Lecture notes on topological insulators

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(Dated: May 5, 2025)

CONTENTS

I. 1D p-WAVE SUPERCONDUCTOR

Now we study p-wave SC in 1D with non-trivial topology. There is few material with p-wave pairing, but as we will see in later chapters, effective p-wave pairing could exist in hybrid structures consisting of s-wave SC material and material with spin-orbital coupling.

In this chapter, the fermions are either spinless, or spin-polarized, so that the spin degree of freedom can be ignored. We first consider a continuum version, then a lattice version of the 1D p-wave SC (the Kitaev model). The main references we use on topological superconductor are Bernevig and Hughes, 2013 and Nomura, 2013.

A. Continuum model

The Hamiltonian of the 1D p-wave SC is given as

$$H = \sum_k \left[\varepsilon_k c_k^\dagger c_k + \frac{1}{2}\left(\Delta_k c_k^\dagger c_{-k}^\dagger + \Delta_k^* c_{-k} c_k\right)\right] \quad (1.1)$$

$$= \frac{1}{2}\sum_k (c_k^\dagger c_{-k}) \begin{pmatrix} \varepsilon_k & \Delta_k \\ \Delta_k^* & -\varepsilon_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \quad (1.2)$$

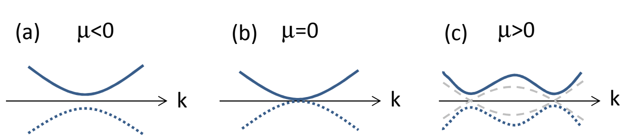

FIG. 1 The energies $\pm E_k$ plotted for 3 chemical potentials.

in which $\varepsilon_k = \hbar^2 k^2/2m - \mu$, and $\Delta_k = \Delta_0 k$ (k can be positive or negative). The eigen-energies are,

$$E_\pm(k) = \pm\sqrt{\varepsilon_k^2 + |\Delta_k|^2}. \quad (1.3)$$

The eigenstate (u_k, v_k) for energy E_k has the components,

$$u_k = \sqrt{\frac{1}{2}\left(1 + \frac{\varepsilon_k}{E_k}\right)}; \quad v_k = \sqrt{\frac{1}{2}\left(1 - \frac{\varepsilon_k}{E_k}\right)}\frac{\Delta_k^*}{|\Delta_k|}. \quad (1.4)$$

The particle-hole symmetry has been discussed in Sec. **??**.

Near $k = 0$, $\varepsilon_k \simeq -\mu$, and $E_k \simeq |\mu|$. In Fig. 1, we show the dependence of the energy spectrum on the chemical potential. For both $\mu < 0$ and $\mu > 0$, the spectra are gapped. If $\mu = 0$, then $E_+(k)$ and $E_-(k)$ touch at $k = 0$.

If $\mu < 0$, then for a small k, $u_k \simeq 1, v_k \simeq \Delta_k^*/2|\mu|$. It is known that the Fourier transform of the Cooper pair wave function $g(x)$ is $g(k) = v_k/u_k$ (de Gennes, 1989). So for $\mu < 0$, $g(k) \propto k$. The function $g(k)$ being analytic near $k = 0$ implies that its Fourier transform falls off exponentially at large distance, $g(x) \simeq e^{-x/x_0}$. The phase with $\mu < 0$ is thus called the strong-coupling phase (Read and Green, 2000).

On the other hand, if $\mu > 0$, then for small k, $u_k \simeq |\Delta_k|/2\mu, v_k \simeq 1$ (we have ignored the phase of Δ_k), and $g(k) \propto 1/k$. The sharp peak near small k implies a slow decay of $g(x)$ at large distance. So the phase with $\mu > 0$ is called the weak-coupling phase.

These two phases cannot be adiabatically connected to each other. Furthermore, the weak-coupling phase has non-trivial topology, as we'll show below.

1. Edge state

We now assume $\mu(x > 0) > 0$, and $\mu(x < 0) < 0$, so the 1D space is separated to a weak-coupling phase and a strong-coupling phase (e.g., $\mu(x) = \mu_0 \tanh x$). Ignore terms of order k^2, the BdG equation is,

$$\begin{pmatrix} -\mu(x) & \Delta_0 k \\ \Delta_0 k & \mu(x) \end{pmatrix} \psi = E\psi. \quad (1.5)$$

We are only interested in the edge-state solution. Let $k \to \partial/i\partial x$, and try

$$\psi(x) = \psi_0 e^{-\frac{1}{\Delta_0}\int_0^x dx' \mu(x')}, \quad (1.6)$$

then

$$\begin{pmatrix} -\mu(x)-E & i\mu(x) \\ i\mu(x) & \mu(x)-E \end{pmatrix}\psi_0 = 0. \tag{1.7}$$

At $E=0$, we have a solution,

$$\psi_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{1.8}$$

This is a zero mode localized at the interface.

The Bogoliubov quasiparticle (QP) operator for the zero mode is,

$$\begin{aligned}\gamma_0 &= \int dx \left[u^*(x)\psi(x) + v^*(x)\psi^\dagger(x)\right] \qquad (1.9)\\ &= \frac{e^{i\pi/4}}{\sqrt{2}}\int dx e^{-\frac{1}{\Delta_0}\int_0^x dx'\mu(x')}\left[e^{-i\pi/4}\psi(x) + e^{i\pi/4}\psi^\dagger(x)\right].\end{aligned}$$

Removing the overall phase of $\pi/4$, we have

$$\gamma_0^\dagger = \gamma_0. \tag{1.10}$$

That is, the anti-particle of this QP is the same as the QP. Such a fermion is called a **Majorana fermion** (MF). The zero mode is protected by the PH symmetry, as long as it's non-degenerate: to shift away from zero energy, 2 levels are required due to PH symmetry. For a general introduction to the Majorana fermion, one can read Wilczek, 2009.

B. Kitaev model

We now study the Kitaev model of a 1D p-wave SC (Kitaev, 2001). It is the lattice version of the continuum model introduced in previous section. With the lattice and its compact Brillouin zone, the topological number can be defined naturally.

Consider a 1D lattice with N sites (lattice constant $a=1$) under periodic BC, $c_{N+1}=c_1$. The Hamiltonian is,

$$\begin{aligned}H &= \sum_{j=1}^{N}\left[-\frac{t}{2}(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}) - \mu c_j^\dagger c_j \right. \qquad (1.11)\\ &\left. + \frac{\Delta_0}{4}\left(c_{j+1}^\dagger c_j^\dagger - c_j^\dagger c_{j+1}^\dagger + h.c\right)\right], t>0, \Delta_0\in R.\end{aligned}$$

The minus sign in the second line is related to the p-wave pairing, $\Delta(k)=\Delta_0\sin k$ (see below). With the Fourier transformation,

$$c_j^\dagger = \frac{1}{\sqrt{N}}\sum_k e^{ijk}c_k^\dagger, \tag{1.12}$$

one has

$$\begin{aligned}H &= \sum_k \left[-t\cos k c_k^\dagger c_k - \mu c_k^\dagger c_k \right. \qquad (1.13)\\ &+ \left.\frac{\Delta_0}{4}\left(e^{ik}c_k^\dagger c_{-k}^\dagger - e^{-ik}c_k^\dagger c_{-k}^\dagger + h.c\right)\right].\end{aligned}$$

It can be written in matrix form,

$$\begin{aligned}H &= \frac{1}{2}\sum_k (c_k^\dagger, c_{-k})\begin{pmatrix} -t\cos k-\mu & i\Delta_0\sin k \\ -i\Delta_0\sin k & t\cos k+\mu \end{pmatrix}\begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}\\ &+ \frac{1}{2}\sum_k t\cos k + \mu. \qquad (1.14)\end{aligned}$$

Recall that the PH operator for p-wave superconductor is $P=\tau_x K$. You may check that the Hamiltonian does have the PH symmetry, $P\mathsf{H}(k)P^{-1} = -H_{-k}$.

The eigen-energies are,

$$E_\pm(k) = \pm\sqrt{(t\cos k+\mu)^2 + \Delta_0^2\sin^2 k}. \tag{1.15}$$

The energy gap closes when both

$$\begin{cases} \sin k = 0, \\ t\cos k + \mu = 0. \end{cases} \tag{1.16}$$

The gap at $k=0$ closes at $\mu_c=-t$, and the gap at $k=\pi$ closes at $\mu_c = t$. So there are 3 quantum phases within the ranges $\mu<-t, |\mu|<t$, and $\mu>t$.

After the diagonalization, we have

$$\begin{aligned}H &= \frac{1}{2}\sum_k (\gamma_k^\dagger, \gamma_{-k})\begin{pmatrix} E(k) & 0 \\ 0 & -E(k) \end{pmatrix}\begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix}\\ &= \sum_k E(k)\gamma_k^\dagger\gamma_k. \qquad (1.17)\end{aligned}$$

That is, $\gamma_k^\dagger$ creates a quasi-particle with excitation energy $E(k)$. Note that in the spinless p-wave model, the negative-energy branch is redundant.

1. Topological number

The 2×2 Hamiltonian matrix can be written in the standard form, $\mathsf{H}(k) = \mathbf{d}\cdot\boldsymbol{\sigma}$, where

$$\mathbf{d} = -(0, \Delta_0\sin k, t\cos k + \mu). \tag{1.18}$$

When k moves from $-\pi$ to π, the tip of $\mathbf{d}(k)$ moves around an ellipse on the $d_y - d_z$ plane with a center at $d_z = -\mu$. If $\mu > t$, then the origin is outside of the elliptical loop, and the winding number of the map $k\to\mathbf{d}(k)$ is zero. If $|\mu|<t$, then the origin is inside the loop, and the winding number is 1. If $\mu<-t$, then the winding number is again 0. To change from 1 to 0, or 0 to 1, the loop needs to cross the origin – at that point the energy gap closes.

This indicates that the region $|\mu|<t$ is the topologically non-trivial phase, while the region $|\mu|>t$ is the trivial phase. These two phases with different winding numbers can be distinguished by a Z_2 topological number ν, which is defined as (see Nomura, 2013),

$$(-1)^\nu = \text{sgn}[d_z(0)]\text{sgn}[d_z(\pi)]. \tag{1.19}$$

In our case, $d_z(0) = -t-\mu$, $d_z(\pi) = t-\mu$. It's not difficult to see that the origin can be inside the loop only when these two quantities have opposite signs.

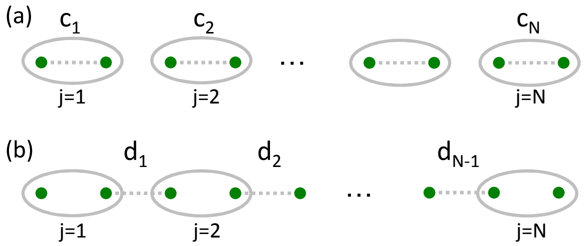

FIG. 2 A Kitaev chain with N sites and open ends. (a) A fermion operator c_j is composed of two MF operators. (b) Using a pair of MFs from different physical fermions to build a fermion operator d_j.

2. Edge states in Kitaev chain

To study the edge states of a Kitaev chain with open ends, it's convenient to introduce the **Majorana fermion representation**. The usual fermions satisfy the anti-commutation relation,

$$\{c_j, c_{j'}^\dagger\} = \delta_{jj'}. \tag{1.20}$$

Let a_j be Majorana fermion operators, with $a_j^\dagger = a_j$. Define their anti-commutation relations as,

$$\{a_j, a_{j'}^\dagger\} = 2\delta_{jj'}, \tag{1.21}$$
$$\{a_j, a_{j'}\} = 2\delta_{jj'}. \tag{1.22}$$

Then we have $a_j^2 = 1$ (not zero!).

Decompose a fermion into 2 MFs (see Fig. 2(a)),

$$c_j = \frac{1}{2}(a_{2j-1} + i a_{2j}), \tag{1.23}$$
$$\text{then} \quad c_j^\dagger = \frac{1}{2}(a_{2j-1} - i a_{2j}). \tag{1.24}$$

This is analogous to the decomposition of a complex number to two real numbers. Given Eq. (1.21), one can verify that they do satisfy Eq. (1.20).

Now consider a Kitaev chain with two open ends and N lattice sites ($j = 1, \cdots, N$). The Hamiltonian is,

$$\begin{aligned} H &= -\frac{t}{2}\sum_{j=1}^{N-1} c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} - \mu \sum_{j=1}^{N} c_j^\dagger c_j \\ &+ \frac{\Delta_0}{2}\sum_{j=1}^{N-1} c_{j+1}^\dagger c_j^\dagger + c_j c_{j+1} \qquad (1.25) \\ &= \frac{i}{4}\sum_{j=1}^{N-1}(t+\Delta_0) a_{2j}a_{2j+1} + (-t+\Delta_0) a_{2j-1}a_{2j+2} \\ &- \frac{1}{2}\sum_{j=1}^{N} \mu\left(i a_{2j-1}a_{2j} + 1\right). \qquad (1.26) \end{aligned}$$

Notice that since $(a_{2j}a_{2j+1})^\dagger = -a_{2j}a_{2j+1}$ (anti-hermitian), so the factor i is required for H to be hermitian. In the Heisenberg equation, this would cancel with the i in $i\partial/\partial t$ so that there is no imaginary number in the equation of motion.

For simplicity, consider the case with $\Delta_0 = t$ and ignore a constant term, then

$$H = \frac{i}{2}\sum_{j=1}^{N-1} t a_{2j}a_{2j+1} - \frac{i}{2}\sum_{j=1}^{N} \mu a_{2j-1}a_{2j}. \tag{1.27}$$

We know that it is a trivial SC when $|\mu| > t$. When $|\mu| < t$, it is a topological SC with Majorana edge states (see Fig. 3). For the latter, consider a special case with $\mu = 0$: the 2nd term vanishes, and thus the 2 Majorana operators on the ends decouple from the rest of the operators. That is, there is a lone Majorana operator on each end of the chain (Fig. 2(b)). This would be the edge states in the topological phase (details later). For $\mu \neq 0$, the edge states are less localized and would decay into the bulk Kitaev, 2001.

On the other hand, for the trivial phase ($|\mu| > t$), one can choose $t = 0$ to simplify the Hamiltonian. Then every Majorana operator in the chain is coupled with its neighbour, and there is no lone operator at the ends.

3. Majorana zero mode

For a chain with N sites, there are N energy eigenstates. The Bogoliubov quasiparticle operators associated with excitation energies E_α ($\alpha = 1, 2, \cdots, N$) are,

$$\gamma_\alpha = \sum_{j=1}^{N}\left(u_{\alpha j}^* c_j + v_{\alpha j}^* c_j^\dagger\right). \tag{1.28}$$

In the topological phase, there is one zero-energy eigenstate (zero mode) localized near both ends. If $\mu = 0$, then one can show that for the zero mode ($E_1 = 0$), $\mathbf{u}_1 = \frac{1}{2}(1, 0, \cdots, 0, 1)$, and $\mathbf{v}_1 = \frac{1}{2}(1, 0, \cdots, 0, -1)$ (D'Abbruzzo and Rossini, 2021). That is,

$$\gamma_1 = \frac{1}{2}(c_1 + c_N) + \frac{1}{2}(c_1^\dagger - c_N^\dagger) \tag{1.29}$$
$$= \frac{1}{2}(a_1 + i a_{2N}). \tag{1.30}$$

This quasiparticle is a highly non-local fermion since it consists of the Majorana fermions on two ends.

The superconducting ground state of the Kitaev chain is annihilated by γ_α, in particular γ_1,

$$\gamma_1|\Psi_0\rangle = 0. \tag{1.31}$$

The zero mode in the topological phase can either be empty or filled with one fermion, $|\Psi_0\rangle$ or $\gamma_1^\dagger|\Psi_0\rangle$. Both states have zero energy, so the ground states are two-fold degenerate. Henceforth they are designated as $|0_+\rangle$ and $|0_-\rangle$ for short (the subscripts $\pm$ would be explained

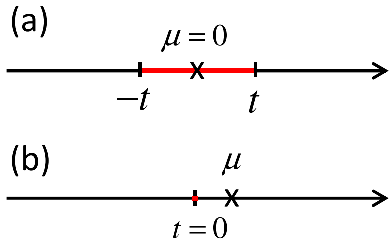

FIG. 3 (a) For $t > 0$, $\mu = 0$ is in the topological phase. (b) For $t = 0$, $\mu > 0$ is in the trivial phase.

later). It follows that,

$$\gamma_1^\dagger \gamma_1 |0_+\rangle = 0|0_+\rangle, \tag{1.32}$$
$$\gamma_1^\dagger \gamma_1 |0_-\rangle = 1|0_-\rangle. \tag{1.33}$$

Some comments on the realization of Majorana zero mode (MZM) in real 1D systems. There is no known p-wave SC so far, but one can combine s-wave superconductivity with spin-orbit (SO) coupling to produce an effective p-wave SC (Fu and Kane, 2008). In practice, one can put a metal wire on top of a s-wave SC. The wire or the SC needs to have the SO coupling. As a result, the electrons in the wire interact with both the SO coupling and the superconductivity (through the proximity effect). Furthermore, since spin degeneracy could double the number of zero modes and destablize the MZMs, magnetic material or magnetic field has to be introduced to break the spin degeneracy. For example, Nadj-Perge *et al.*, 2014 uses ferromagnetic iron atomic chains on top of a superconducting lead (which has strong SO coupling). Some more discussions can be found in Alicea, 2012.

A brief summary of 1D models with nontrivial topology: In Chap **??**, we have studied the SSH model of polyacetylene. In Chap **??**, we introduced the Fu-Kane spin pump. In this Chap we have the Kitaev model of p-wave SC. Other topological 1D models not covered in this course are, for example, the AKLT model of spin-1 chain, and the Lubensky-Kane model of mechanical chain. They all have robust edge states due to nontrivial topology.

4. Fermion parity of the ground state

As we have explained above, for the topological phase, there are two MZMs at the ends of the chain. These two Majorana operators have the same degree of freedom as one ordinary fermion, which can either be occupied or unoccupied. This two ground states can be characterized by **fermion parity**.

The fermion parity of site-j with fermion number n_j is defined as,

$$(-1)^{n_j} = \begin{cases} +1 \text{ if } n_j = 0, \\ -1 \text{ if } n_j = 1. \end{cases} \tag{1.34}$$

When written as operators, $n_j = c_j^\dagger c_j$, one has

$$P_j \equiv (-1)^{n_j} = e^{i\pi n_j} \quad (n_j^2 = n_j)$$
$$= 1 - 2n_j \tag{1.35}$$
$$= -ia_{2j-1}a_{2j}. \tag{1.36}$$

We'd like to know if the fermion parity operator P_j commutes with the Hamiltonian. First, one can show that

$$\sum_j [c_j^\dagger c_{j+1}, c_i^\dagger c_i] + h.c. = 0. \tag{1.37}$$

Second,

$$\sum_j [c_j^\dagger c_{j+1}^\dagger, c_i^\dagger c_i] + h.c. = -c_{i-1}^\dagger c_i^\dagger - c_i^\dagger c_{i+1}^\dagger + h.c.$$
$$= 0 \mod 2. \tag{1.38}$$

The result is not zero, but since the SC ground state does not have a definite number of Cooper pairs, so when the fermion parity operator is acting within the subspace of the SC ground state, the commutator can be considered as 0. That is, the fermion parity operator "effectively" commutes with the Hamiltonian.

We thus define the fermion parity operator for the whole system as,

$$P_F \equiv \prod_{j=1}^N (1 - 2c_j^\dagger c_j) \tag{1.39}$$
$$= \prod_{j=1}^N (-ia_{2j-1}a_{2j}), \quad P_F^2 = 1. \tag{1.40}$$

Its eigenvalue can only be ± 1: for the trivial phase, it is always $+1$; for the non-trivial phase, it can be $+1$ or -1. This is demonstrated below.

The SC state of the Kitaev model is trivial when $|\mu| > t$. To study its fermion parity, for simplicity, let $\Delta_0 = t$, and just pick up a particular set of parameters with $t = 0, \mu < 0$. Then

$$H = -\frac{1}{2}\sum_{j=1}^N \mu \left(ia_{2j-1}a_{2j} + 1\right). \tag{1.41}$$

Rewrite $ia_{2j-1}a_{2j} = 2c_j^\dagger c_j - 1$, then

$$H = |\mu| \sum_{j=1}^N c_j^\dagger c_j. \tag{1.42}$$

The ground state is annihilated by c_j, $c_j|\Psi_0\rangle = 0$. Its fermion parity is $+1$, since

$$P_F|\Psi_0\rangle = \prod_{j=1}^N (1 - 2c_j^\dagger c_j)|\Psi_0\rangle \tag{1.43}$$
$$= |\Psi_0\rangle. \tag{1.44}$$

On the other hand, for the non-trivial phase in $|\mu| < t$ $(t > 0)$, let $\Delta_0 = t$ and $\mu = 0$, then

$$H = \frac{i}{2}\sum_{j=1}^{N-1} t a_{2j} a_{2j+1}. \tag{1.45}$$

Since the Majorana edge fermions decouple from the bulk, instead of Eq. (1.24), for fermions in the bulk $(j = 1, \cdots, N-1)$, define (see Fig. 2(b))

$$\begin{aligned} d_j &= \frac{1}{2}(a_{2j} + i a_{2j+1}), && (1.46) \\ \text{then } d_j^\dagger &= \frac{1}{2}(a_{2j} - i a_{2j+1}). && (1.47) \end{aligned}$$

Rewrite $i a_{2j} a_{2j+1} = 2 d_j^\dagger d_j - 1$, then

$$H = t\sum_{j=1}^{N-1}\left(d_j^\dagger d_j - \frac{1}{2}\right). \tag{1.48}$$

The ground state is annihilated by d_j, $d_j|\Psi_0\rangle = 0$ for $j = 1, 2, \cdots, N-1$. To calculate its fermion parity, first rewrite

$$\begin{aligned} P_F &= -i a_1 \prod_{j=1}^{N-1}(-i a_{2j} a_{2j+1}) a_{2N} && (1.49) \\ &= -i a_1 a_{2N} \prod_{j=1}^{N-1}(1 - d_j^\dagger d_j). && (1.50) \end{aligned}$$

Since $d_j|\Psi_0\rangle = 0$, we have

$$P_F|\Psi_0\rangle = (-i a_1 a_{2N})|\Psi_0\rangle. \tag{1.51}$$

Furthermore, from Eq. (1.30), one has

$$-i a_1 a_{2N} = 1 - 2\gamma_1^\dagger \gamma_1. \tag{1.52}$$

As we have explained earlier, the ground state can have zero or one γ_1 quasiparticle (designated as $|0_+\rangle$ or $|0_-\rangle$). Therefore,

$$\begin{aligned} P_F|0_\pm\rangle &= (1 - 2\gamma_1^\dagger \gamma_1)|0_\pm\rangle && (1.53) \\ &= \pm|0_\pm\rangle. && (1.54) \end{aligned}$$

That is, the ground states can be labelled by the fermion parity. It can be considered as a 2-state system that could store 1 qubit of information. Being nonlocal, such a qubit is robust against local perturbations.

Since the fermion operator γ_1 operates within a 2D Hilbert space, they can be represented by Pauli matrices under the $(|0_+\rangle, |0_-\rangle)$ basis,

$$\begin{aligned} \gamma_1 &\simeq \sigma_+, && (1.55) \\ \gamma_1^\dagger &\simeq \sigma_-, && (1.56) \\ 1 - 2\gamma_1^\dagger \gamma_1 &\simeq \sigma_z. && (1.57) \end{aligned}$$

Or,

$$\begin{aligned} a_1 &\simeq \sigma_x, && (1.58) \\ a_{2N} &\simeq \sigma_y, && (1.59) \\ -i a_1 a_{2N} &\simeq \sigma_z. && (1.60) \end{aligned}$$

A different spin representation of the fermionic chain can be found in Prob. 1.

Exercise

1. One can write fermion operators in terms of spin operators,

$$\begin{aligned} a_{2j-1} &= \left(\prod_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^x, && (1.61) \\ a_{2j} &= \left(\prod_{k=1}^{j-1}\sigma_k^z\right)\sigma_j^y. && (1.62) \end{aligned}$$

This is called the **Jordan-Wigner transformation**. Show that, using this transformation, the Hamiltonian of the Kitaev chain,

$$H = \frac{i}{2}\sum_{j=1}^{N-1} t a_{2j} a_{2j+1} - \frac{i}{2}\sum_{j=1}^{N}\mu a_{2j-1} a_{2j},$$

can be transformed to

$$H = -J_x \sum_{j=1}^{N-1}\sigma_j^x \sigma_{j+1}^x + h\sum_{j=1}^{N}\sigma_j^z, \tag{1.63}$$

where $J_x = t/2, h = \mu/2$. This is the Hamiltonian of an Ising spin chain in a transverse magnetic field h. Also, show that

$$P_F = \prod_{j=1}^{N}(-i a_{2j-1} a_{2j}) = \prod_{j=1}^{N}\sigma_j^z. \tag{1.64}$$

Note: There are two possible quantum phases in the Ising chain above: (a) When $|h| < J_x$, the ground state has all the spins either parallel or anti-parallel to the x-axis (two-fold degenerate). (b) When $|h| > J_x$, the ground state has all the spins anti-parallel to the magnetic field (non-degenerate).

2. Show that the inverse of the Jordan-Wigner transformation is given as,

$$\begin{aligned} \sigma_j^+ &= c_j \prod_{k=1}^{j-1}(-1)^{c_k^\dagger c_k}, && (1.65) \\ \sigma_j^- &= c_j^\dagger \prod_{k=1}^{j-1}(-1)^{c_k^\dagger c_k}. && (1.66) \end{aligned}$$

Also,

$$\sigma_j^z = 1 - 2c_j^\dagger c_j. \tag{1.67}$$

With these, one can transform the Hamiltonian of a spin chain to that of a chain with fermions.

REFERENCES

Alicea, Jason (2012), “New directions in the pursuit of majorana fermions in solid state systems,” Reports on Progress in Physics **75** (7), 076501.

Bernevig, B Andrei, and Taylor L. Hughes (2013), *Topological Insulators and Topological Superconductors* (Princeton University Press).

D’Abbruzzo, Antonio, and Davide Rossini (2021), “Topological signatures in a weakly dissipative kitaev chain of finite length,” Phys. Rev. B **104**, 115139.

Fu, Liang, and C. L. Kane (2008), “Superconducting proximity effect and majorana fermions at the surface of a topological insulator,” Phys. Rev. Lett. **100**, 096407.

de Gennes, Pierre-Gilles (1989), *Superconductivity of metals and alloys* (Addison-Wesley Publishing Co.).

Kitaev, A Yu (2001), “Unpaired majorana fermions in quantum wires,” Physics-Uspekhi **44** (10S), 131.

Nadj-Perge, Stevan, Ilya K. Drozdov, Jian Li, Hua Chen, Sangjun Jeon, Jungpil Seo, Allan H. MacDonald, B. Andrei Bernevig, and Ali Yazdani (2014), “Observation of majorana fermions in ferromagnetic atomic chains on a superconductor,” Science **346** (6209), 602–607.

Nomura, K (2013), “Fundamental theory of topological insulator,” Unpublished, written in Japanese.

Read, N, and Dmitry Green (2000), “Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum hall effect,” Phys. Rev. B **61**, 10267–10297.

Wilczek, Frank (2009), “Majorana returns,” Nat Phys **5** (9), 614–618.