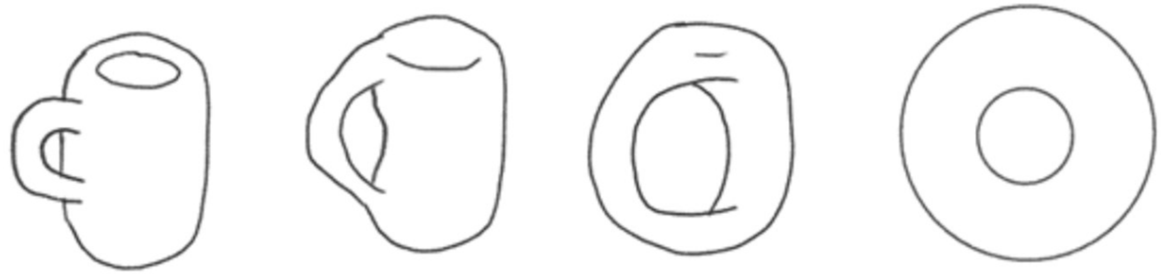
Basics of differential topology

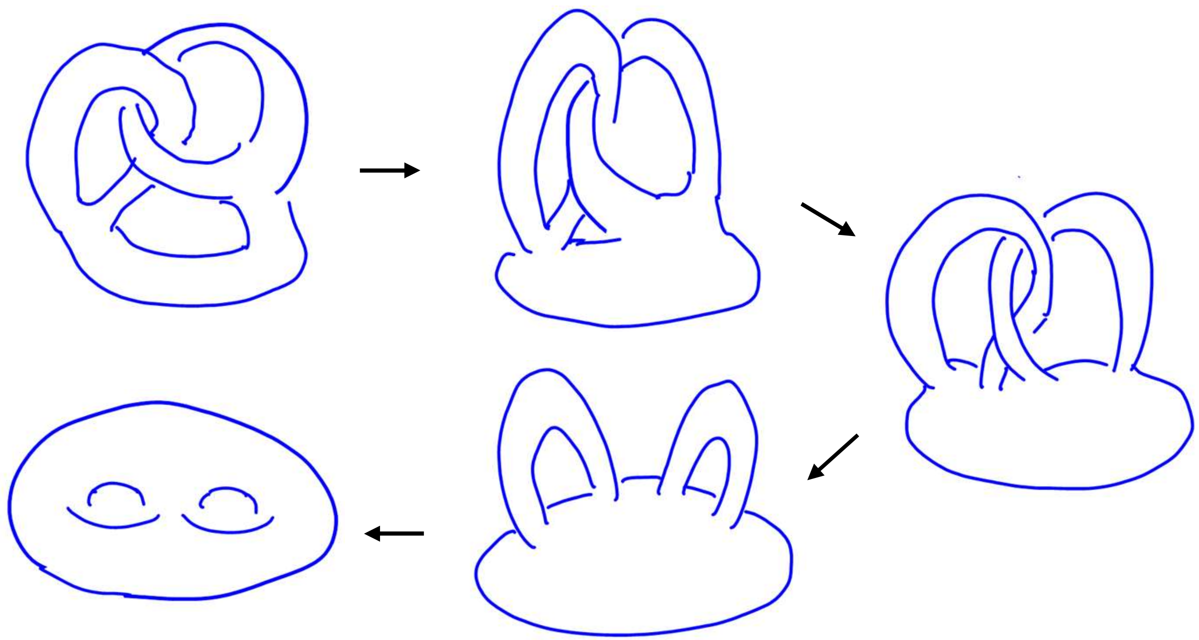
- Winding number
- Euler characteristics
- Gaussian curvature
- Parallel transport
- Gauss-Bonnet theorem
- Hopf-Poincare theorem

Topology in one hour

Topology:

The property of a geometric object that is invariant under *continuous deformation*





If a physics system has topological property, then it is stable against small change of physical condition.

Youtube course: Topology & Geometry, by Tadashi Tokieda

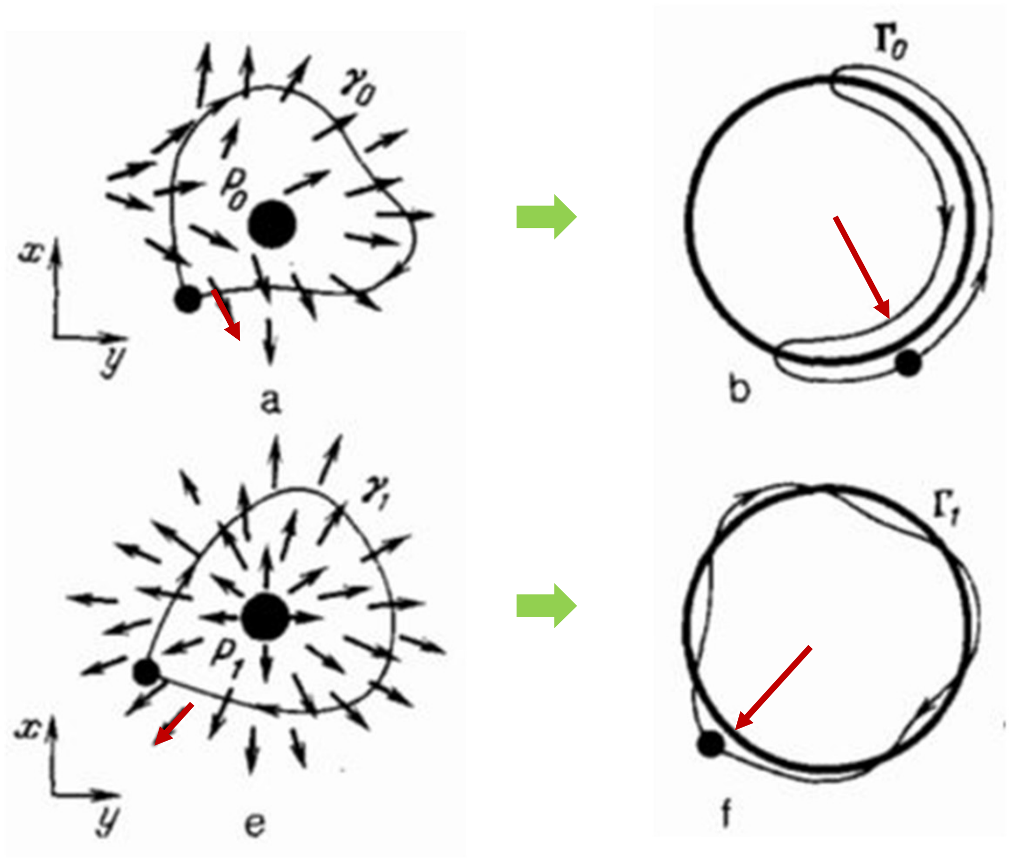
Winding number I

Pattern of flow (fluid, EM…) near a “zero” (point with ***v***=0)

Consider a map from vectors along a closed path to the circle depicted by the direction of vectors

$$f : \vec{r} \to \vec{v}(\vec{r})$$

$$S^1 \to S^1$$



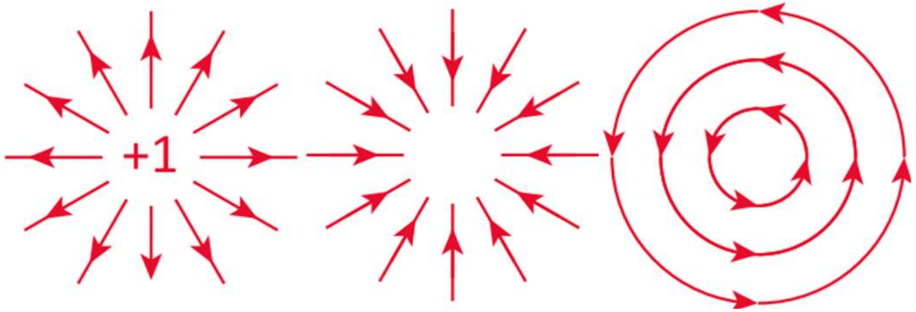
Winding number

w=0

w=1

Fig from Kurik and Lavrentovich, 1988

Source, sink, vortex:



$$\vec{v}(\vec{r}) = v_0\left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

What's their winding numbers?

They all have w=1 and are continuously deformable to each other (in 2D).

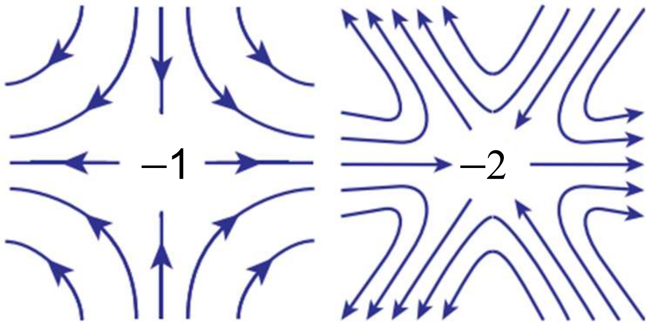


Fig from W. Chen et al PRL 2019

$$d\theta = \mathrm{d}\tan^{-1} v_y/v_x$$

➡ Winding number

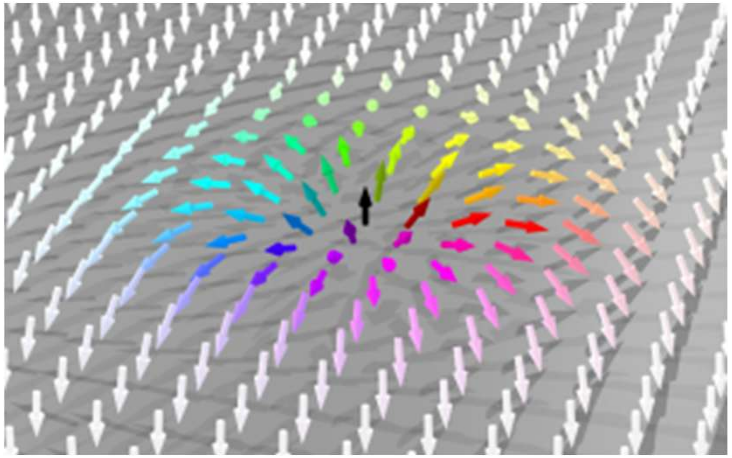
$$w = \frac{1}{2\pi}\oint_c \frac{\vec{v} \times d\vec{v} \cdot \hat{z}}{v^2}$$

Check that the vortex flow above has w=1

Winding number II

Now the vectors on a plane can point out of the plane $f:\vec{r}\rightarrow\vec{v}(\vec{r})$

$$R^2 \rightarrow S^2$$

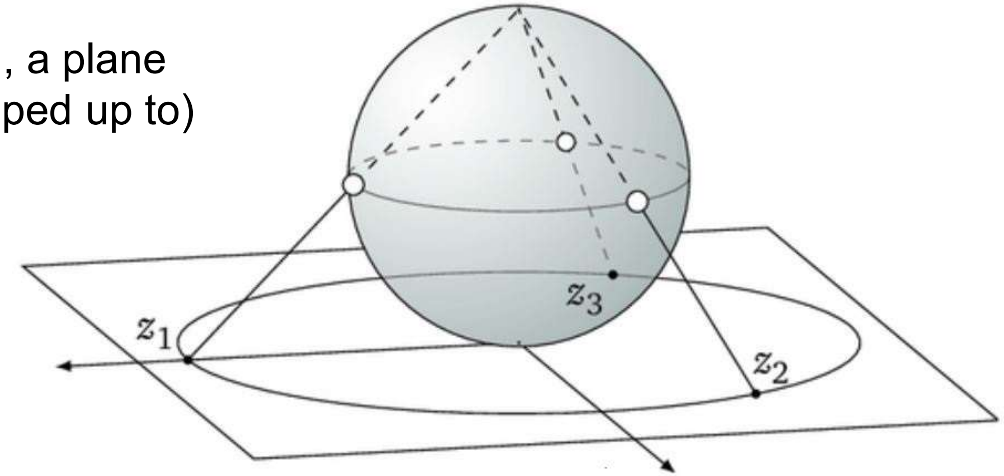


Such a localized texture of vectors is called a skyrmion 史科子

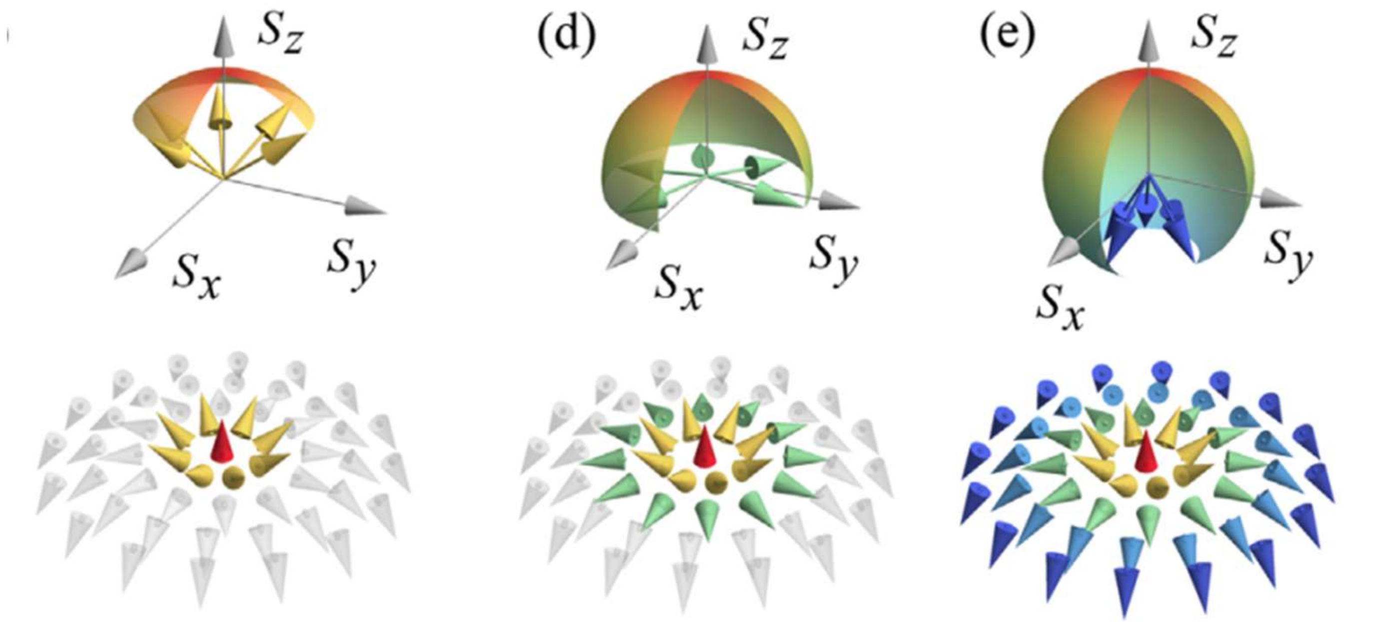
Hypothetical structure of nucleons proposed by Skyrme, 1962

Fig from Karin and Matthias Sitte, JAP 2014

By stereographic projection, a plane can be identified with (wrapped up to) a sphere



A map from this sphere to the direction of vectors $f : S^2 \rightarrow S^2$



w=1

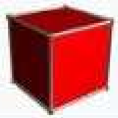
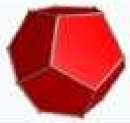
J. Junquera et al, Rev. Mod. Phys. 2024

Winding number
(details later)

$$\mathrm{w} = \frac{1}{4\pi}\int_A \mathbf{m}\cdot\left(\frac{\partial \mathbf{m}}{\partial x}\times\frac{\partial \mathbf{m}}{\partial y}\right)\mathrm{d}A.$$

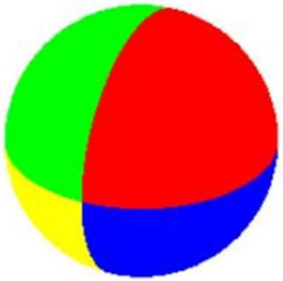
Another topological number: Euler characteristics

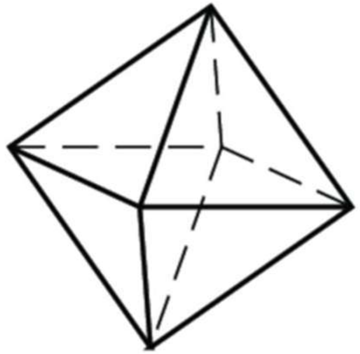
First, platonic solids (F. Maurolico, 1537) 正多面體

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

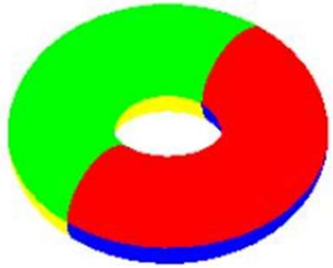
Beyond regular polyhedron, Euler (1758)

sphere





torus

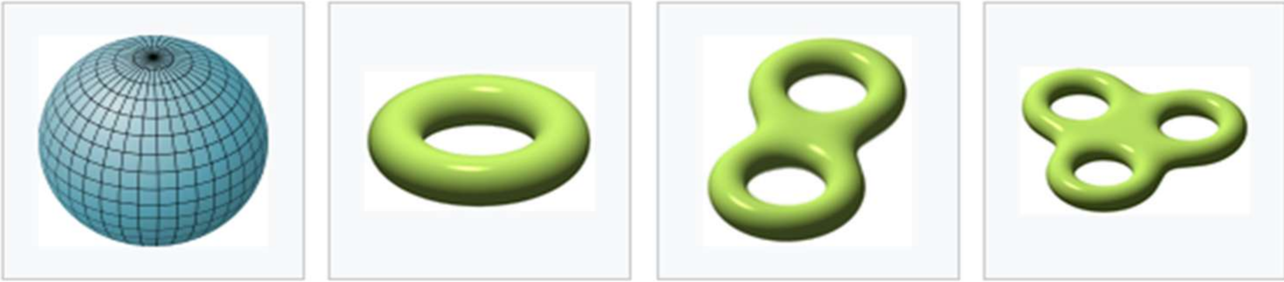


$$\chi = V - E + F = 2 - 4 + 4 = 2$$

$$\chi = V - E + F = 4 - 8 + 4 = 0$$

- Divide a surface into patches.
- This number χ is independent of the ways of division, so it's a property of the surface itself.
- Furthermore, it does not change under continuous deformation, so it's a topological invariant.

Euler characteristic of a closed surface



$\chi = 2$ $\chi = 0$ $\chi = -2$ $\chi = -4$

k-simplex
(*k*-單體)

$$\chi(M) = \beta_0 - \beta_1 + \beta_2$$

β_k is the number of k-**simplexes**

Euler characteristic $\chi(M) = 2(1-g)$

# of holes

Q: Verify that the Euler characteristics of a disk is 1.

How would χ change if you punch a hole in the disk?

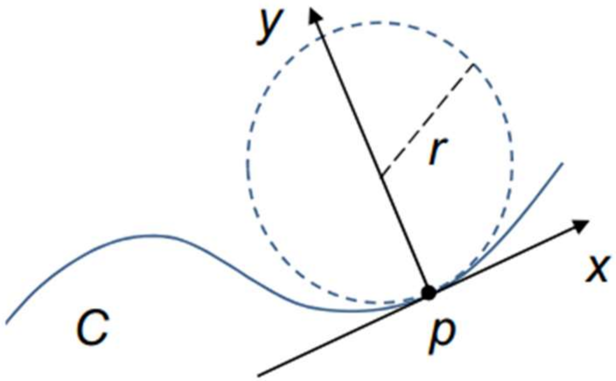
https://math.stackexchange.com/questions/2270687/variation-on-gauss-bonnet-theorem-disjoint-discs

Differential geometry in one hour

Curvature of a surface

- First, how do we quantfy the curvature of a line (at point p)?

$$y = kx^2 + O(x^3)$$



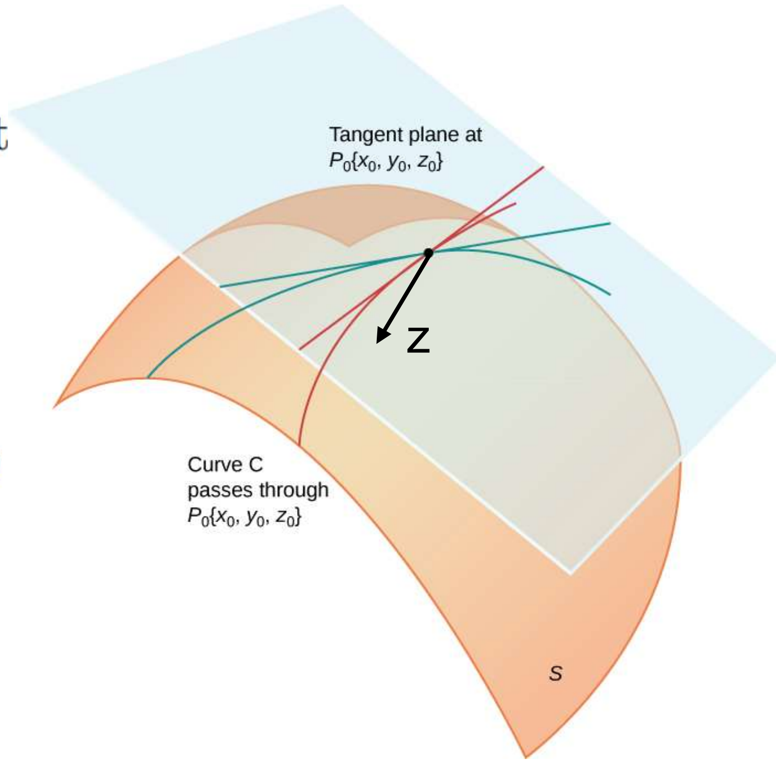
Curvature k at p: $k = \frac{1}{r}$

- How do we quantify the curvature of a surface?

$$\begin{aligned} z &= ax^2 + 2bxy + cy^2 + O(x^3, y^3 \text{ et} \\ &\simeq (x, y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \mathbf{r}^T \mathsf{M} \mathbf{r}. \end{aligned}$$

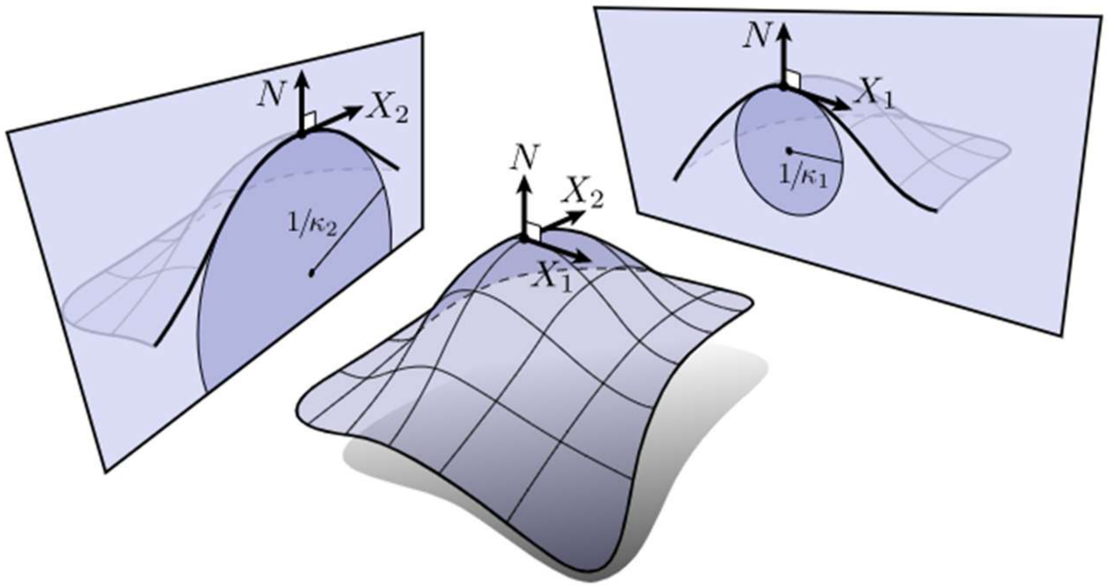
Rotation of xy- coordinate $\mathsf{M} = \mathsf{O}^T \mathsf{D} \mathsf{O}$

$$\begin{aligned} z &= \mathbf{r}'^T \mathsf{D} \mathbf{r}', \ \mathbf{r}' = \mathsf{O}\mathbf{r} \\ &= k_1 x'^2 + k_2 y'^2. \end{aligned}$$



One can fit the surface near p by a quadratic surface (ellipsoid, paraboloid, hyperboloid)

- Along 2 perpendicular directions (x'- and y'- axes), we can define curvatures k_1,k_2. They are called principle curvatures. 主曲率
- They are max and min curvatures among all directions.
- Osculating circles along 2 principle directions have radii $r_1 = \frac{1}{k_1}$ $r_2 = \frac{1}{k_2}$



Two invariants of matrix M under rotation: tr M and det M

- **Mean curvature** 平均曲率 $$H = k_1 + k_2 = \frac{1}{r_1} + \frac{1}{r_2}$$
- **Gaussian curvature** $$G = k_1 k_2 = \frac{1}{r_1\, r_2}$$

- Bending a surface can change H, but not G
- *Without* stretching/squeezing a surface (i.e., the shortest distance between any 2 points remain the same), its G will not change.
- *H* is extrinsic curvature 外在, *G* is Intrinsic curvature 內在

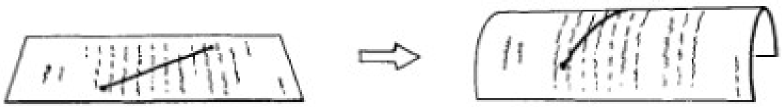
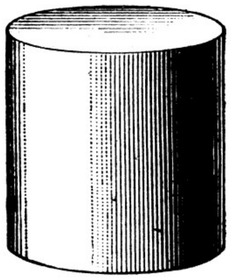


Figure 3.6 Bending a sheet of paper changes its extrinsic—but not its intrinsic—geometry.

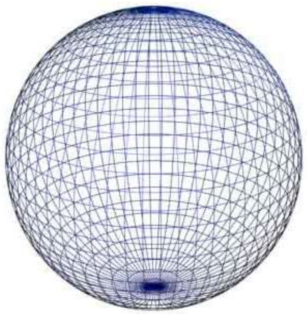
H≠0

G=0



H≠0

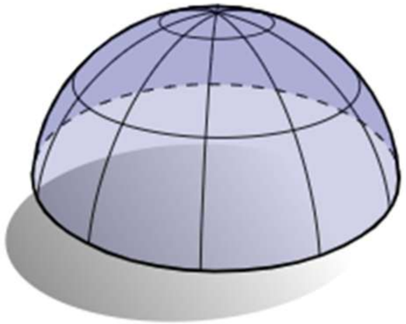
G=0



H≠0

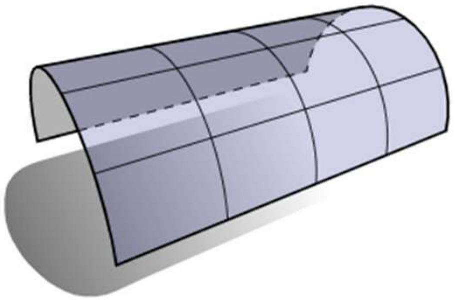
G≠0

Gaussian curvature: *Positive* and *negative*



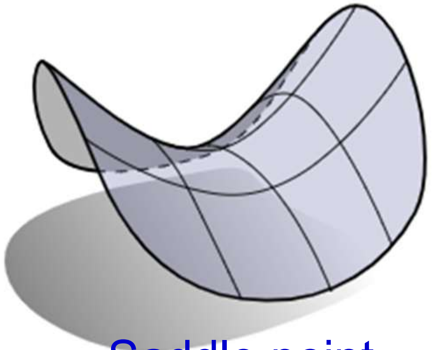
$k_1, k_2 > 0$

G>0



$k_1 > 0, k_2 = 0$

G=0

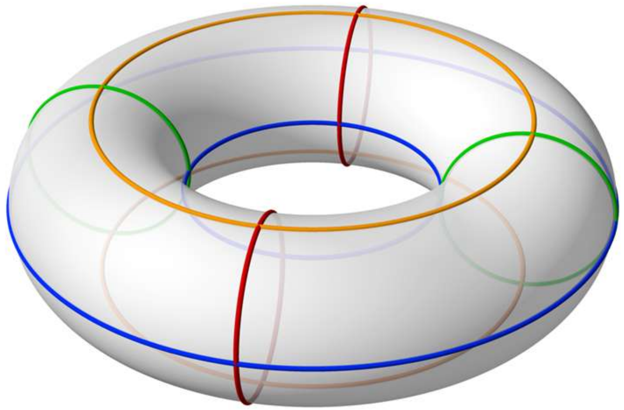


$k_1 > 0, k_2 < 0$

G<0

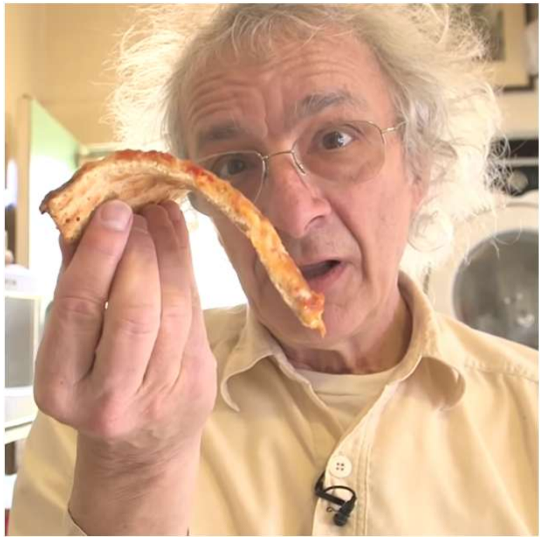
Saddle point

A torus　環面

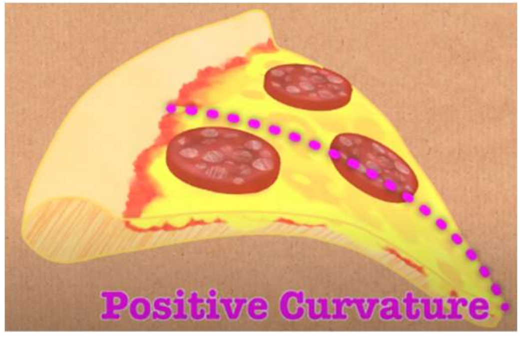


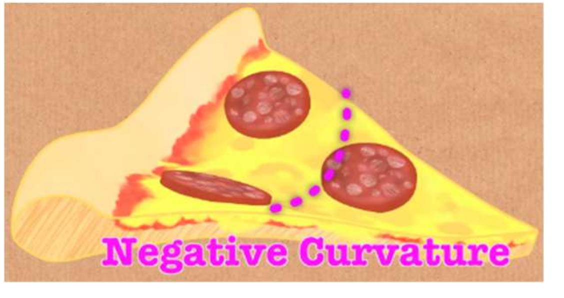
wordpress.discretization.de/geometryprocessingandapplicationsws19/a-quick-and-dirty-introduction-to-the-curvature-of-surfaces/

The Remarkable Way We Eat Pizza -
Youtube: *Numberphile*



- You cannot change Gaussian curvature without stretching/squeezing the surface.
- That is, without stretching your pizza, its G must remain zero, and one of the $k_{1,2}$ must be zero.





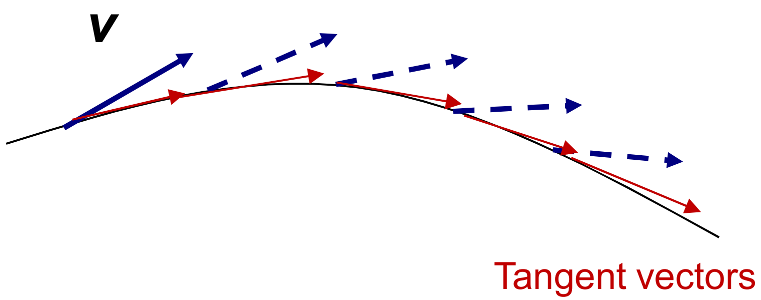
Theorema Egregium, i.e. the most remarkable theorem
(Gauss, 1827)

Gaussian curvature can be determined entirely by measuring angles and distances on a surface.

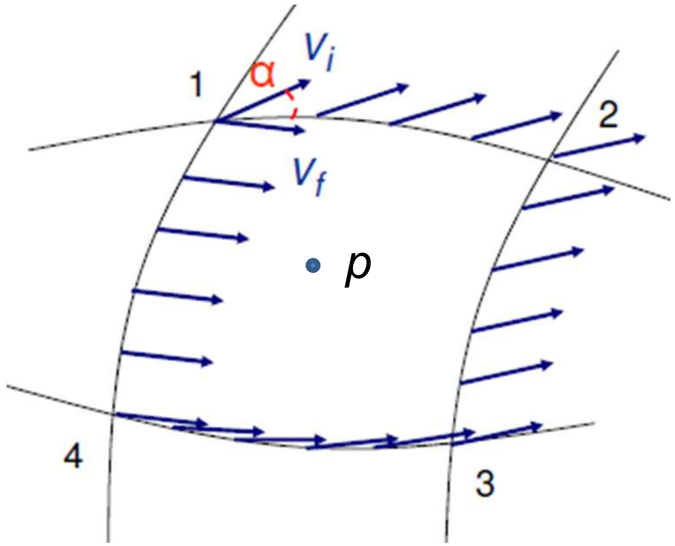
To define G this way, first we need to introduce Parallel Transport

- How do we compare two vectors at different locations on a curved surface?
- *Parallel transport* a vector ***v*** along a geodesic curve

That is, keep the angles between ***v*** and the tangent vectors of the curve fixed.



- Parallel transport a vector around a loop of geodesics.
- After circling a loop on a curved surface, ***v*** would not come back to its initial state, but is rotates by an angle α
- This kind of behavior is called anholonomy (incomplete) or holonomy, and the angle α is called an anholonomy angle (or holonomy angle, defect angle) 虧角

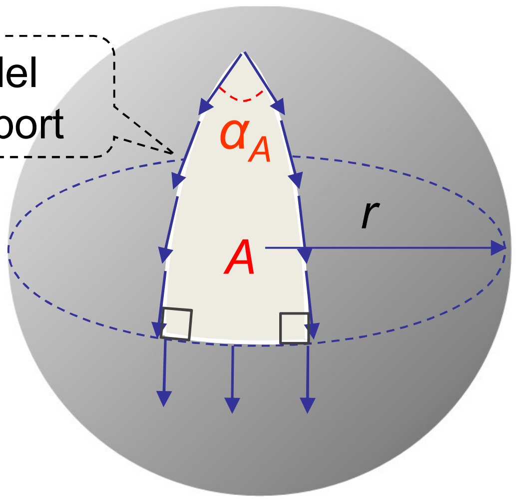


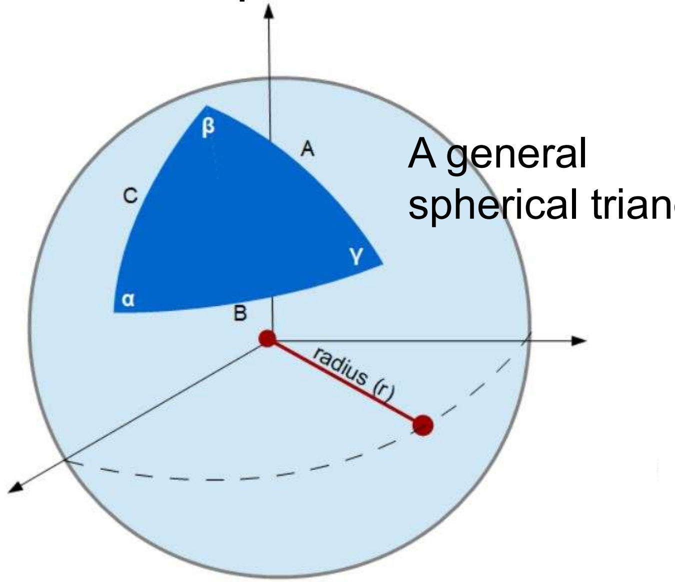
Gaussian curvature at *p* can be defined as

$$G \equiv \lim_{A \to 0} \frac{\alpha_A}{A}$$

Intrinsic definition of Gaussian curvature

Anholonomy angle and curvature of a sphere





A general spherical triangle,

Girard theorem (1626)

area $A = r^2(\alpha + \beta + \gamma - \pi)$

The gravity acts downward, so it cannot affect the orientation of the pendulum. Hence, only the angles between two segments contribute to the anholonomy.

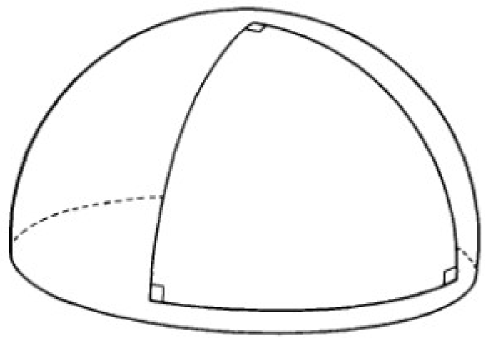
(From Satija's note)

Imagine you're carrying a pendulum walking slowly around the triangle, then

$$\alpha_A = \alpha + \beta + \gamma - \pi$$

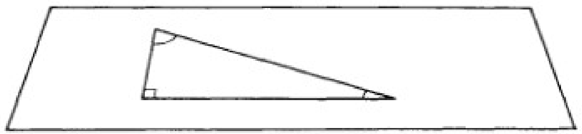
$$\Rightarrow \quad G \equiv \lim_{A \to 0} \frac{\alpha_A}{A} = \frac{1}{r^2}$$

Anholonomy angle and curvature



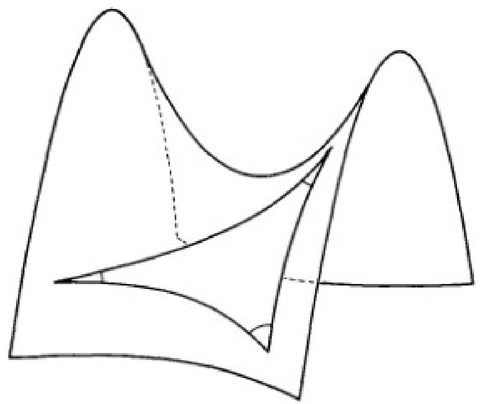
$\alpha_A > 0$

G>0



$\alpha_A = 0$

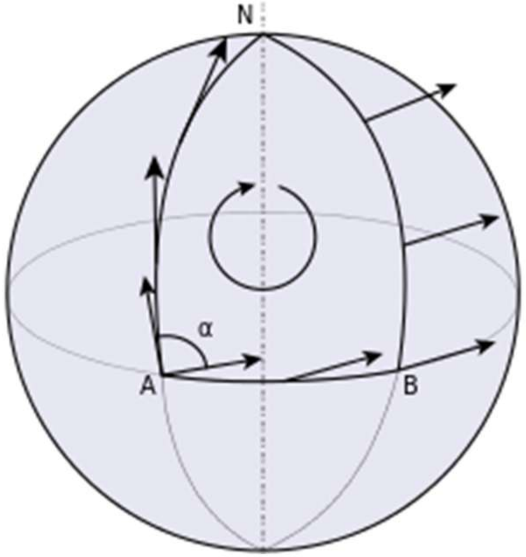
G=0



$\alpha_A < 0$

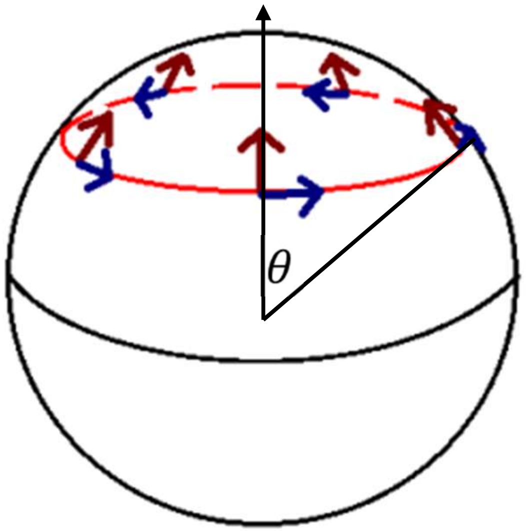
G<0

Great circle



$$\alpha_A = \frac{A}{r^2}$$

Small circle



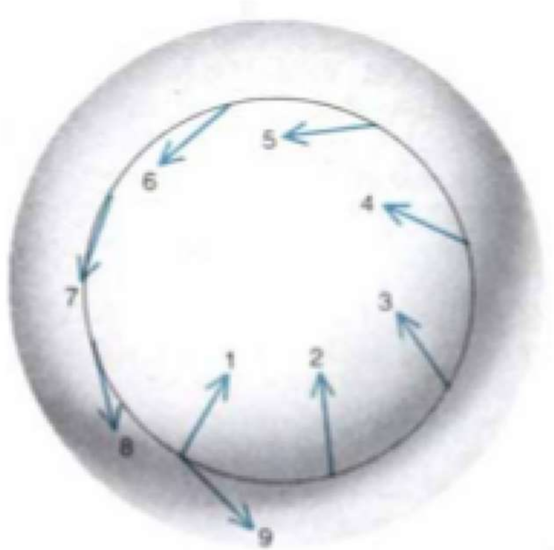
$\alpha_A = 0$?

PT condition along a general curve:
The earlier definition of PT cannot be right (e.g., transporting a vector along a curve on a flat surface).

New definition:
v does not twist around the local vertical axis (normal vector **n**) as we move along a curve *C.*

Parallel transport around a *small* circle on a sphere

v does not twist around the local vertical axis (normal vector **n**) as we move along a curve *C.*



What is the anholonomy angle?

$\alpha = 2\pi(1\text{-}\cos\theta).$

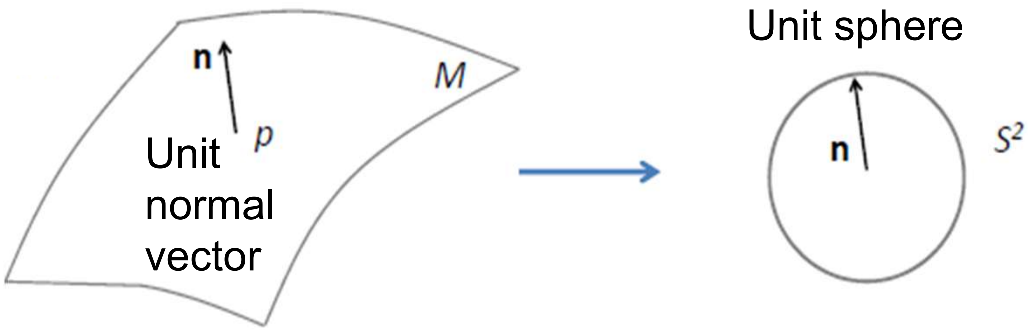
p.234, Intro Diff geometry and Riemannian geometry, by Kreyszig

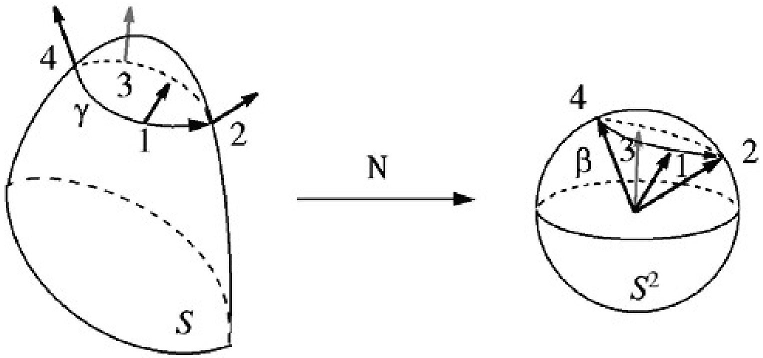
Gauss map and Gaussian curvature

Gauss map

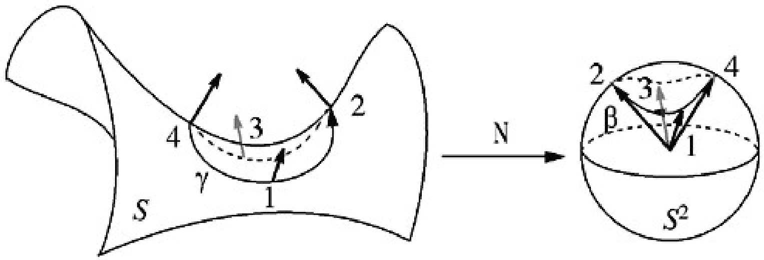
n: M → S^2

From normal vector to a sphere





(a)



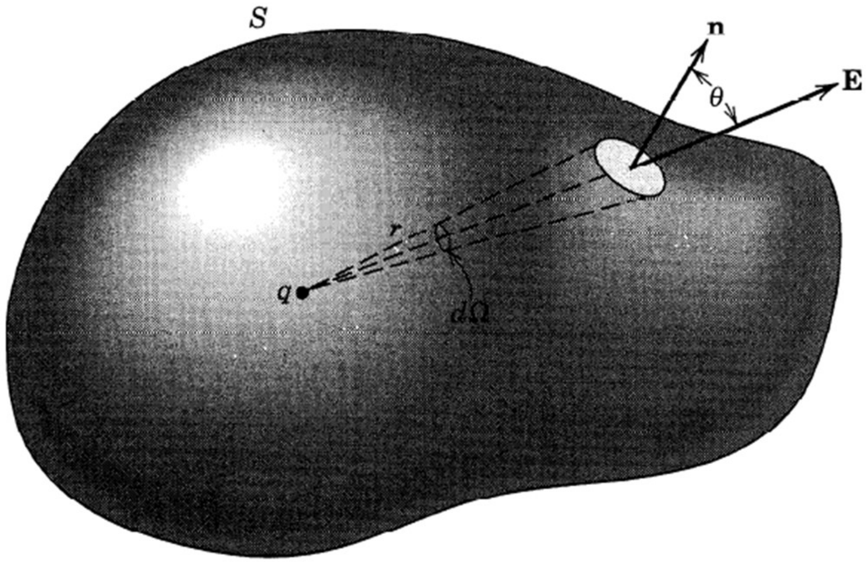
(b)

$$G \equiv \lim_{A \to 0} \frac{S_A}{A}$$

(Ratio between two areas)

Elli Angelopoulou and Lawrence B. Wolff, IEEE Transactions 1998

Total curvature of a closed surface



Total curvature of a closed surface is 4π

(=solid angle of the unit sphere), *no matter how the surface is deformed*

$$\int_M da\, G = \int_M da\, \frac{dS_a}{da} = 4\pi$$

Total curvature is a topological invariant

Topology and differential geometry

Gauss-Bonnet theorem (for 2D surface)

– connecting *local curvature* with *global topology*

- Closed surface

$$\frac{1}{2\pi}\int_M da\, G = \chi(M)$$

The most beautiful theorem in differential topology

- Open surface

$$\frac{1}{2\pi}\left[\int_M da\, G + \int_{\partial M} d\ell \kappa_g\right] = \chi(M, \partial M)$$

p.211, Intro Diff geometry and Riemannian geometry, by Kreyszig

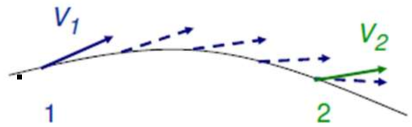
$\chi = 1$

Similar theorem exists in electronic states

$$\frac{1}{2\pi}\int_M da\, \Omega = C$$

Berry curvature   Chern number

Anholonomy in geometry and quantum state

	Geometry	**Quantum state**
PT condition		$i\langle\psi\vert\dot{\psi}\rangle = 0$
anholonomy	Anholonomy angle	Berry phase
curvature	Gaussian curvature	Berry curvature
Topo number	Euler characteristic $\chi = \frac{1}{2\pi}\int_S da\, G$	Chern number $C = \frac{1}{2\pi}\int_M da\, \Omega$

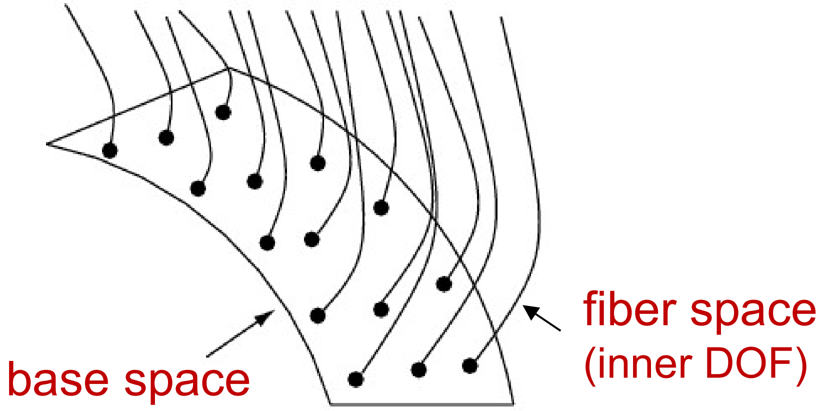
- For a condensed matter system, the geometry and topology are often hidden in momentum space, not real space.

- Chern number refers to the topological number of *fiber bundle space* (Chern, 1946) 纖維束

Ref: *Fiber bundles and quantum theory*, by Bernstein and Phillips, Sci. Am. 1981

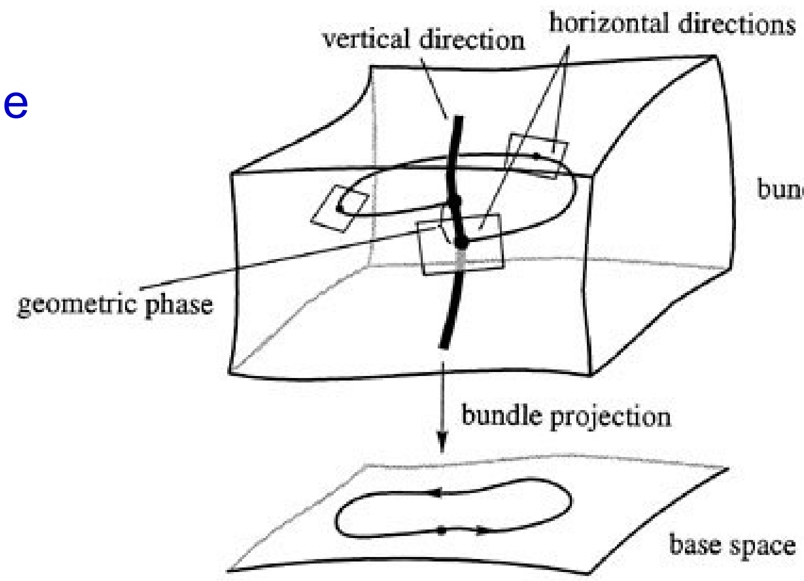
Fiber bundle space *locally* = base space x fiber space

Tangent vector, phase, spin … etc



Fiber bundle

Anholonomy in fiber bundle



Fiber bundles in physics

Gauss-Bonnet theorem:
- Surface
- Tangent space

System	Base space	Fiber space
Electromagnetism	Spacetime	U(1)
Electro-weak theory	Spacetime	U(1)xSU(2)
QCD	Spacetime	SU(3)
Abelian Berry phase	Parameter space	U(1)
Non-Abelian Berry phase	Parameter space	U(N)

Space, Brillouin zone … etc

Winding number again

Index of a point defect in a 2D vector field

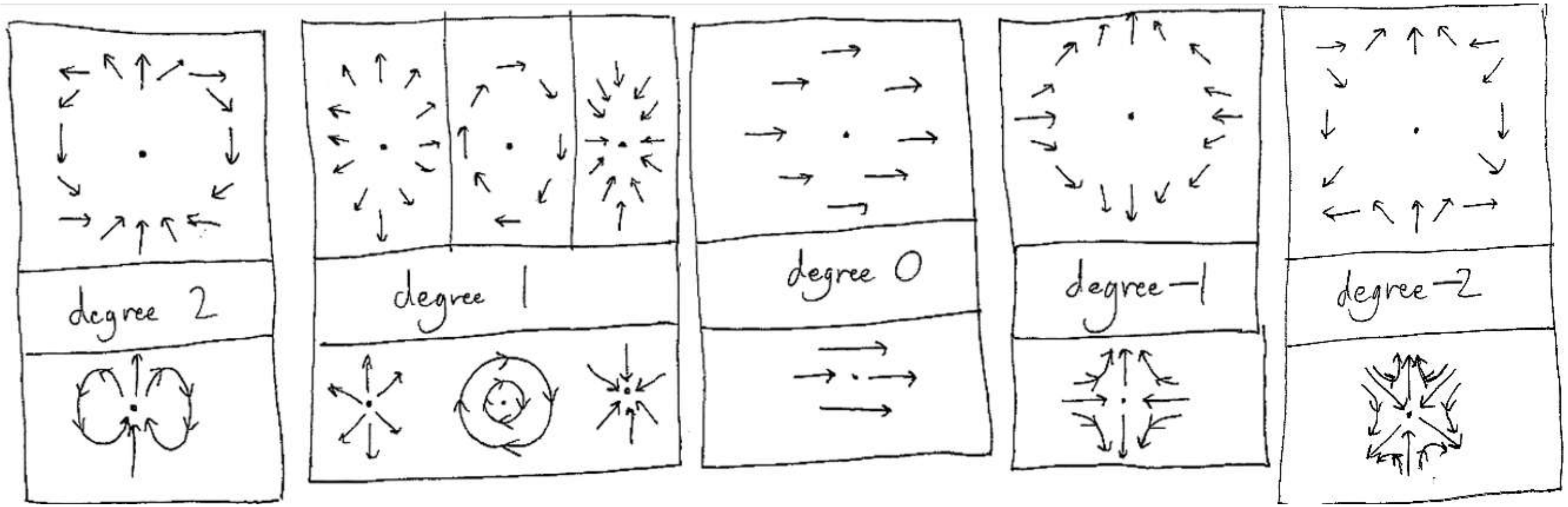


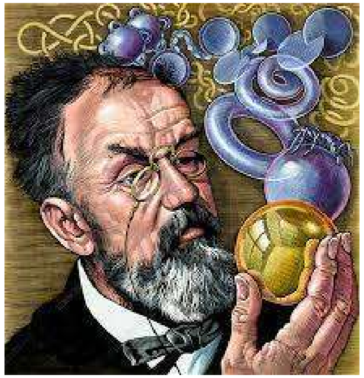
Fig from Jonas Kibelbek

Hopf-Poincare theorem

Winding number

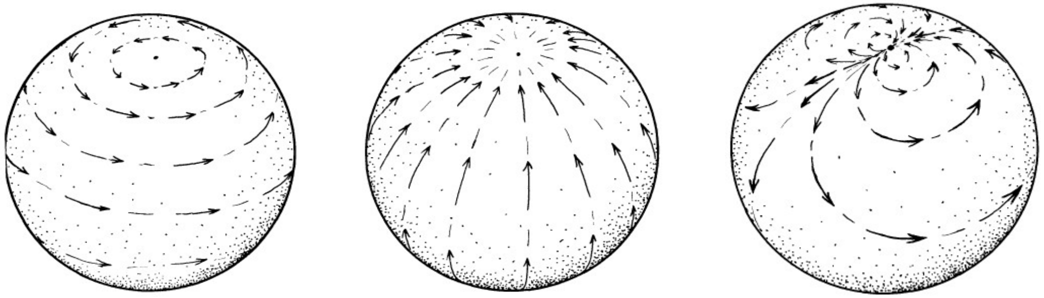
- Connecting index of point defect with topology

$$\sum_i ind(v_i) = \chi(M)$$

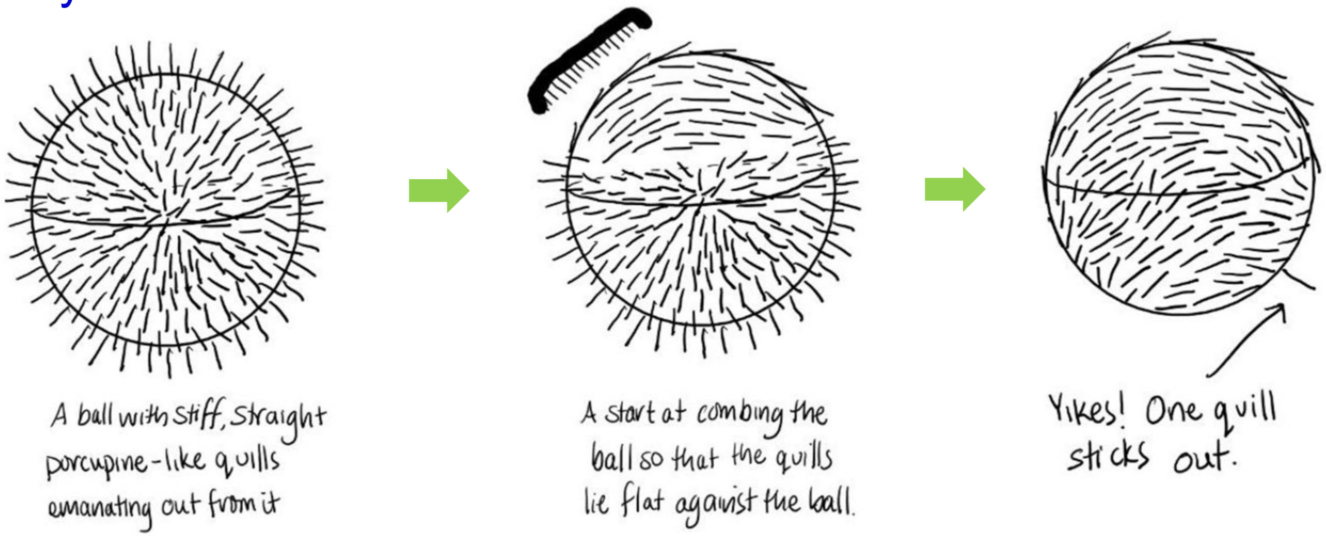


e.g., on a sphere

$$\sum_i ind(v_i) = 2$$



aka Hairy ball theorem

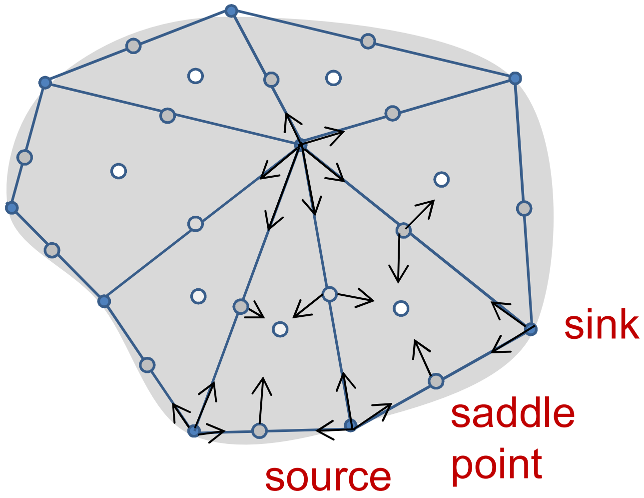


A heuristic explanation of the Hopf-Poincare theorem

Youtube course: *Topology & Geometry*, by Tadashi Tokieda 時枝正

- Put a source on a vertex, a saddle point on an edge,

  and a sink on a face,

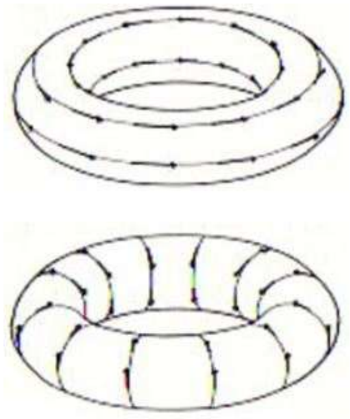


- Then the sum of the indices equals the Euler characteristics

$$\sum_i \text{ind}(\mathbf{v}_i) = (+1)\beta_0 + (-1)\beta_1 + (+1)\beta_2$$
$$= \chi(M)$$

$$\chi(M) = \sum_{k=0}^{D} (-1)^k \beta_k$$

e.g., vector field on a torus $\sum_i ind(v_i) = \chi(T^2) = 0$



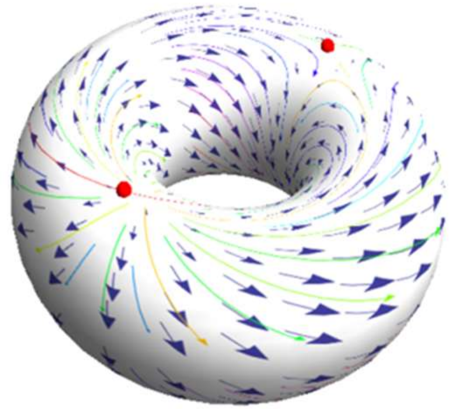
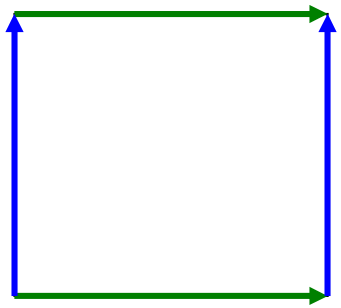
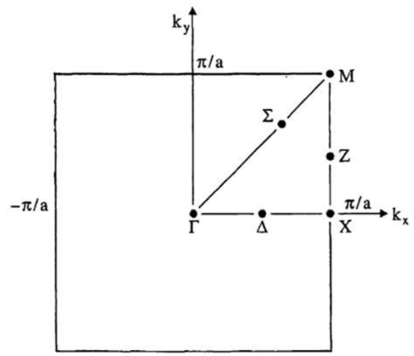


Fig from arxiv.org/pdf/1201.1162.pdf

Application: Brillouin zone (BZ) as a torus (1D, 2D, 3D)





- Berry curvature ***F***(***k***) as a vector field in BZ
- Degenerate point as a source/sink of the ***F*** field