partial_{\ell} A'_k - i[A'_k, A'_{\ell}] \\ &= \mathbf{U}^{\dagger} F_{k\ell} \mathbf{U}. \end{aligned}$$