

Lecture notes on topological insulators

Ming-Che Chang

Department of Physics,
National Taiwan Normal University, Taipei,
Taiwan

(Dated: May 26, 2025)

CONTENTS

I. 2D p -wave superconductor	1
A. Lattice model	1
B. Edge state	1
C. Vortex and its bound states	2
1. Zero-energy bound state	3
D. Topological qubit	3
1. Braiding 2 Majorana fermions	3
2. Braiding 4 Majorana fermions	4
3. Quantum logic gate	5
References	6

I. 2D p -WAVE SUPERCONDUCTOR

We now consider spinless p -wave SC in 2D. Two pioneering works on this topic are [Read and Green, 2000](#) and [Ivanov, 2001](#), from which many of the discussions in this lecture are based. *Spinful* p -wave SC in 1D and higher dimensions will be investigated in later lectures.

A. Lattice model

Consider the following lattice model with real Δ_0 ,

$$\begin{aligned}
 H = & \sum_{mn} \left[-t(c_{m+1,n}^\dagger c_{mn} + c_{m,n+1}^\dagger c_{mn}) + h.c. \right. \\
 & - (\mu - 4t)c_{mn}^\dagger c_{mn} \\
 & + \frac{\Delta_0}{2} c_{m+1,n}^\dagger c_{mn}^\dagger + i \frac{\Delta_0}{2} c_{m,n+1}^\dagger c_{mn}^\dagger + h.c. \\
 & \left. - \frac{\Delta_0}{2} c_{m-1,n}^\dagger c_{mn}^\dagger - i \frac{\Delta_0}{2} c_{m,n-1}^\dagger c_{mn}^\dagger + h.c. \right]. \quad (1.1)
 \end{aligned}$$

With the Fourier transform,

$$c_{mn}^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{i(k_x m + k_y n)} c_{k_x, k_y}^\dagger, \quad (1.2)$$

where N is the total number of lattice sites, one gets ($c_{\mathbf{k}}$ is simply written as c_k)

$$H = \frac{1}{2} \sum_{\mathbf{k}} (c_{\mathbf{k}}^\dagger, c_{-\mathbf{k}}) \mathbf{H}(\mathbf{k}) \begin{pmatrix} c_{\mathbf{k}} \\ c_{-\mathbf{k}}^\dagger \end{pmatrix}, \quad (1.3)$$

where

$$\begin{aligned}
 \mathbf{H} &= \begin{pmatrix} \varepsilon(\mathbf{k}) & 2i\Delta_0(\sin k_x + i \sin k_y) \\ -2i\Delta_0(\sin k_x - i \sin k_y) & -\varepsilon(\mathbf{k}) \end{pmatrix}, \\
 \varepsilon(\mathbf{k}) &= -2t(\cos k_x + \cos k_y) - (\mu - 4t).
 \end{aligned}$$

The Hamiltonian matrix $\mathbf{H}(\mathbf{k})$ has the same form as that of the QWZ model in Eq. (??). One only needs to identify

$$2t = -t_{QWZ}, \quad \mu = -m, \quad \text{and} \quad 2i\Delta_0 = \lambda. \quad (1.4)$$

For example, the QWZ model is gapless at $m = 0, 2, 4$. Therefore, here the gap closes at $\mu = 0, -2, -4$.

Choose $t = 1/2$, then

$$\begin{aligned}
 \mathbf{H}(\mathbf{k}) &= \overbrace{(2 - \mu - \cos k_x - \cos k_y)}^{\equiv M(\mathbf{k})} \tau_z \\
 &\quad - 2\Delta_0(\sin k_x \tau_y + \sin k_y \tau_x). \quad (1.5)
 \end{aligned}$$

It has the eigen-energies,

$$E_{\pm}(\mathbf{k}) = \pm \sqrt{M(\mathbf{k})^2 + 4\Delta_0^2(\sin^2 k_x + \sin^2 k_y)}. \quad (1.6)$$

Based on our understanding of the QWZ model, we know that the system has 3 distinct quantum phases: When $\mu > 0$ or $\mu < 4$, it is a trivial phase. When $-1 < \mu < 0$, it is a topological SC phase. When $-4 < \mu < -2$, it is another topological SC phase with opposite chirality.

The topological number is characterized by the first Chern number. Given

$$\mathbf{H}(\mathbf{k}) = \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}, \quad (1.7)$$

one has

$$C_1 = \frac{1}{4\pi} \int_{BZ} d^2k \frac{1}{h^3} \mathbf{h} \cdot \frac{\partial \mathbf{h}}{\partial k_x} \times \frac{\partial \mathbf{h}}{\partial k_y}. \quad (1.8)$$

However, since the electric charge is not conserved, the topological phases have no quantized Hall conductance.

B. Edge state

For simplicity, we study the edge state in the continuum limit. In the small- k limit, the Hamiltonian matrix reduces to

$$\mathbf{H}(\mathbf{k}) = \begin{pmatrix} tk^2 - \mu & 2i\Delta_0(k_x + ik_y) \\ -2i\Delta_0(k_x - ik_y) & -tk^2 + \mu \end{pmatrix}. \quad (1.9)$$

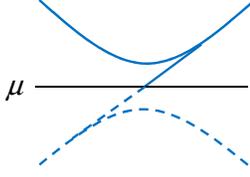


FIG. 1 The energy dispersion of the chiral edge state inside a SC gap.

Assume the chemical potential has a profile similar to $\mu(x) = \tanh x$, then the topological SC occupies the space with $x < 0$. Because of the translation symmetry along y , the eigenstate is of the form $\psi(x)e^{ik_y y}$. We now substitute k_x by $(1/i)(d/dx)$, neglect k^2 terms, and solve for

$$\begin{pmatrix} -\mu & 2i\Delta_0 \left(\frac{1}{i} \frac{d}{dx} + ik_y \right) \\ -2i\Delta_0 \left(\frac{1}{i} \frac{d}{dx} - ik_y \right) & \mu \end{pmatrix} \psi(x) = \varepsilon_{k_y} \psi(x). \quad (1.10)$$

Again it's easier to take a guess at the edge eigenstate. Try

$$\psi(x) = e^{-\frac{1}{2\Delta_0} \int_0^x dx' \mu(x')} \psi_0, \quad (1.11)$$

then we will get

$$\psi_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (1.12)$$

with the eigen-energy $\varepsilon_{k_y} = 2\Delta_0 k_y$ (see Fig. 1).

The energy dispersion of the edge state is roughly linear at small k . Furthermore, it is chiral. Therefore, the 2D p -wave SC is sometimes called the **chiral superconductor**.

The Bogoliubov QP for the edge state is

$$\begin{aligned} \gamma_{k_y} &= \int d^2 r [u^*(\mathbf{r})\psi(\mathbf{r}) + v^*(\mathbf{r})\psi^\dagger(\mathbf{r})] \\ &= \int d^2 r e^{-ik_y y} e^{-\frac{1}{2\Delta_0} \int_0^x dx' \mu(x')} \left[e^{-i\pi/2} \psi + e^{i\pi/2} \psi^\dagger \right], \end{aligned} \quad (1.13)$$

where we have removed an overall phase $e^{i\pi/2}$. Therefore,

$$\gamma_{-k_y}^\dagger = \gamma_{k_y}. \quad (1.14)$$

When $k_y = 0$, $\gamma_0^\dagger = \gamma_0$ and the zero mode is a Majorana mode. However, it is not gapped from the edge states at higher energy, and can be easily damaged by thermal excitation. In the next Sec, we'll see that the Majorana mode inside a vortex is gapped, thus can avoid this problem (to some extent).

C. Vortex and its bound states

In the Ginzberg-Landau (GL) theory of SC, the SC state is described by a macroscopic wave function $\Psi(\mathbf{r})$.

This effective theory works near the SC transition, and can be derived from the microscopic BCS theory. In fact, one can show that $\Psi(\mathbf{r}) \simeq \Delta(\mathbf{r})$ (e.g., see [Fetter and Walecka, 1971](#)), differing only by a multiplicative factor. The current density in the GL theory is given as ($q^* = -2e, m^* = 2m$),

$$\mathbf{J} = \frac{q^*}{2m^*} \left[\Psi^* \left(\frac{\hbar}{i} \nabla - q^* \mathbf{A} \right) \Psi + c.c. \right] \quad (1.15)$$

$$= -\frac{e\hbar}{2mi} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) - \frac{2e^2}{m} |\Psi|^2 \mathbf{A}. \quad (1.16)$$

Therefore, if $\Delta(\mathbf{r}) = |\Delta(\mathbf{r})|e^{-i\xi(\mathbf{r})}$, where $\xi(\mathbf{r})$ is a *single-valued* function, then

$$\mathbf{J} \propto \frac{e\hbar}{2m} \nabla \xi - \frac{2e^2}{m} \mathbf{A}. \quad (1.17)$$

A vortex in a superconductor has a core of normal state surrounded by superconducting state. A magnetic flux can penetrate through the core, which is surrounded by circulating superconducting current on the outside. Far away from a vortex, the circulating current density $\mathbf{J}(\mathbf{r})$ reduces to zero exponentially, such that for a large loop C ,

$$\oint_C d\mathbf{r} \cdot \mathbf{J} = 0. \quad (1.18)$$

Therefore,

$$\oint_C d\mathbf{r} \cdot \mathbf{A} = -\frac{\hbar}{2e} [\xi(2\pi) - \xi(0)] \quad (1.19)$$

$$= \frac{\hbar}{2e} n, \quad n \in Z, \quad (1.20)$$

in which $\xi(2\pi) - \xi(0) = 2\pi n$, since ξ is single-valued. Thus the magnetic flux through a SC vortex needs be quantized in units of $h/2e$. In the case of $n = 1$, one can choose $\xi = \theta$, the polar angle.

The phase of Δ is adjustable via a gauge transformation. For example, if

$$\Delta \rightarrow \Delta' = \Delta e^{i\chi}, \quad (1.21)$$

$$\text{then } \xi \rightarrow \xi' = \xi - \chi, \quad (1.22)$$

$$\text{and choose } \mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} - \frac{\hbar}{2e} \nabla \chi. \quad (1.23)$$

The current density would be gauge invariant. Also, you can check that the BdG equation is invariant under the gauge transformation, if

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} e^{i\chi/2} u \\ e^{-i\chi/2} v \end{pmatrix}. \quad (1.24)$$

We now choose $\chi = \xi (= n\theta)$ to remove the SC phase, so that $\Delta' = |\Delta|$. After a 2π rotation of θ , we have

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} e^{i\xi/2} u \\ e^{-i\xi/2} v \end{pmatrix} \quad (1.25)$$

$$= (-1)^n \begin{pmatrix} u \\ v \end{pmatrix}. \quad (1.26)$$

To keep track of the extra sign, one can add a “branch-cut” emanating from the vortex, so that after circling a vortex (and crossing the branch-cut) once, a phase factor $(-1)^n$ is added.

1. Zero-energy bound state

We now study the bound states inside a vortex by re-quantize the equation in Eq. (1.9). First recall that

$$\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \quad (1.27)$$

$$\frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}. \quad (1.28)$$

Therefore, upon quantization,

$$i(k_x + ik_y) \rightarrow \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \quad (1.29)$$

$$= e^{i\theta} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right). \quad (1.30)$$

Neglecting the k^2 terms in Eq. (1.9), then we have $(\Delta_0(r) \in R)$,

$$\begin{pmatrix} -\mu & 2\Delta_0 e^{i\theta} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right) \\ 2\Delta_0 e^{-i\theta} \left(-\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right) & \mu \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = E_n \begin{pmatrix} u_n \\ v_n \end{pmatrix}. \quad (1.31)$$

One can verify that the following is an zero-energy solution for $E_0 = 0$,

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \underbrace{\frac{i}{\sqrt{r}} e^{-\frac{1}{2} \int_0^r dr' \frac{\mu}{\Delta_0(r')}}}_{\equiv ig(r)} \begin{pmatrix} -e^{i\theta/2} \\ e^{-i\theta/2} \end{pmatrix}. \quad (1.32)$$

The corresponding Bogoliubov QP is,

$$\begin{aligned} \gamma_0 &= \int d^2r [u_0^*(\mathbf{r})\psi(\mathbf{r}) + v_0^*(\mathbf{r})\psi^\dagger(\mathbf{r})] \\ &= \int d^2r ig(r) [e^{-i\theta/2}\psi(\mathbf{r}) - e^{i\theta/2}\psi^\dagger(\mathbf{r})]. \end{aligned} \quad (1.33)$$

Such a zero-mode bound state is a Majorana mode, $\gamma_0^\dagger = \gamma_0$.

A few remarks: First, for a p -wave SC, near the core of a vortex, $E_n \simeq n\hbar\omega_0$ at low energy, where $\omega_0 \simeq \Delta_0^2/\varepsilon_F \ll \Delta_0$, and n is the angular momentum of the QP (see, e.g., Tewari *et al.*, 2007). However, for a s -wave SC, $E_n \simeq (n+1/2)\hbar\omega_0$ for low-energy bound states (see p. 155 of de Gennes, 1989). The lowest one has energy $\hbar\omega_0/2$, thus there is no zero mode.

Second, candidate host materials for Majorana fermions are: the A -phase of superfluid He-3, and the fractional quantum Hall phase with filling fraction $\nu = 5/2$ (the Moore-Read state). It is also possible to find them in the hybrid structure of 2D TI surface state+ s -wave SC (Fu and Kane, 2008), or 2D Rashba+ s -wave SC (Alicea, 2010; Sau *et al.*, 2010).

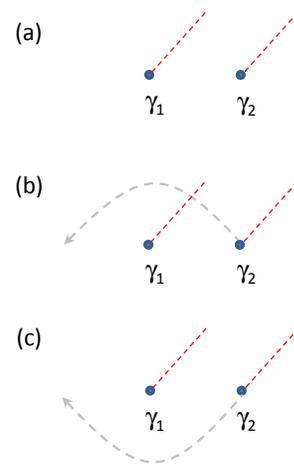


FIG. 2 (a) Attach a branch cut to each vortex. The position of the branch cut is gauge dependent. (b) Exchange the locations of 2 MZMs by moving γ_2 counter-clockwise around γ_1 . (c) Exchange the locations of 2 MZMs by moving γ_2 clockwise around γ_1 .

D. Topological qubit

Like the MZMs in a Kitaev chain, two MZMs (γ_1, γ_2) in the p -wave SC can store one qubit of information. Such a qubit composed of 2 spatially separated MZMs is robust against local decoherence. In order to understand how to manipulate such qubits, we now consider a system with multiple MZMs. For a MZM located at \mathbf{R}_j , we have (Nayak *et al.*, 2008),

$$\gamma_j = \int d^2r [h_j(\mathbf{r})e^{-i\theta_j/2+i\Gamma_j/2}\psi_j + h_j^*(\mathbf{r})e^{i\theta_j/2-i\Gamma_j/2}\psi_j^\dagger],$$

where $h_j(\mathbf{r}) = ig(\mathbf{r} - \mathbf{R}_j)$, and

$$\theta_j = \arg(\mathbf{r} - \mathbf{R}_j), \quad (1.34)$$

$$\Gamma_j = \sum_{\ell \neq j} \arg(\mathbf{R}_j - \mathbf{R}_\ell). \quad (1.35)$$

The phase Γ_j arises because of the anti-periodicity in Eq. (1.26). For example, consider only 2 MZMs. If we move γ_2 around γ_1 once, then $\Gamma_2 = \arg(\mathbf{R}_2 - \mathbf{R}_1)$ changes by 2π , and γ_2 changes sign. To register such a change of sign, we add a branch cut to each vortex, as shown in Fig. 2(a).

1. Braiding 2 Majorana fermions

To exchange the locations of 2 MZMs, γ_1 and γ_2 , we can move γ_2 around γ_1 in 2 ways (see Figs. 2(b),(c)). The results are different, because of the branch cut. In Fig. 2(b), γ_2 crossed the branch cut of γ_1 , and we have

$$\gamma_1 \rightarrow \gamma_2, \quad (1.36)$$

$$\gamma_2 \rightarrow -\gamma_1. \quad (1.37)$$

In Fig. 2(c), γ_2 does not cross the branch cut of γ_1 , but γ_1 crossed the branch cut of γ_2 , and we have

$$\gamma_1 \rightarrow -\gamma_2, \quad (1.38)$$

$$\gamma_2 \rightarrow \gamma_1. \quad (1.39)$$

For clockwise rotation, we can write

$$\gamma_j \rightarrow B_{12}\gamma_j B_{12}^\dagger, \quad (1.40)$$

where

$$B_{12} = \frac{1}{\sqrt{2}}(1 + \gamma_1\gamma_2) \quad (1.41)$$

is called the **braiding operator**. It is an unitary operator and

$$B_{12}B_{12}^\dagger = B_{12}^\dagger B_{12} = 1. \quad (1.42)$$

A full circle is composed of 2 half circles, and

$$B_{12}^2 = \gamma_1\gamma_2. \quad (1.43)$$

It follows that,

$$\gamma_j \rightarrow B_{12}^2\gamma_j(B_{12}^\dagger)^2 = -\gamma_j. \quad (1.44)$$

Both MZMs change sign since each of them crossed a branch cut once.

Similar to a Kitaev chain, when there are two MZMs in a p -wave SC, the ground states are two-fold degenerate. They can be labelled by fermion parities, $|0_+\rangle, |0_-\rangle$, or fermion occupations, $|0\rangle, |1\rangle$. Two Majorana fermions equal one ordinary fermion, thus we define

$$f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2), \quad (1.45)$$

$$f_1^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2), \quad (1.46)$$

$$\rightarrow f_1^\dagger f_1 = \frac{1 + i\gamma_1\gamma_2}{2} \sim 0, 1. \quad (1.47)$$

The states $|0\rangle, |1\rangle$ have fermion numbers 0, 1, hence

$$|1\rangle = f_1^\dagger|0\rangle, \quad (1.48)$$

$$\begin{cases} f_1^\dagger f_1|0\rangle = 0, \\ f_1^\dagger f_1|1\rangle = |1\rangle. \end{cases} \quad (1.49)$$

Recall that $-i\gamma_1\gamma_2$ is the fermion parity operator, and

$$\begin{cases} -i\gamma_1\gamma_2|0\rangle = +|0\rangle, \\ -i\gamma_1\gamma_2|1\rangle = -|1\rangle. \end{cases} \quad (1.50)$$

It follows that,

$$B_{12}|0\rangle = \frac{1}{\sqrt{2}}(1 + i)|0\rangle = e^{+i\pi/4}|0\rangle, \quad (1.51)$$

$$B_{12}|1\rangle = \frac{1}{\sqrt{2}}(1 - i)|1\rangle = e^{-i\pi/4}|1\rangle. \quad (1.52)$$

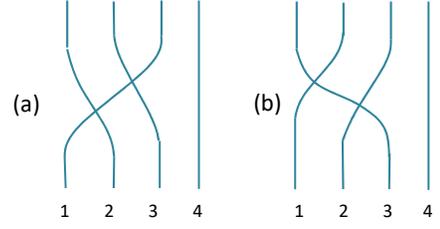


FIG. 3 (a) First exchange γ_1 with γ_2 , then exchange γ_2 with γ_3 . (b) First exchange γ_2 with γ_3 , then exchange γ_1 with γ_2 .

That is, the braiding operator does not switch the states $|0\rangle, |1\rangle$, it only shifts the phases of the states. We can also write

$$B_{12} = e^{\frac{\pi}{4}\gamma_1\gamma_2} = e^{i\frac{\pi}{4}\sigma_z} \quad (1.53)$$

$$= \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}. \quad (1.54)$$

Note that $B_{12}^4 = -1$. A related property is that

$$B_{12}fB_{12}^\dagger = if \text{ or } e^{i\pi/2}f, \quad (1.55)$$

so that after 4 operations,

$$B_{12}^4f(B_{12}^\dagger)^4 = f. \quad (1.56)$$

2. Braiding 4 Majorana fermions

We now consider a 2-qubit system with 4 MZMs,

$$f_1 = \frac{1}{2}(\gamma_1 + i\gamma_2), \quad (1.57)$$

$$f_2 = \frac{1}{2}(\gamma_3 + i\gamma_4). \quad (1.58)$$

The basis of the Hilbert space are $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. For *intra*-fermion braiding, we have, for example,

$$B_{12}|00\rangle = \frac{1}{\sqrt{2}}(1 + i)|00\rangle, \quad (1.59)$$

$$B_{34}|00\rangle = \frac{1}{\sqrt{2}}(1 + i)|00\rangle. \quad (1.60)$$

For *inter*-fermion braiding, we have, for example,

$$\begin{aligned} B_{23}|00\rangle &= \frac{1}{\sqrt{2}}(1 + \gamma_2\gamma_3)|00\rangle \\ &= \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle). \end{aligned} \quad (1.61)$$

Note that the fermion parity, $n_1 + n_2 \pmod{2}$, is not changed by these braidings.

In general, since $(\gamma_k\gamma_l)^2 = -1$ for $k \neq l, k, l = 1, \dots, 4$, which is similar to the property of $i^2 = -1$, we thus have

$$B_{kl} = \frac{1}{\sqrt{2}}(1 + \gamma_k\gamma_l), \quad (k \neq l) \quad (1.62)$$

$$= e^{\frac{\pi}{4}\gamma_k\gamma_l}. \quad (1.63)$$

Two braiding operations that act on different fermions would commute,

$$[B_{12}, B_{34}] = 0. \quad (1.64)$$

However,

$$[B_{j-1,j}, B_{j,j+1}] = \gamma_{j-1}\gamma_{j+1}. \quad (1.65)$$

Because the commutator does not vanish, the order of braiding matters, as shown in Fig. 3.

Under the basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)^T$, these braiding operators can be written as,

$$\begin{aligned} B_{12} &= e^{\frac{\pi}{4}\gamma_1\gamma_2} = e^{i\frac{\pi}{4}\sigma_z \otimes 1} \\ &= \begin{pmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 & 0 \\ 0 & 0 & e^{-i\pi/4} & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \end{pmatrix}, \end{aligned} \quad (1.66)$$

$$\begin{aligned} B_{34} &= e^{\frac{\pi}{4}\gamma_3\gamma_4} = e^{i\frac{\pi}{4}1 \otimes \sigma_z} \\ &= \begin{pmatrix} e^{i\pi/4} & 0 & 0 & 0 \\ 0 & e^{-i\pi/4} & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 0 & e^{-i\pi/4} \end{pmatrix}, \end{aligned} \quad (1.67)$$

$$\begin{aligned} B_{23} &= e^{\frac{\pi}{4}\gamma_2\gamma_3} = e^{i\frac{\pi}{4}\sigma_x \otimes \sigma_x} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \quad (1.68)$$

Since the braiding operations do not change the fermion parity. They could only connect $|00\rangle$ and $|11\rangle$ (or $|10\rangle$ and $|01\rangle$) in a 2D Hilbert sub-space. That is, with braiding only, one can only access 1 qubit out of the two-qubit system. Define the new qubit as,

$$|\bar{0}\rangle \equiv |00\rangle, \quad |\bar{1}\rangle \equiv |11\rangle, \quad (1.69)$$

then within this subspace

$$B_{12}|_+ = B_{34}|_+ = e^{i\frac{\pi}{4}\tau_z}, \quad (1.70)$$

$$B_{23}|_+ = e^{i\frac{\pi}{4}\tau_x}, \quad (1.71)$$

in which $|_+$ projects a 4×4 braiding matrix to the one spanned by the bases with even fermion-parity. They rotate a single qubit by an angle $\pi/2$ around either z -axis or x -axis (see Fig. 4).

3. Quantum logic gate

We can build quantum gates with the braiding of Majorana fermions. For example,

$$B_{12}^2|_+ = i\tau_z, \quad B_{23}^2|_+ = i\tau_x. \quad (1.72)$$

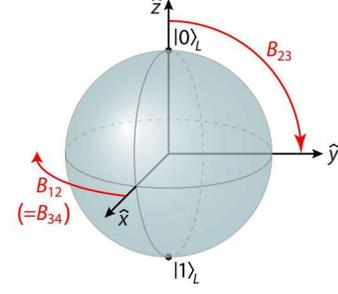


FIG. 4 Inter-fermion braiding B_{12} and intra-fermion braiding B_{23} lead to $\pi/2$ -rotations around different axes. Fig. from the thesis Engineering the Kitaev chain, by A. Bordin.

These are known as the Pauli-Z gate and the Pauli-X gate (the i is a phase shift that is not crucial). Also, you may check that

$$B_{12}B_{23}B_{12}|_+ = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = iH, \quad (1.73)$$

which is known as the Hadamard gate H (again the constant i is not crucial).

Quantum computation can be realized with a set of a few *universal quantum logic gates*. This set can be the Hadamard gate H, the phase-shift gate T, and the Controlled-NOT gate CNOT. They have the following matrix representations,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (1.74)$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}, \quad (1.75)$$

$$\text{and CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (1.76)$$

The CNOT gate changes $|00\rangle$ to $|00\rangle$, $|01\rangle$ to $|01\rangle$, $|10\rangle$ to $|11\rangle$, and $|11\rangle$ to $|10\rangle$. That is, when the first bit is 0, don't flip the second bit; when the first bit is 1, flip the second bit. Since it changes $|10\rangle$ to $|11\rangle$ etc, the fermion parity is not conserved and the H gate cannot be realized with the braiding of 4 Majorana fermions.

Nevertheless, it is still possible to build the CNOT gate with 6 Majorana fermion. You may check that

$$\text{CNOT} = B_{34}^{-1}B_{45}B_{34}B_{12}B_{56}B_{45}B_{34}^{-1}|_+. \quad (1.77)$$

The braiding sequence is shown in Fig. 5.

Therefore, the braiding of Majorana fermions helps to build two universal quantum gates. Unfortunately, it fails to simulate the phase-shift gate T. To achieve this, different quasiparticles in systems with richer topology are required. For example, the one in $\nu = 12/5$ fractional quantum Hall effect (Nayak *et al.*, 2008).

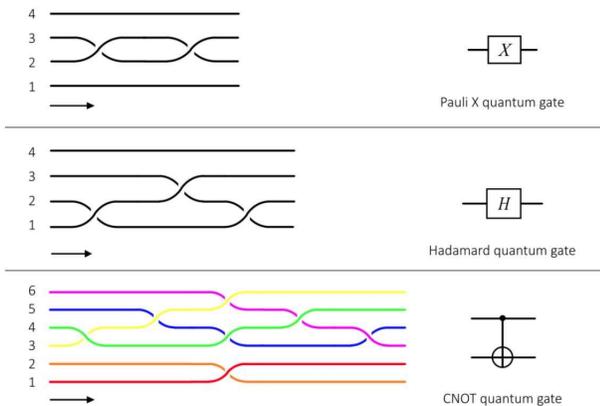


FIG. 5 Realization of the Pauli-X gate, the Hadamard gate, and the CNOT gate using different braiding sequences. Figs. from www.quantum-bits.org/?p=2226.

Exercise

- Starting from Eq. (1.31), verify that the state in Eq. (1.32) is a zero-energy solution.
- Show that under the $(|0\rangle, |1\rangle)^T$ basis, Majorana fermion operators have the Pauli-matrix representation,

$$\gamma_1 = \sigma_x, \quad (1.78)$$

$$\gamma_2 = \sigma_y, \quad (1.79)$$

$$-i\gamma_1\gamma_2 = \sigma_z, \text{ and } B_{12} = e^{i\frac{\pi}{4}}\sigma_z. \quad (1.80)$$

- Based on relations like Eqs. (1.59), (1.61), show that the intra-fermion braiding operators and inter-fermion

braiding operator have the 4×4 matrix representations in Eqs. (1.66), (1.67), and (1.68).

REFERENCES

- Alicea, Jason (2010), “Majorana fermions in a tunable semiconductor device,” *Phys. Rev. B* **81**, 125318.
- Fetter, Alexander L, and John Dirk Walecka (1971), *Quantum Theory of Many-Particle Systems* (McGraw-Hill Book Co.).
- Fu, Liang, and C. L. Kane (2008), “Superconducting proximity effect and majorana fermions at the surface of a topological insulator,” *Phys. Rev. Lett.* **100**, 096407.
- de Gennes, Pierre-Gilles (1989), *Superconductivity of metals and alloys* (Addison-Wesley Publishing Co.).
- Ivanov, D A (2001), “Non-abelian statistics of half-quantum vortices in p -wave superconductors,” *Phys. Rev. Lett.* **86**, 268–271.
- Nayak, Chetan, Steven H. Simon, Ady Stern, Michael Freedman, and Sankar Das Sarma (2008), “Non-abelian anyons and topological quantum computation,” *Rev. Mod. Phys.* **80**, 1083–1159.
- Read, N, and Dmitry Green (2000), “Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum hall effect,” *Phys. Rev. B* **61**, 10267–10297.
- Sau, Jay D, Roman M. Lutchyn, Sumanta Tewari, and S. Das Sarma (2010), “Generic new platform for topological quantum computation using semiconductor heterostructures,” *Phys. Rev. Lett.* **104**, 040502.
- Tewari, Sumanta, S. Das Sarma, Chetan Nayak, Chuanwei Zhang, and P. Zoller (2007), “Quantum computation using vortices and majorana zero modes of a $p_x + ip_y$ superfluid of fermionic cold atoms,” *Phys. Rev. Lett.* **98**, 010506.