2D p-wave superconductor

A. Lattice model

B. Edge state

C. Vortex and its bound states

D. Topological qubit

   1. Braiding 2 Majorana fermions
   2. Braiding 4 Majorana fermions

A. Lattice model (spinless electrons)

$$
\begin{aligned}
H &= \sum_{mn} \Big[-t(c^\dagger_{m+1,n} c_{mn} + c^\dagger_{m,n+1} c_{mn}) + h.c. \\
&- (\mu - 4t) c^\dagger_{mn} c_{mn} \\
&+ \frac{\Delta_0}{2} c^\dagger_{m+1,n} c^\dagger_{mn} + i \frac{\Delta_0}{2} c^\dagger_{m,n+1} c^\dagger_{mn} + h.c. \\
&- \frac{\Delta_0}{2} c^\dagger_{m-1,n} c^\dagger_{mn} - i \frac{\Delta_0}{2} c^\dagger_{m,n-1} c^\dagger_{mn} + h.c. \Big]
\end{aligned}
$$

$$
c^\dagger_{mn} = \frac{1}{\sqrt{N}} \sum_k e^{i(k_x m + k_y n)} c^\dagger_{k_x k_y}
$$

➡ $$
H = \frac{1}{2} \sum_{\mathbf{k}} (c^\dagger_k \ c_{-k}) \mathsf{H}(\mathbf{k}) \begin{pmatrix} c_k \\ c^\dagger_{-k} \end{pmatrix}
$$

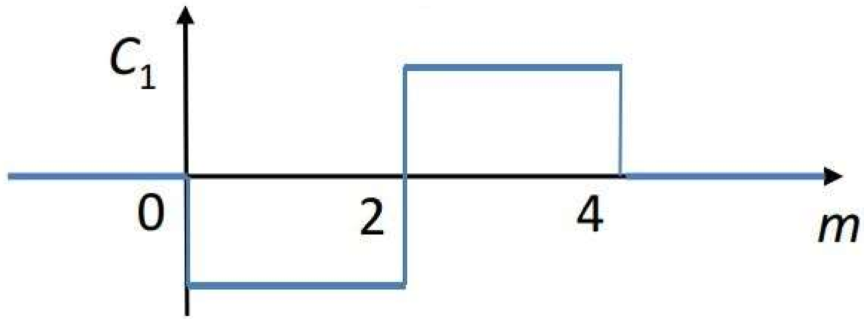
p-wave gap function
(aka $p_x + ip_y$ SC)

$$
\mathsf{H} = \begin{pmatrix} \varepsilon(\mathbf{k}) & 2i\Delta_0(\sin k_x + i \sin k_y) \\ -2i\Delta_0(\sin k_x - i \sin k_y) & -\varepsilon(\mathbf{k}) \end{pmatrix}
$$

$$
\varepsilon(\mathbf{k}) = -2t(\cos k_x + \cos k_y) - (\mu - 4t)
$$

Recall QWZ model:

$$
\begin{aligned}
\mathsf{H}(\mathbf{k}) &= \mathsf{H}_0 + \mathsf{H}_m + \mathsf{H}_{so}, \qquad (1.1)\\
\mathsf{H}_0 &= \varepsilon_0(\mathbf{k}) - \\
t &\begin{pmatrix} 2-\cos k_x a - \cos k_y a & 0 \\ 0 & -(2-\cos k_x a - \cos k_y a) \end{pmatrix}, \quad \begin{matrix}\uparrow \\ \downarrow\end{matrix}\\
\mathsf{H}_m &= m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\\
\mathsf{H}_{so} &= \lambda \begin{pmatrix} 0 & \sin k_x a - i \sin k_y a \\ \sin k_x a + i \sin k_y a & 0 \end{pmatrix}.
\end{aligned}
$$



Compare QWZ model with p-wave model:

$$2t = -t_{QWZ}, \ \mu = -m, \ \text{and} \ 2i\Delta_0 = \lambda.$$

Choose t=1/2

$$\begin{aligned} \mathsf{H}(\mathbf{k}) &= \overbrace{(2-\mu-\cos k_x - \cos k_y)}^{\equiv M(\mathbf{k})}\tau_z \\ &- 2\Delta_0(\sin k_x \tau_y + \sin k_y \tau_x). \end{aligned}$$

$$E_\pm(\mathbf{k}) = \pm\sqrt{M(\mathbf{k})^2 + 4\Delta_0^2(\sin^2 k_x + \sin^2 k_y)}$$

$$\mathsf{H}(\mathbf{k}) = \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}$$

$$C_1 = \frac{1}{4\pi}\int_{BZ} d^2k \frac{1}{h^3}\mathbf{h} \cdot \frac{\partial \mathbf{h}}{\partial k_x} \times \frac{\partial \mathbf{h}}{\partial k_y}$$

B. Edge state

$$\mathsf{H}(\mathbf{k}) = \begin{pmatrix} tk^2 - \mu & 2i\Delta_0(k_x + ik_y) \\ -2i\Delta_0(k_x - ik_y) & -tk^2 + \mu \end{pmatrix}$$

$\mu(x) \simeq \tanh \mathrm{x}$

$$\begin{pmatrix} -\mu & 2i\Delta_0\left(\frac{1}{i}\frac{d}{dx} + ik_y\right) \\ -2i\Delta_0\left(\frac{1}{i}\frac{d}{dx} - ik_y\right) & \mu \end{pmatrix}\psi(x) = \varepsilon_{k_y}\psi(x)$$

➡ $\psi(x) = e^{-\frac{1}{2\Delta_0}\int_0^x dx'\mu(x')}\psi_0 \qquad \psi_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

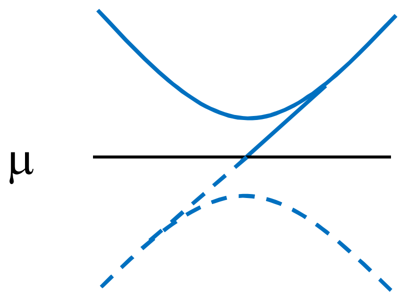
$$\begin{aligned} \gamma_{k_y} &= \int d^2r\left[u^*(\mathbf{r})\psi(\mathbf{r}) + v^*(\mathbf{r})\psi^\dagger(\mathbf{r})\right] \\ &= \int d^2r e^{-ik_y y} e^{-\frac{1}{2\Delta_0}\int_0^x dx'\mu(x')}\left[e^{-i\pi/2}\psi + e^{i\pi/2}\psi^\dagger\right] \end{aligned}$$

➡ $\gamma^\dagger_{-k_y} = \gamma_{k_y}$

➡ $\gamma_0^+ = \gamma_0$

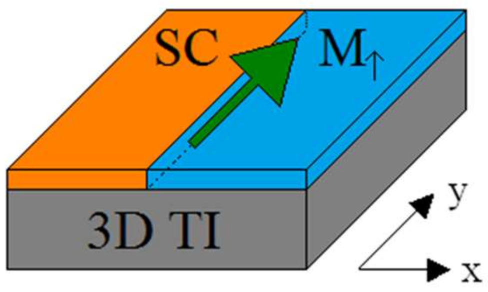
Majorana zero mode (MZM)

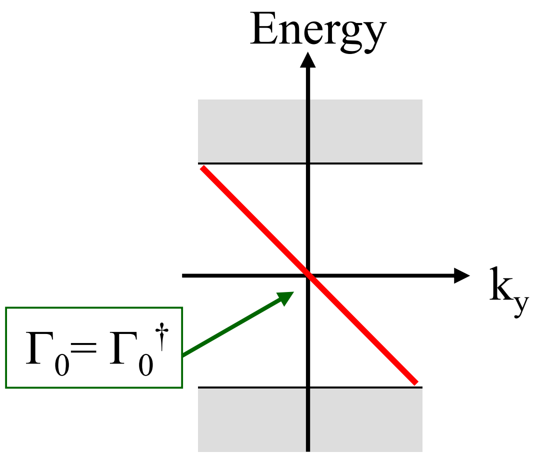
Not isolated



Chiral superconductor

Edge

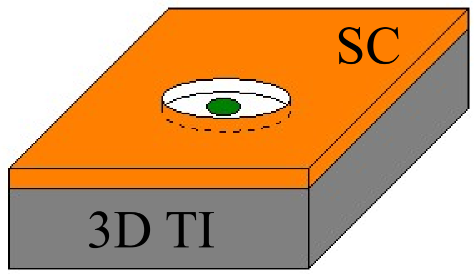


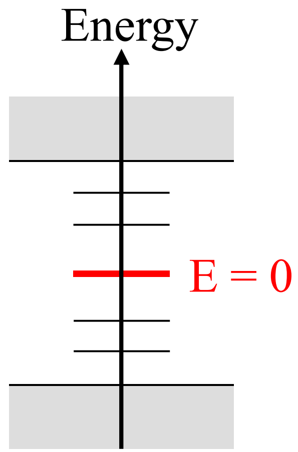


Chiral MF

$\Gamma_E = \Gamma_{-E}^\dagger$

vortex



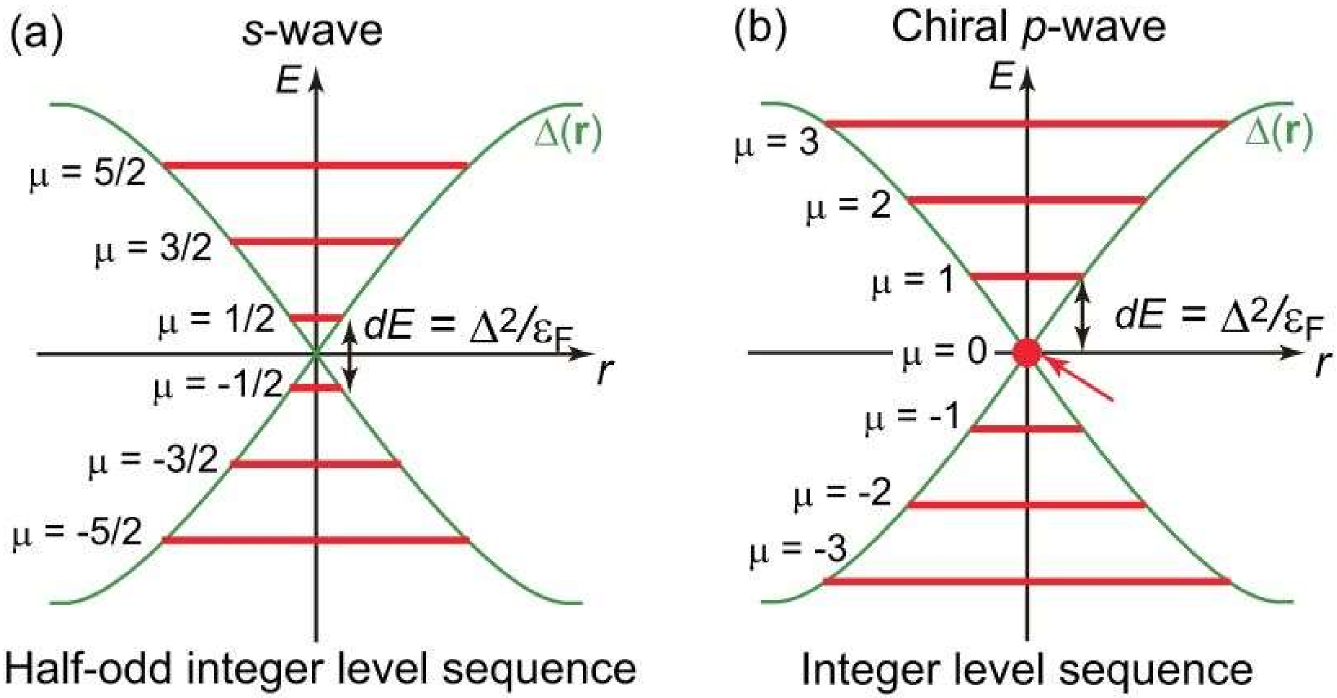


Localized MF

$\Gamma_0 = \Gamma_0^\dagger$

From Fleury's ppt

Bound states in the vortex of *s*-wave and *p*-wave SC



Figs from Machida and Hanaguri,
Prog. Theor. Exp. Phys 2024

- dE ~ 1 μeV in conventional SC

  dE ~ 0.1 meV in iron-based SC

  (Δ~ 2 meV, E_F~10-20 meV)

Brief Review of the Ginzburg-Landau theory (1950)

A phenomenological theory of the order parameter Ψ(r) (App. I of Kittel)

(Later this can be derived from the BCS theory, Gor'kov 1959)

- ψ(r) is complex-valued. It can be seen as a macroscopic wave function.
  ψ(r)=0 for normal state.
- Near *Tc*, it satisfies the Ginzburg-Landau eq

$$\left[-\frac{\hbar^2}{2m^*}\left(\nabla - i\frac{q^*}{\hbar}\mathbf{A}(\mathbf{r})\right)^2 + a(T) + b\,|\psi(\mathbf{r})|^2\right]\psi(\mathbf{r}) = 0$$

$$q^* = -2e$$
$$m^* = 2m$$

- Connection with the gap function

$$\Psi(\mathbf{r}) \simeq \Delta(\mathbf{r})$$

Fetter and Walecka

$$\Psi(\mathbf{x}) \equiv \left[\frac{7\zeta(3)\,n}{8(\pi k_B T_c)^2}\right]^{\frac{1}{2}} \Delta(\mathbf{x})$$

- Connection with SC electron density and SC current density

SC electron density $\rho_S(\vec{r}) = |\psi(\vec{r})|^2$

Current density

$$\vec{j} = \frac{q^*}{2m^*}\left[\psi^*\left(\frac{\hbar}{i}\nabla - q^*\vec{A}\right)\psi + \psi\left(\frac{\hbar}{i}\nabla - q^*\vec{A}\right)^*\psi^*\right]$$

C. Vortex and its bound states

$$\mathbf{j} = \frac{q^*}{2m^*}\left[\Psi^*\left(\frac{\hbar}{i}\nabla - q^*\mathbf{A}\right)\Psi + c.c.\right]$$

$$= -\frac{e\hbar}{2mi}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{2e^2}{m}|\Psi|^2\mathbf{A}$$

$$\Psi(\mathbf{r}) \simeq \Delta(\mathbf{r}) \quad \text{if } \Delta(\mathbf{r}) = |\Delta(\mathbf{r})|e^{-i\xi(\mathbf{r})}$$

$$\Rightarrow \quad \mathbf{j} \propto \frac{\hbar}{2e}\nabla\xi - \mathbf{A}$$

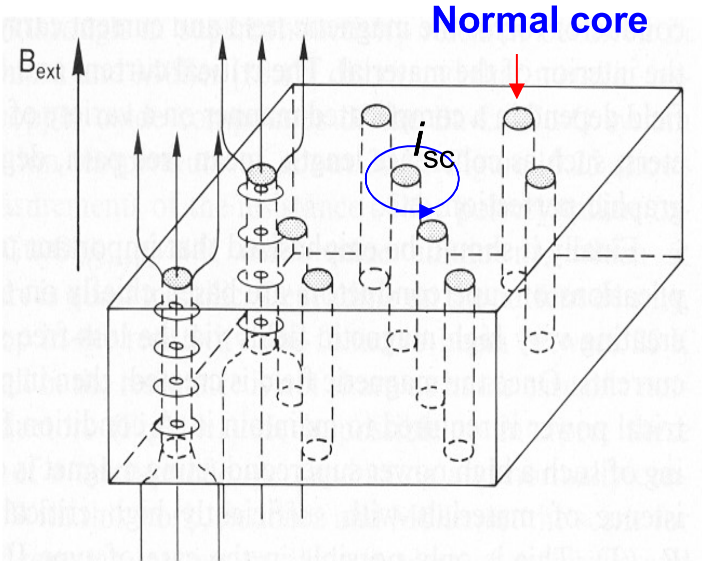
- SC current around a vortex

Far way from a vortex, $\boldsymbol{j}$=0

$$\Rightarrow \quad \oint_C d\mathbf{r}\cdot\mathbf{j} = 0$$

$$\Rightarrow \quad \oint_C d\mathbf{r}\cdot\mathbf{A} = -\frac{\hbar}{2e}[\xi(2\pi) - \xi(0)]$$

$$= \frac{h}{2e}n, \quad n \in Z,$$



The magnetic flux through a vortex is quantized!

Gauge transformation

The phase of Δ is adjustable under the gauge transformation

$$\Delta \;\rightarrow\; \Delta' = \Delta e^{i\chi},$$

$$\text{then}\;\; \xi \;\rightarrow\; \xi' = \xi - \chi,$$

choose $$\mathbf{A} \;\rightarrow\; \mathbf{A}' = \mathbf{A} - \frac{\hbar}{2e}\nabla\chi.$$ so that ***j*** is not changed

BdG equation:

$$\begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0^* \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = E_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

It is invariant under the GT if

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} e^{i\chi/2}u \\ e^{-i\chi/2}v \end{pmatrix}$$

choose $\chi = \xi \;(= n\theta)$ to remove the SC phase, $\Delta' = |\Delta|$

Consider *n*=1,
after a 2π rotation of θ,

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} e^{i\xi/2}u \\ e^{-i\xi/2}v \end{pmatrix}$$

$$= (-1)^n \begin{pmatrix} u \\ v \end{pmatrix}$$

We now study the bound states inside a vortex

Requantizing

$$\mathrm{H}(\mathbf{k}) = \begin{pmatrix} tk^2 - \mu & 2i\Delta_0(k_x + ik_y) \\ -2i\Delta_0(k_x - ik_y) & -tk^2 + \mu \end{pmatrix}$$

Polar coordinate

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y} &= \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \end{aligned}$$

$$\begin{aligned} i(k_x + ik_y) &\to \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \\ &= e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) \end{aligned}$$

➡

$$\begin{pmatrix} -\mu & 2\Delta_0 e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) \\ 2\Delta_0 e^{-i\theta}\left(-\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right) & \mu \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = E_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

Zero-energy solution (zero mode):

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \underbrace{\frac{i}{\sqrt{r}} e^{-\frac{1}{2}\int_0^r dr' \frac{\mu}{\Delta_0(r')}}}_{\equiv ig(r)} \begin{pmatrix} -e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix}$$

Bogoliubov QP for the zero mode

$$
\begin{aligned}
\gamma_0 &= \int d^2r \left[u_0^*(\mathbf{r})\psi(\mathbf{r}) + v_0^*(\mathbf{r})\psi^\dagger(\mathbf{r})\right] \\
&= \int d^2r \; ig(r) \left[e^{-i\theta/2}\psi(\mathbf{r}) - e^{i\theta/2}\psi^\dagger(\mathbf{r})\right]
\end{aligned}
$$

➡ $\gamma_0^+ = \gamma_0$ Majorana zero mode

D. Topological qubit

Two Majorana fermions make an ordinary fermion

$$f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2),$$
$$f_1^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2),$$

Fermion number op $\rightarrow$ $f_1^\dagger f_1 = \frac{1 + i\gamma_1\gamma_2}{2} \sim 0, 1.$

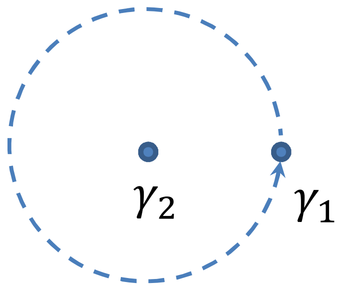
$-i\gamma_1\gamma_2$ is the fermion parity operator

For a MZM located at $\boldsymbol{R}_j$

$$\gamma_j = \int d^2r \left[h_j(\mathbf{r})e^{-i\theta_j/2+i\Gamma_j/2}\psi_j + h_j^*(\mathbf{r})e^{i\theta_j/2-i\Gamma_j/2}\psi_j^\dagger\right]$$

$$h_j(\mathbf{r}) = ig(\mathbf{r} - \mathbf{R}_j),$$

$$\theta_j = \arg(\mathbf{r} - \mathbf{R}_j),$$
$$\Gamma_j = \sum_{\ell \neq j} \arg(\mathbf{R}_j - \mathbf{R}_\ell).$$



The phase Γ_j arises because of a sign change under a 360-degree rotation.

For example, consider only 2 MZMs. If we move γ_1 around γ_2 once, then

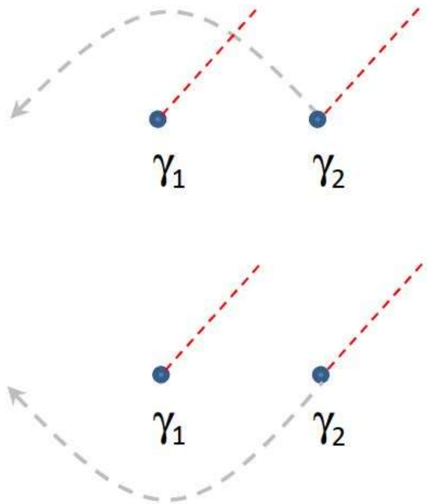
$\Gamma_1 = \arg(\mathbf{R}_1\text{-}\mathbf{R}_2)$ changes by 2π, and γ_1 changes sign.

1. Braiding 2 Majorana fermions

Two ways to exchange γ_1, γ_2

(a) $\begin{cases} \gamma_1 \rightarrow \gamma_2, \\ \gamma_2 \rightarrow -\gamma_1. \end{cases}$

(b) $\begin{cases} \gamma_1 \rightarrow -\gamma_2, \\ \gamma_2 \rightarrow \gamma_1. \end{cases}$



braiding operator

$$\gamma_j \rightarrow B_{12}\gamma_j B_{12}^\dagger$$

For clockwise rotation,

$$B_{12} = \frac{1}{\sqrt{2}}(1 + \gamma_1\gamma_2)$$ (unitary operator)

$$B_{12}^2 = \gamma_1\gamma_2$$

$$\gamma_j \rightarrow B_{12}^2\gamma_j(B_{12}^\dagger)^2 = -\gamma_j$$

Similar to a Kitaev chain, when there are two MZMs in a p-wave SC, the ground states are two-fold degenerate.

- Two Majorana fermions equal one ordinary fermion

$$
\begin{aligned}
f_1 &= \frac{1}{2}(\gamma_1 + i\gamma_2), \\
f_1^\dagger &= \frac{1}{2}(\gamma_1 - i\gamma_2), \\
\rightarrow \quad f_1^\dagger f_1 &= \frac{1 + i\gamma_1\gamma_2}{2} \sim 0, 1.
\end{aligned}
$$

- Write the states with fermion numbers 0, 1 as $|0\rangle$, $|1\rangle$, then,

$$
|1\rangle = f_1^\dagger |0\rangle,
$$

$$
\begin{cases} f_1^\dagger f_1 |0\rangle = 0, \\ f_1^\dagger f_1 |1\rangle = |1\rangle. \end{cases}
$$

- Fermion parity

$$
\begin{cases} -i\gamma_1\gamma_2|0\rangle = +|0\rangle, \\ -i\gamma_1\gamma_2|1\rangle = -|1\rangle. \end{cases}
$$

It follows that,

$$
\begin{aligned}
B_{12}|0\rangle &= \frac{1}{\sqrt{2}}(1 + i)|0\rangle = e^{+i\pi/4}|0\rangle, \\
B_{12}|1\rangle &= \frac{1}{\sqrt{2}}(1 - i)|1\rangle = e^{-i\pi/4}|1\rangle.
\end{aligned}
$$

The braiding operator does not switch the states $|0\rangle$, $|1\rangle$, it only shifts the phases of the states.

2. Braiding 4 Majorana fermions

Consider a 2-qubit system with 4 MZMs

$$f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2),$$
$$f_2 = \frac{1}{2}(\gamma_3 + i\gamma_4).$$

The basis of the Hilbert space are $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

$$B_{kl} = \frac{1}{\sqrt{2}}(1 + \gamma_k\gamma_l)$$

For *intra*-fermion braiding,

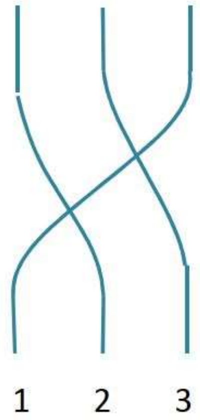
$$B_{12}|00\rangle = \frac{1}{\sqrt{2}}(1+i)|00\rangle,$$
$$B_{34}|00\rangle = \frac{1}{\sqrt{2}}(1+i)|00\rangle.$$

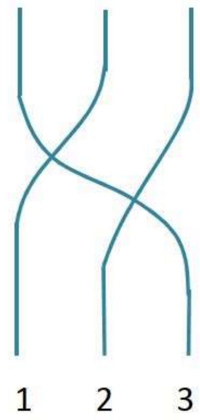
For *inter*-fermion braiding,

$$B_{23}|00\rangle = \frac{1}{\sqrt{2}}(1+\gamma_2\gamma_3)|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle).$$

➡ $[B_{12}, B_{34}] = 0$

$[B_{j-1,j}, B_{j,j+1}] = \gamma_{j-1}\gamma_{j+1}$

(a) 

(b) 

These operations do not change fermion parity of a state.

Matrix representation

$$B_{12} = e^{\frac{\pi}{4}\gamma_1\gamma_2} = e^{i\frac{\pi}{4}\sigma_z\otimes 1}$$
$$= \begin{pmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 & 0 \\ 0 & 0 & e^{-i\pi/4} & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \end{pmatrix},$$

$$B_{34} = e^{\frac{\pi}{4}\gamma_3\gamma_4} = e^{1\otimes i\frac{\pi}{4}\sigma_z}$$
$$= \begin{pmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \end{pmatrix},$$

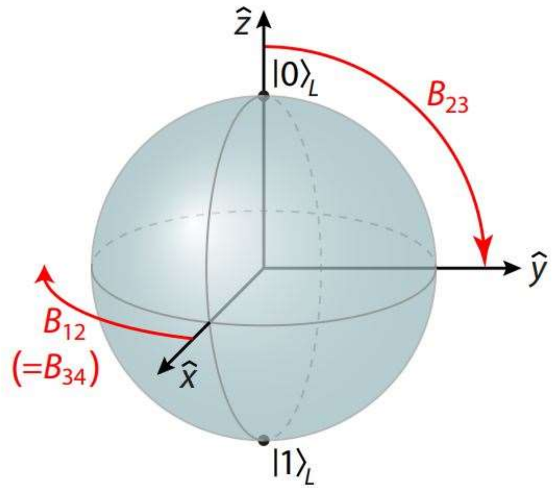
$$B_{23} = e^{\frac{\pi}{4}\gamma_2\gamma_3} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & +i & 1 & 0 \\ +i & 0 & 0 & 1 \end{pmatrix}.$$

Braiding does not change the fermion parity, so one can only access 1 qubit out of the two-qubit system:

Use two states to have 1 qubit

$$|\bar{0}\rangle \equiv |00\rangle,\ |\bar{1}\rangle \equiv |11\rangle$$

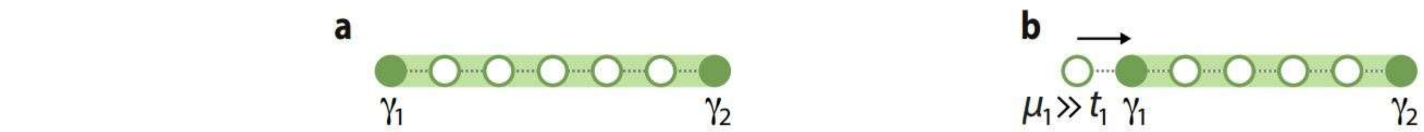
then
$$B_{12}|_+ = B_{34}|_+ = e^{i\frac{\pi}{4}\tau_z}$$
$$B_{23}|_+ = e^{i\frac{\pi}{4}\tau_x}$$



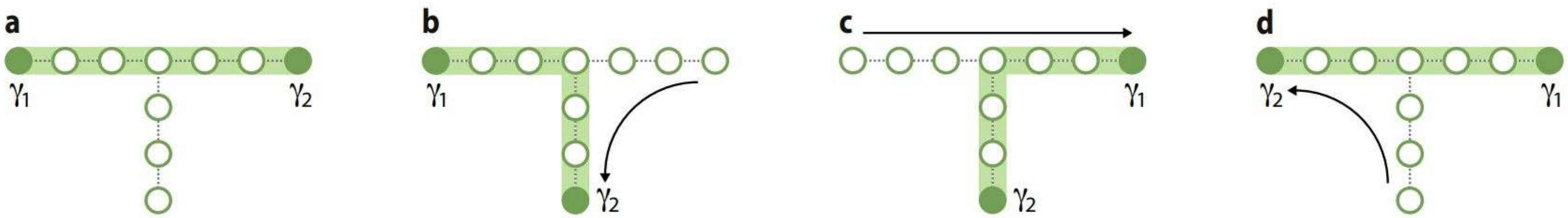
Engineering the Kitaev chain, Bordin

Back to Kitaev chain,

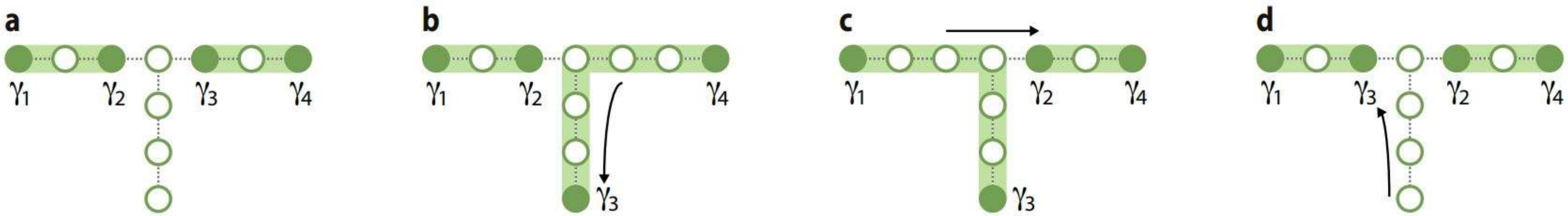
Shuttling Majorana modes by acting on the onsite energies



B_{12} braiding sequence using a Kitaev chain T-junction



B_{23} braiding sequence using a Kitaev chain T-junction



Engineering the Kitaev chain, Bordin

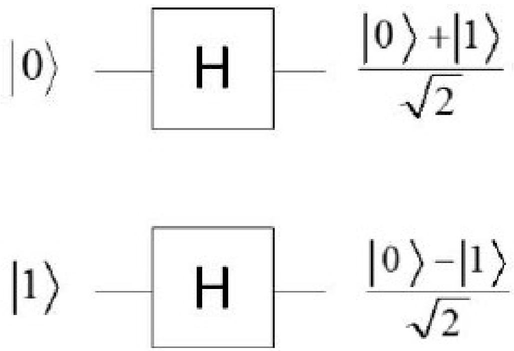
Building quantum gates with the braiding of MZMs

For example,

$$B_{12}^2|_+ = i\tau_z,\ B_{23}^2|_+ = i\tau_x.$$

Pauli-Z gate, Pauli-X gate

Hadamard gate



$$\mathrm{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$B_{12}B_{23}B_{12}|_+ = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = i\mathrm{H}$$

Controlled NOT gate (CNOT)

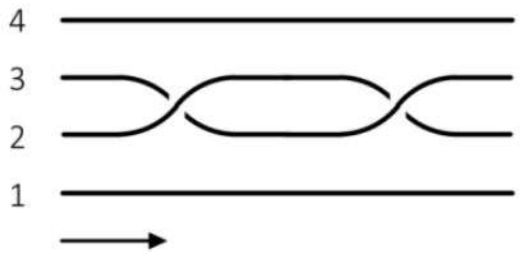
Before		After	
Control	Target	Control	Target
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Since it changes |10⟩ to |11⟩ etc, the fermion parity is not conserved and the H gate cannot be realized with the braiding of 4 Majorana fermions.

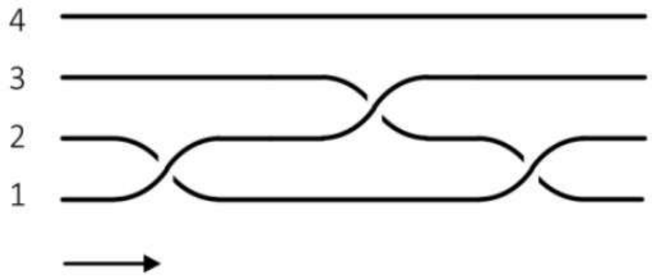
Nevertheless, it is still possible to build the CNOT gate with 6 Majorana fermion.

$$\text{CNOT} = B_{34}^{-1} B_{45} B_{34} B_{12} B_{56} B_{45} B_{34}^{-1} |+$$



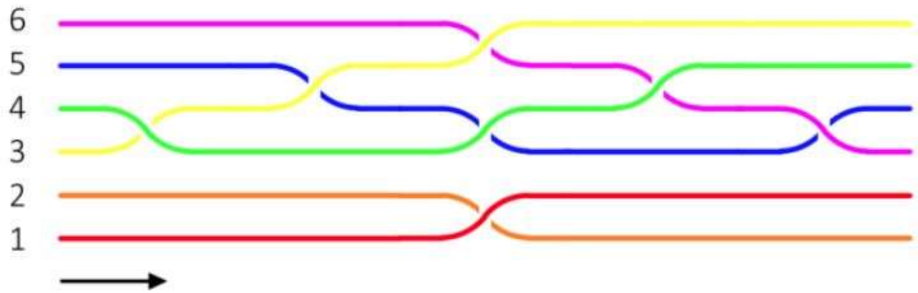
$$X = B_{23}B_{23} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Pauli X quantum gate



$$H = B_{12}B_{23}B_{12}$$

Hadamard quantum gate



$$\text{CNOT} = B_{34}^{-1}B_{45}B_{34}B_{12}B_{56}B_{45}B_{34}^{-1}$$

CNOT quantum gate

https://www.quantum-bits.org/?p=2226

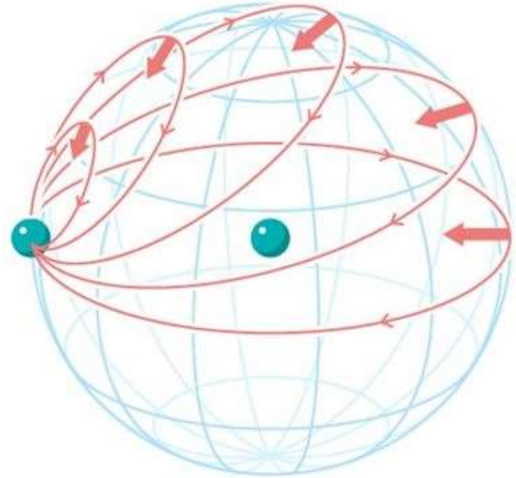
Note: {H, T, CNOT} gates are enough for universal computation

$$\text{T} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Can't be implemented with braiding operations.

In 3D, a loop can shrink to a point.

→ 2 exchanges = no change



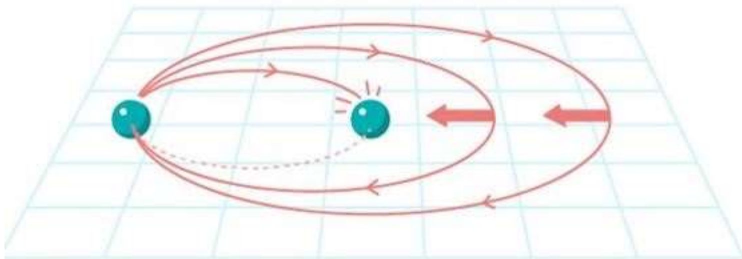
- Exchange operator *E*

$$E\Psi(\vec{r}_1, \vec{r}_2) \equiv \Psi(\vec{r}_2, \vec{r}_1)$$

- For indistinguishable particles $E^2 = 1$

$$\Rightarrow \begin{cases} E = 1 & \text{Boson} \\ E = -1 & \text{fermion} \end{cases}$$

Anyon in 2D (Leinaas and Myrheim 1977, Wilczek 1982)



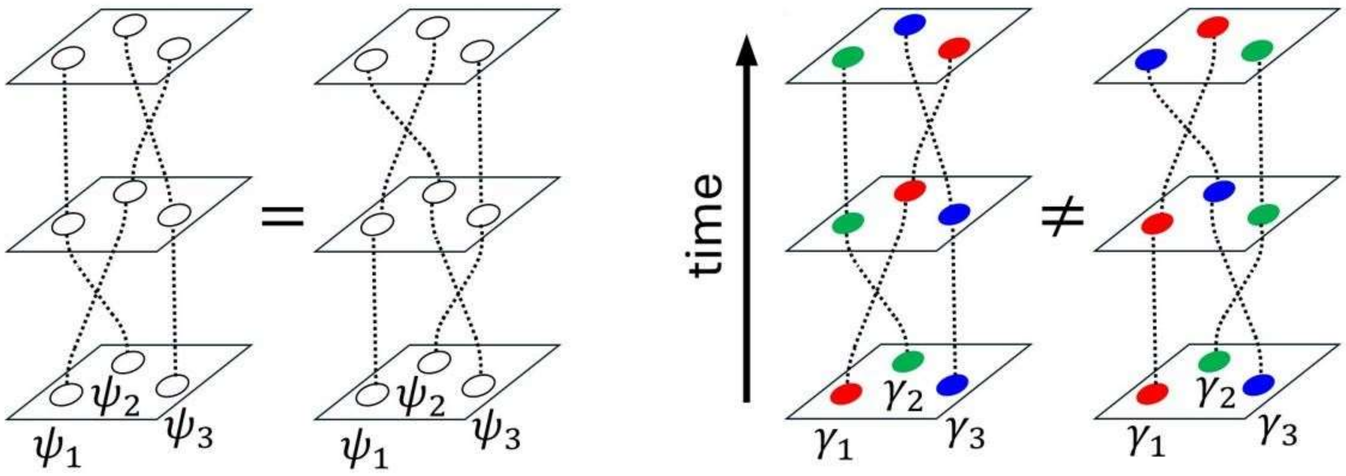
In 2D, a loop can't be unentangled.

→ 2 exchanges ≠ no change

$$\Rightarrow E = e^{i\theta}$$

- *For Majorana fermions, E^4* =1, instead of E^2 =1. So they are anyons.

Abelian anyon | non-Abelian anyon



$$U_{23}U_{12} = U_{12}U_{23} \qquad U_{23}U_{12} \neq U_{12}U_{23}$$

- non-degenerate ground state ~ operations are trivial (hence abelian)
- doubly-degenerate ground state ~ non-abelian anyons
- higher degeneracy ~ universal quantum computation

Real-life examples

Abelian anyon

- QPs in FQHE with ν=1/q (Arovas, Schrieffer, and Wiczek PRL 1984)

$$e^* = \frac{e}{q} \qquad \theta = \frac{\pi}{q}$$

Note: For the 1/q FQHE on a surface with genus g, the ground states are q^g-fold degenerate.

Non-Abelian anyon

- Majorana fermion in the vortex of p-wave SC
- QPs in FQHE with ν=5/2

} Ising anyon

- …
- QPs in FQHE with ν=12/5

Fibonacci anyon (not Majorana)

The simplest one capable of universal quantum computation.

Greiter and Wilczek, Ann Rev CMP 2024