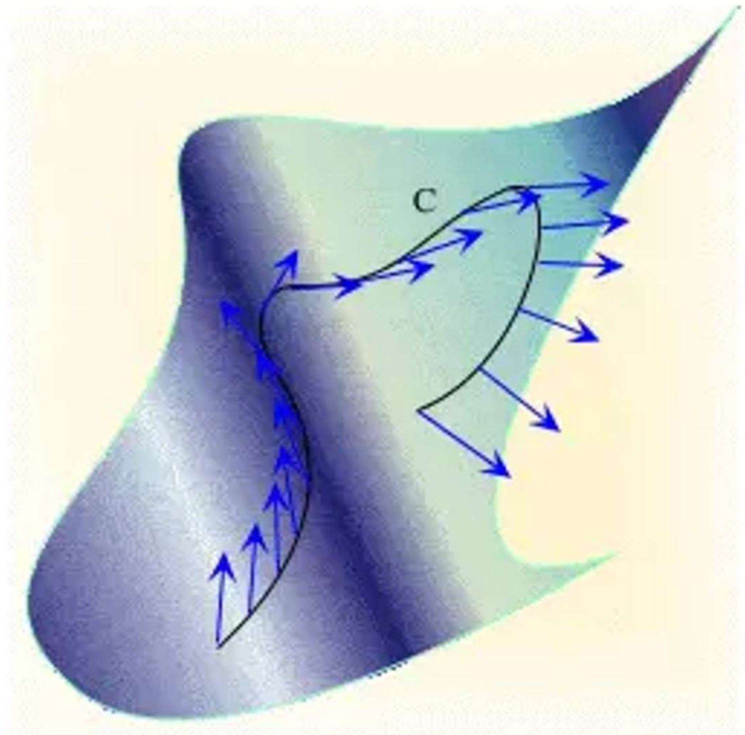
# Topology in two hours





Ming-Che Chang

Department of Physics

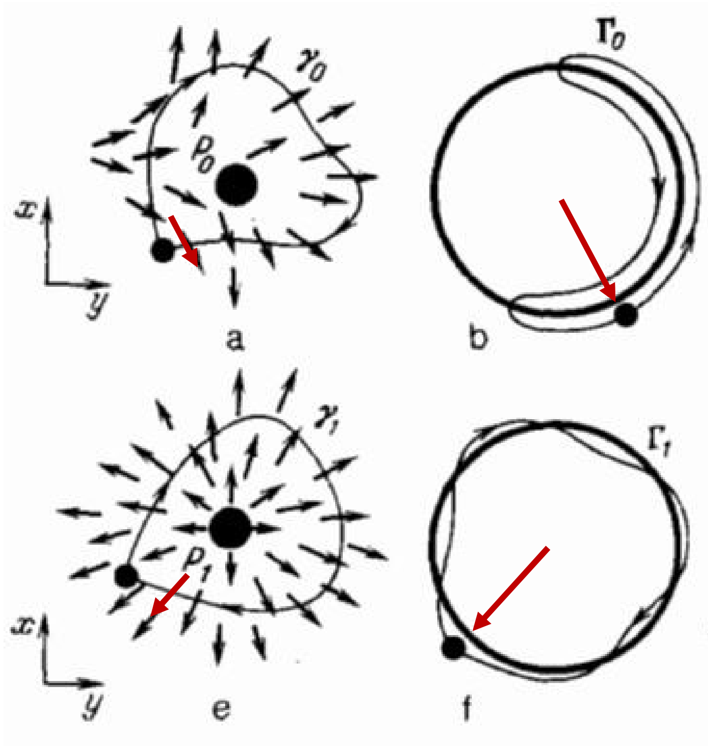
- Winding number
- Gaussian curvature
- Anholonomy
- Euler characteristics
- Gauss-Bonnet theorem
- Hopf-Poincare theorem

# Topology in vector field (Fluid flow, EM field …)

- Winding number

A map from the path to the direction of vectors

$$f : S^1 \rightarrow S^1$$



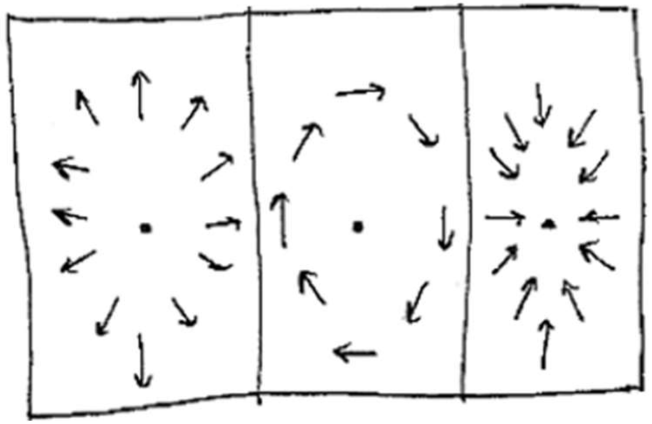

w=0

w=1

Fig from Kurik and Lavrentovich, 1988

- Source, vortex, drain

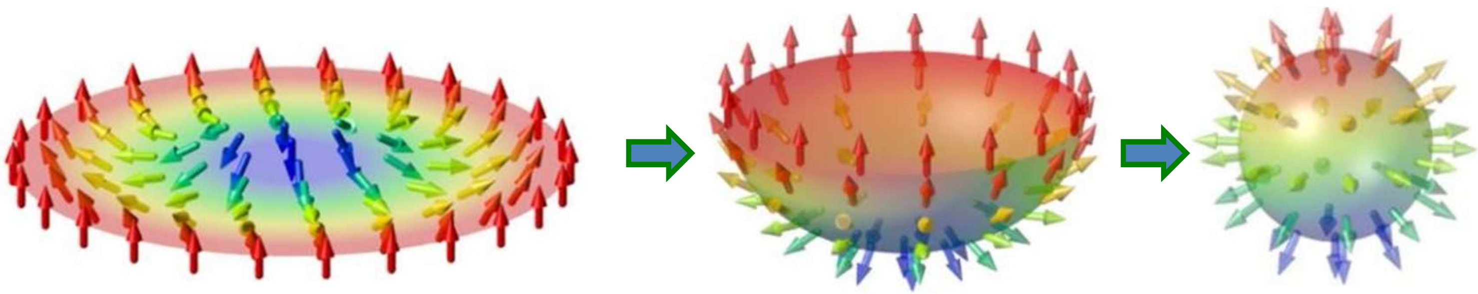
In 2D, they all have w=1 and are deformable to each other (not so in 3D).



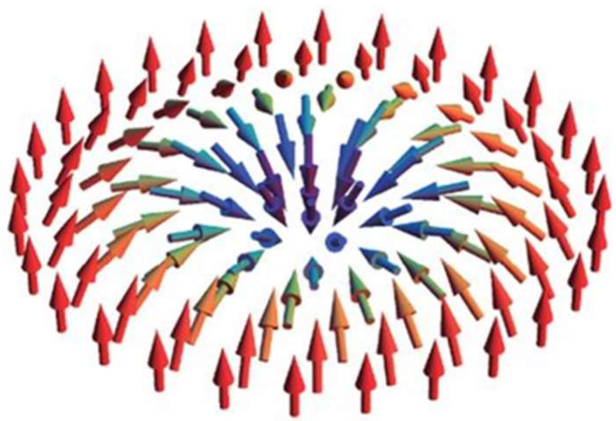
# Winding number again (or, wrapping number)

Now, the vectors on a plane can point out of plane

**w=1**



- By stereographic projection, a plane can be identified with a sphere
- A map from this sphere to the direction of vectors $f: S^2 \rightarrow S^2$



Such a localized spin texture is called a skyrmion

史科子

Hypothetical structure of nucleons (Skyrme, 1962)

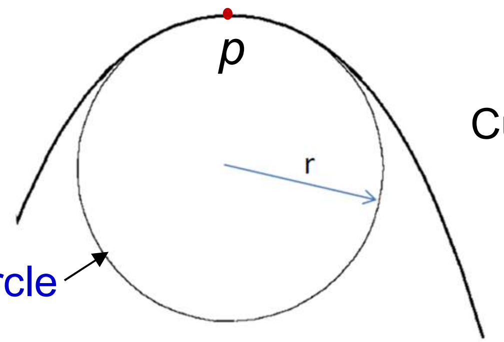
Fig from Karin and Matthias Sitte, JAP 2014

I. Kézsmárki et al, Nature Material 2015

# Basics of differential topology

- How do we define the curvature of a line (at point *p*)?

密接圓
Osculating circle
of point *p*

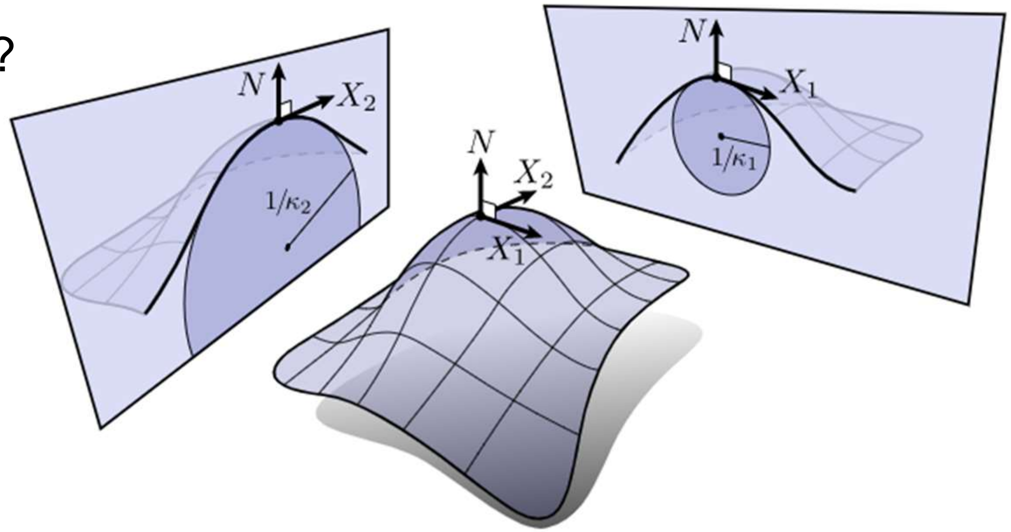



Curvature k at *p*:

$$k = \frac{1}{r}$$

- How do we define the curvature of a surface?

Fit the surface near *p* by a quadratic surface (ellipsoid, paraboloid, hyperboloid)




A quadratic surface must have two principal directions with maximum and minimum radii $r_1, r_2$. They correspond to two **principle curvatures** $k_1 = 1/r_1, k_2 = 1/r_2$ (up to a sign). 主曲率

# Two kinds of curvature

- Mean curvature
  平均曲率

$$H = k_1 + k_2 = \frac{1}{r_1} + \frac{1}{r_2}$$

Extrinsic
外在

- Gaussian curvature

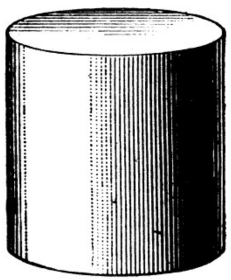
$$G = k_1 k_2 = \frac{1}{r_1 r_2}$$

Intrinsic
内在

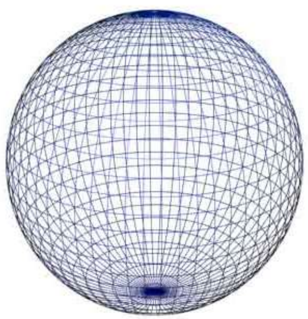

*Without* stretching/squeezing a surface (i.e., the shortest distance between any 2 points remain the same), its G will not change.

**Figure 3.6** Bending a sheet of paper changes its extrinsic—but not its intrinsic—geometry.

H≠0
G=0



H≠0
G=0



H≠0
G≠0

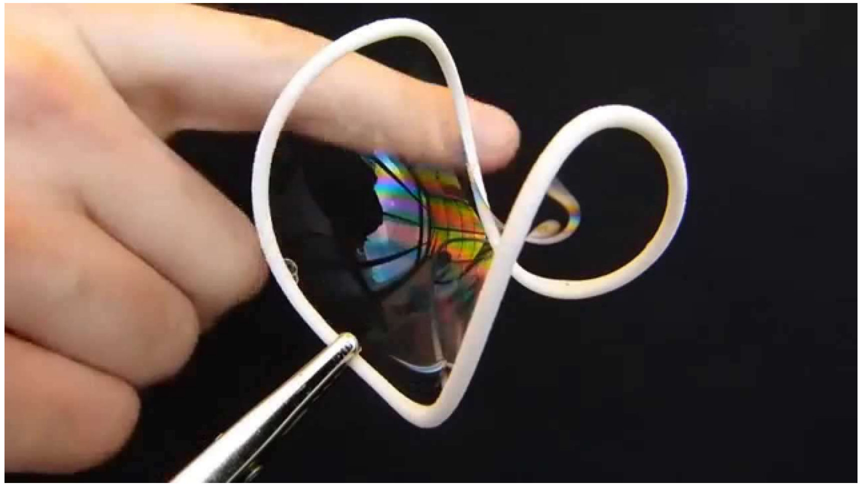
# Mean curvature in physics

- Lagrange (1760)

A surface is bounded by a curve. What is the shape of the surface with the minimum area?

- Plateau (1829)

Such a surface can be simulated by a soap film



www.youtube.com/watch?v=jReQUm9EB9k

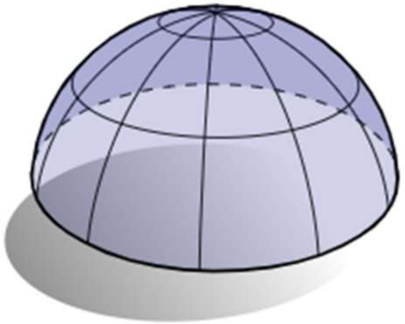
Energy of flim

∝ Surface tension

∝ Surface area

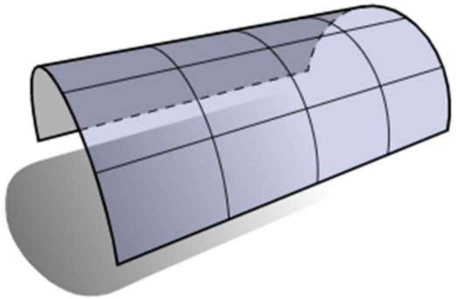
The minimal surface has zero mean curvature at every point!

# *Positive* and *negative* Gaussian curvature

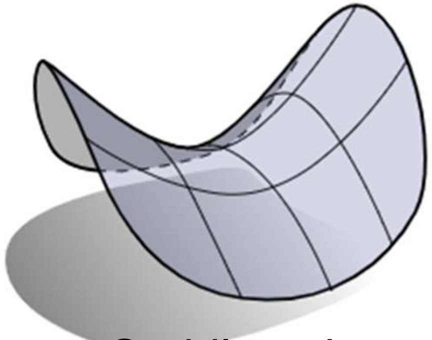
$k_1, k_2 > 0$

G>0



$k_1 > 0, k_2 = 0$

G=0

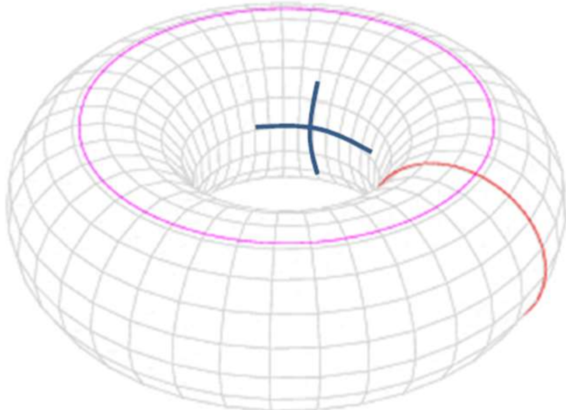


$k_1 > 0, k_2 < 0$

G<0

Saddle point

A torus　環面

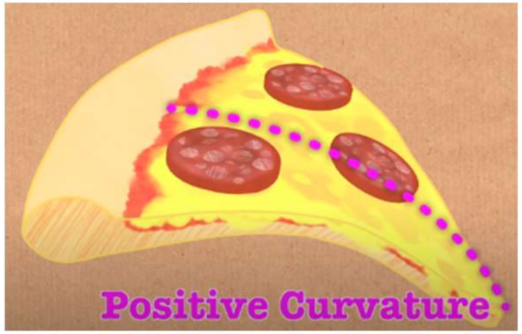


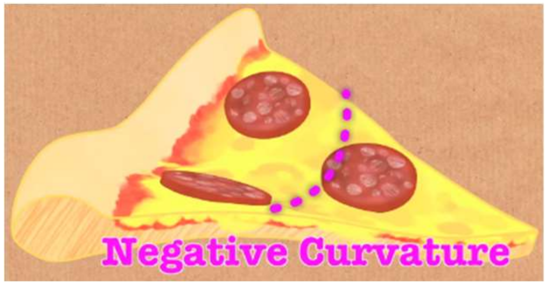


wordpress.discretization.de/geometryprocessingandapplicationsws19/a-quick-and-dirty-introduction-to-the-curvature-of-surfaces/

# The Remarkable Way We Eat Pizza -
Youtube: *Numberphile*

- You cannot change Gaussian curvature without stretching/squeezing the surface.
- That is, without stretching your pizza, its G must remain zero, and one of the $k_{1,2}$ must be zero.



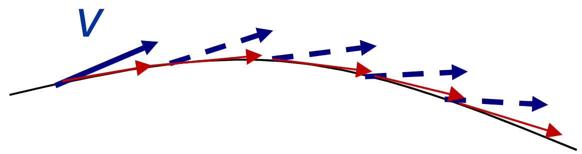



## Theorema Egregium:
(Gauss, 1827) 絕妙定理

Gaussian curvature can be determined entirely by measuring angles, distances and their rates on a surface.

### Intrinsic definition of Gaussian curvature

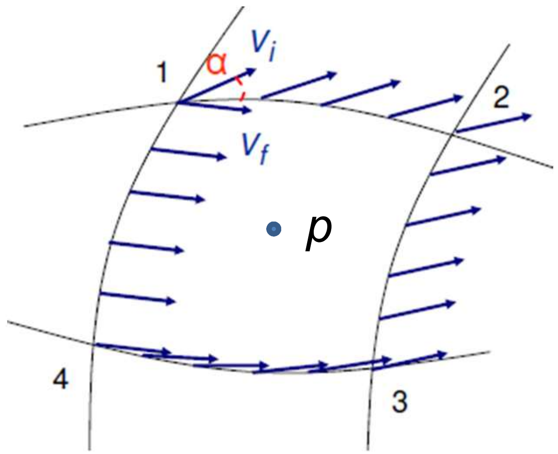
- Parallel transport of a vector ***v*** along a geodesic curve on a curved surface:
  The angles between *v* and tangent vectors remain fixed.




- After circling a loop, ***v*** rotates by an angle

  anholonomy angle (or defect angle) 虧角

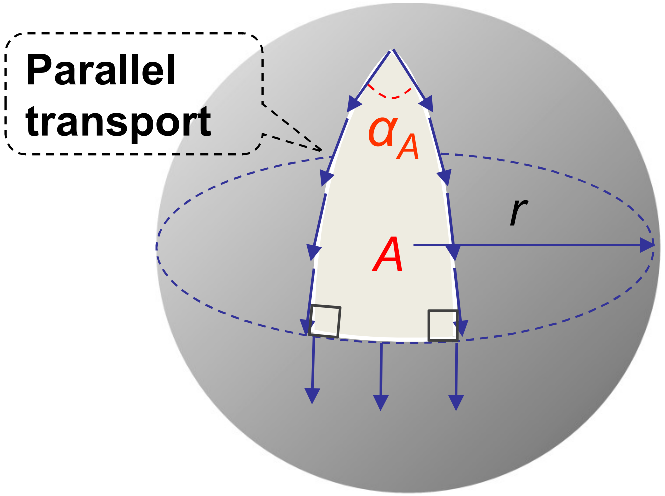
  This kind of behavior is called anholonomy



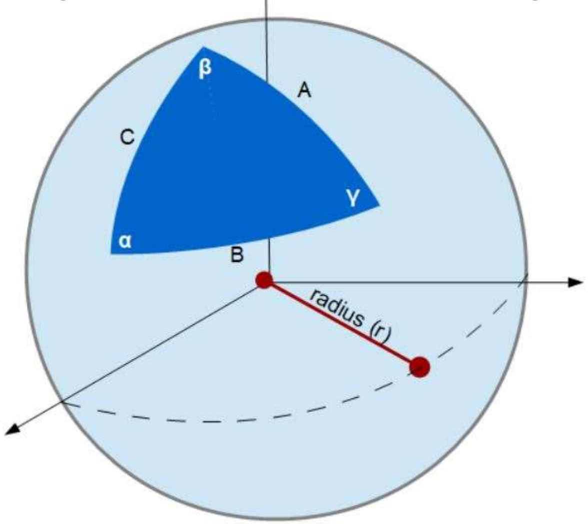

Gaussian curvature at *p* can be defined as

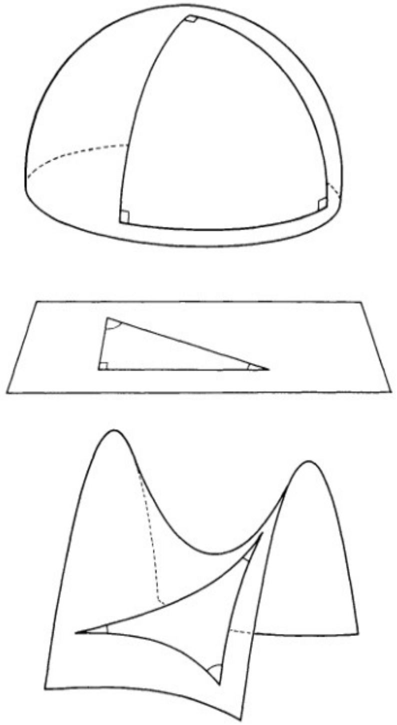
$$G \equiv \lim_{A \to 0} \frac{\alpha_A}{A}$$

# Anholonomy angle on a sphere




A general spherical triangle,



$\alpha_A > 0$

G>0

$\alpha_A = 0$

G=0

$\alpha_A < 0$

G<0

Girard theorem (1626)

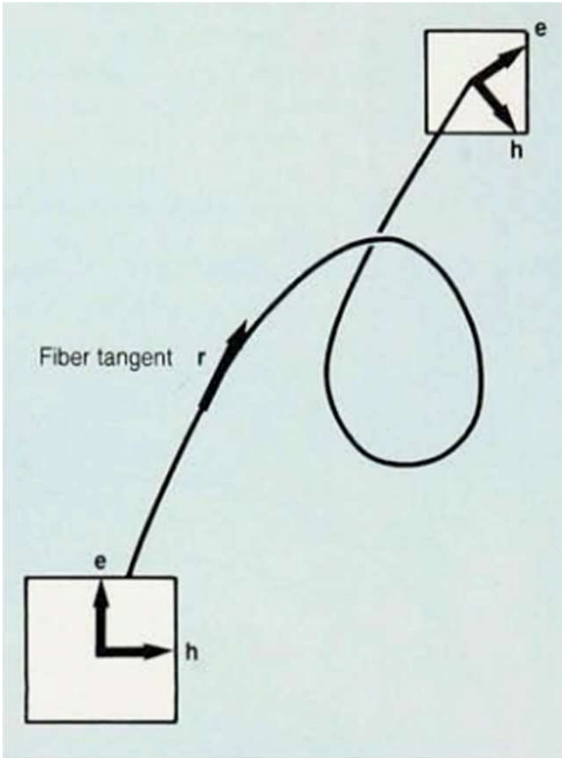
$$A = r^2(\alpha + \beta + \gamma - \pi)$$

$$\Rightarrow \quad \alpha_A = \frac{A}{r^2}$$

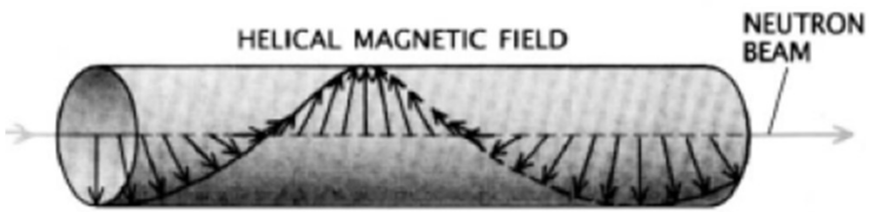
$$G \equiv \lim_{A \to 0} \frac{\alpha_A}{A} = \frac{1}{r^2}$$

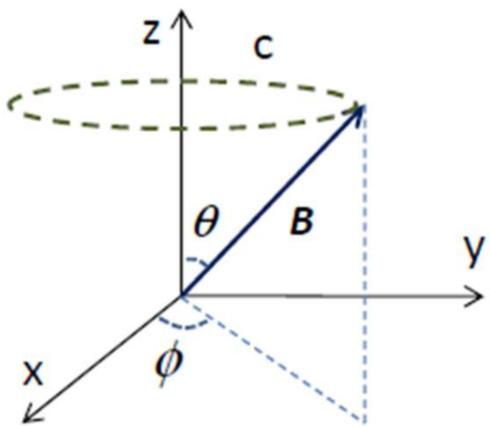
# Anholonomy (or *non-integrability*) in physics

Rotation of polarization in an optical fiber



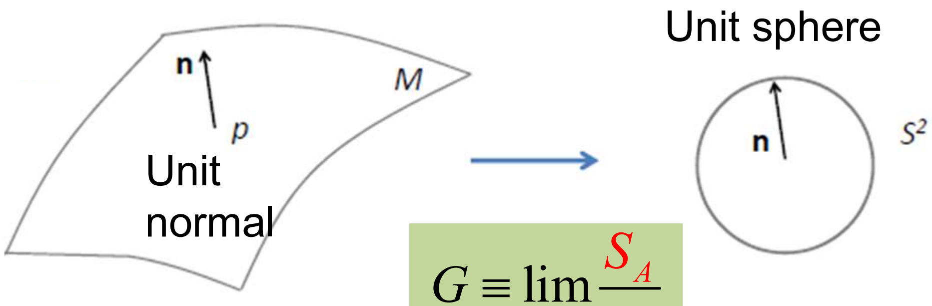

Berry phase of electron spin in a rotating magnetic field



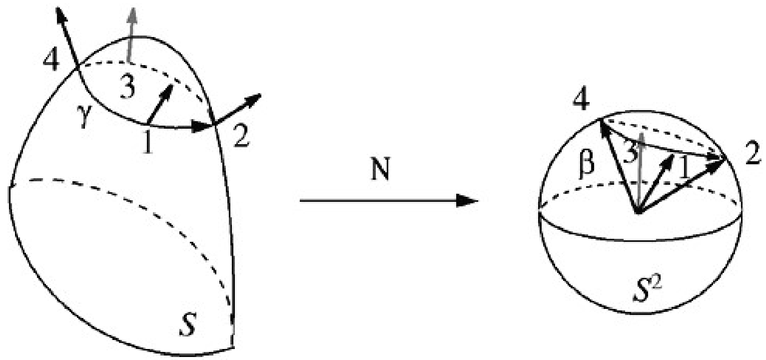
# Gauss map and Gaussian curvature

Gauss map
$n$: M → $S^2$

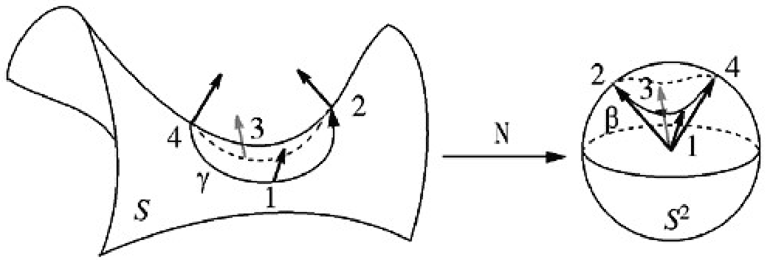



$$G \equiv \lim_{A \to 0} \frac{S_A}{A}$$

(Ratio between two areas)

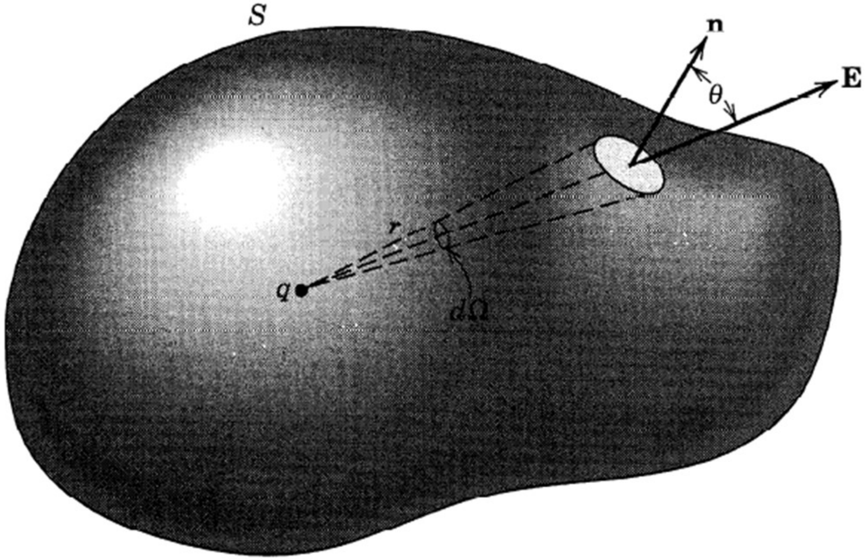

(a)




(b)

Elli Angelopoulou and Lawrence B. Wolff, IEEE Transactions 1998

# Total curvature



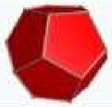

Total curvature of a closed surface is $4\pi$,
*no matter how the surface is deformed*

$$\int_M da\, G = \int_M \cancel{da}\, \frac{dS_a}{\cancel{da}} = 4\pi$$

Total curvature is a topological invariant

## Platonic solids, F. Maurolico (1537)

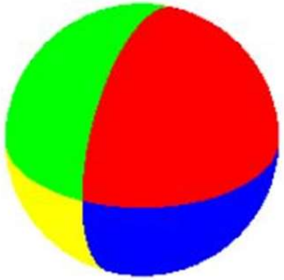
正多面體

| Name | Image | Vertices $V$ | Edges $E$ | Faces $F$ | Euler characteristic: $V - E + F$ |
|---|---|---|---|---|---|
| Tetrahedron |  | 4 | 6 | 4 | **2** |
| Hexahedron or cube |  | 8 | 12 | 6 | **2** |
| Octahedron |  | 6 | 12 | 8 | **2** |
| Dodecahedron |  | 20 | 30 | 12 | **2** |
| Icosahedron |  | 12 | 30 | 20 | **2** |

## Beyond regular polyhedron

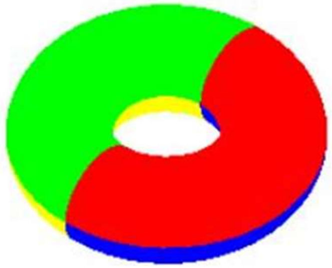
Euler (1758)

sphere



$$\chi = V - E + F = 2 - 4 + 4 = 2$$

torus



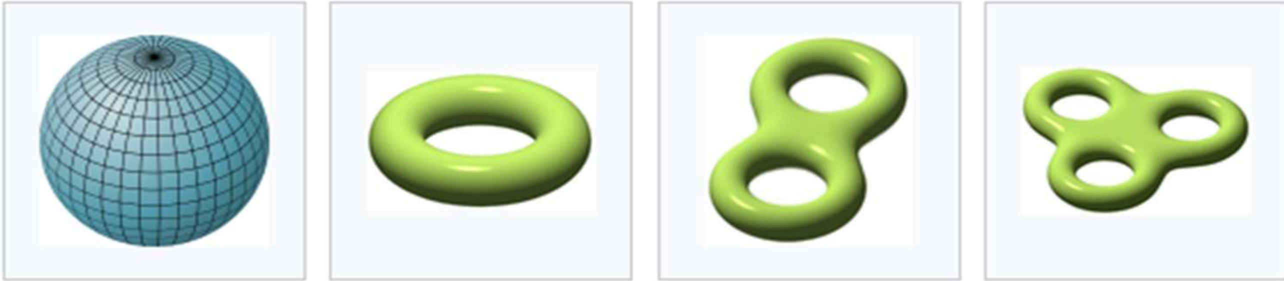
$$\chi = V - E + F = 4 - 8 + 4 = 0$$

- This number is independent of the ways of division, so it's a property of the surface itself.
- Furthermore, it does not change under continuous deformation, so it's a topological invariant.

# Euler characteristic of a surface

$\chi(M) = 2(1-g)$

\# of holes



$\chi = 2$  $\chi = 0$  $\chi = -2$  $\chi = -4$

In general, for a surface $M$ with dimension $D$, we can divide it into a patchwork of cells, and define

$$\chi(M) = \sum_{k=0}^{D} (-1)^k \beta_k, \qquad \text{(B8)}$$

where $\beta_k$ is the number of $k$-**simplexes**. *k*-單體

*k*=0, 1, 2, 3, … = 

, …

For a surface (D=2), $\chi(M) = \beta_0 - \beta_1 + \beta_2$

## Gauss-Bonnet theorem (for 2D surface)

– connecting *local curvature* with *global topology*

- Closed surface

$$\frac{1}{2\pi}\int_M da\, G = \chi(M)$$

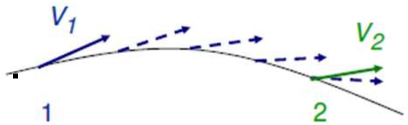
*The most beautiful theorem in differential topology*

- Open surface

$$\frac{1}{2\pi}\left[\int_M da\, G + \int_{\partial M} d\ell\, \kappa_g\right] = \chi(M, \partial M)$$

$\chi = 1$

# Anholonomy in geometry and quantum state

| | Geometry | Quantum state |
|---|---|---|
| PT condition |  | $i\langle\psi\vert\dot{\psi}\rangle = 0$ |
| anholonomy | Anholonomy angle | Berry phase |
| curvature | Gaussian curvature | Berry curvature |
| Topo number | Euler characteristic | Chern number |
| | $\chi = \frac{1}{2\pi}\int_S da\, G$ | $C = \frac{1}{2\pi}\int_M da\, \Omega$ |



- Chern number refers to the topological number of *fiber bundle space*
- Fiber bundle space ≈ inner DOF x spacetime

Spin … etc

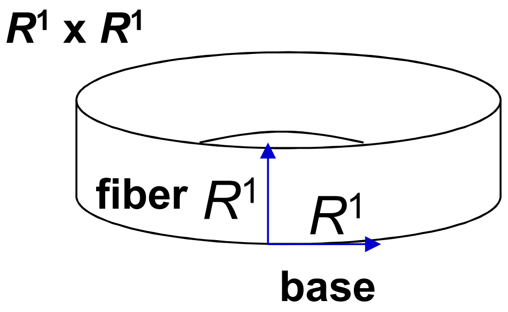
陳省身

# What is a fiber bundle 纖維束

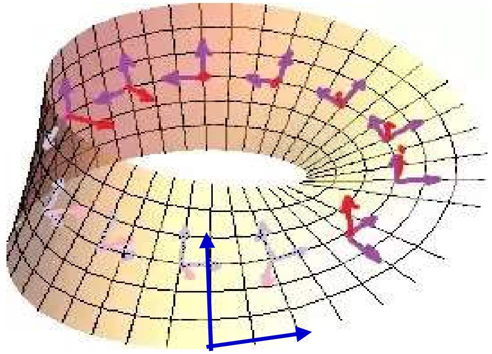
Ref: *Fiber bundles and quantum theory*, by Bernstein and Phillips, Sci. Am. 1981

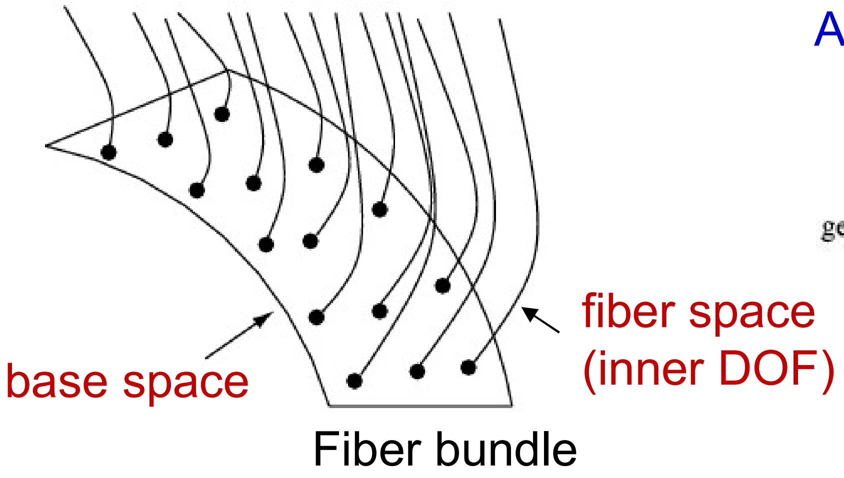
Simplest examples:

- **Trivial** fiber bundle
(= a product space)

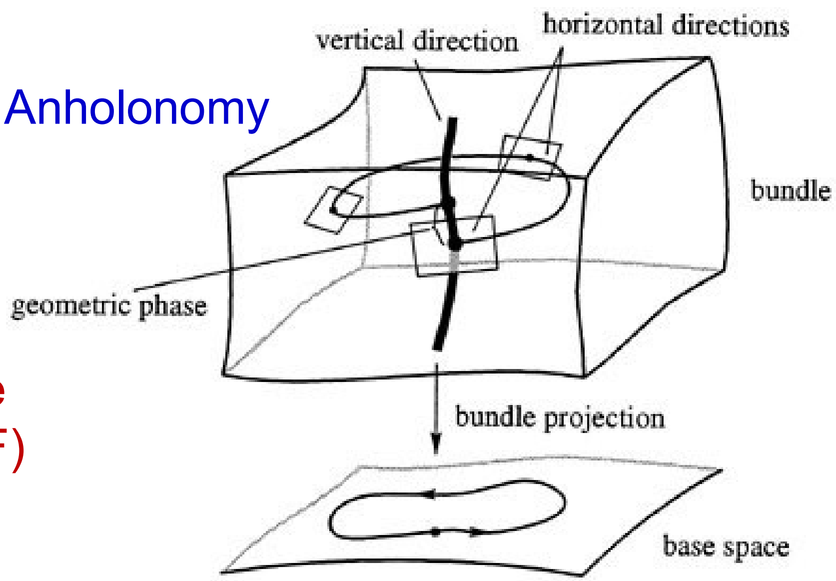



- **Nontrivial** fiber bundle
Möbius band






Anholonomy



# Fiber bundles in physics

| System | Base space | Fiber space |
|---|---|---|
| EM without monopole | Spacetime | U(1) trivial |
| EM with monopole | Spacetime | U(1) nontrivial |
| Electro-weak theory | Spacetime | U(1)xSU(2) |
| QCD | Spacetime | SU(3) |
| Abelian Berry phase | Parameter manifold | U(1) |
| Non-Abelian Berry phase | Parameter manifold | U(N) |

Space,
Brillouin zone
… etc

Lie groups

# Winding number again

Index of a point defect

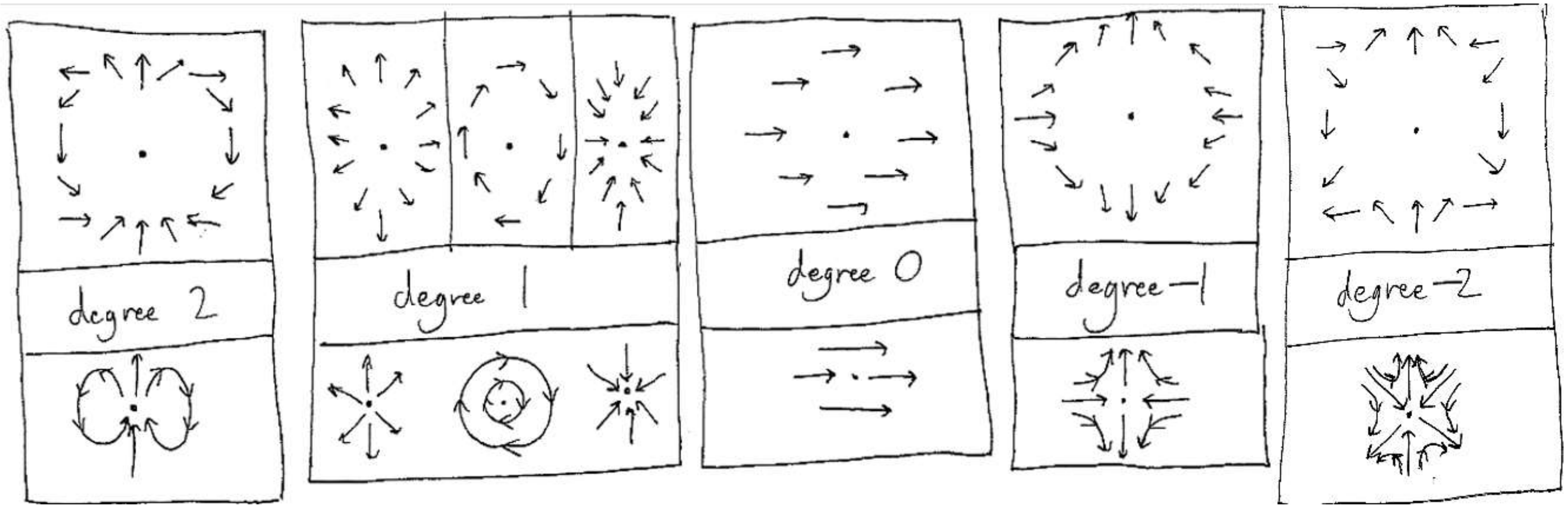


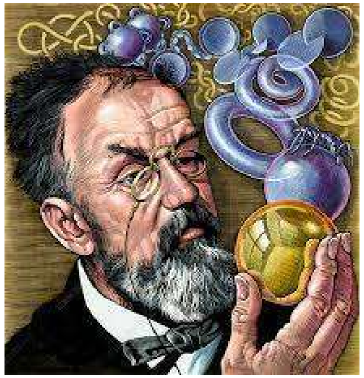

Fig from Jonas Kibelbek

## Hopf-Poincare theorem

- Connecting index of point defect with topology

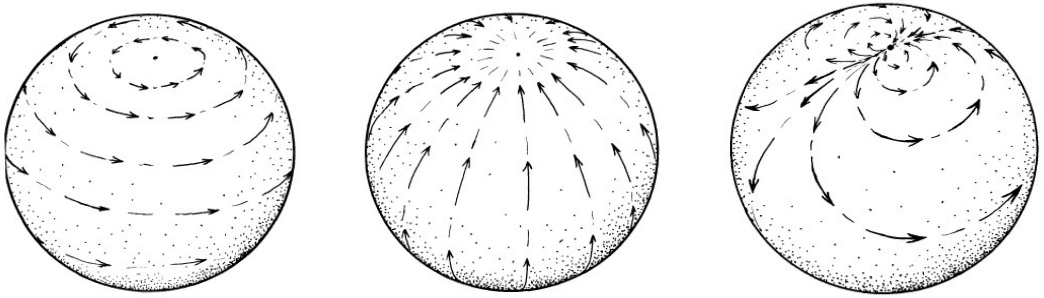
$$\sum_i ind(v_i) = \chi(M)$$

Winding number

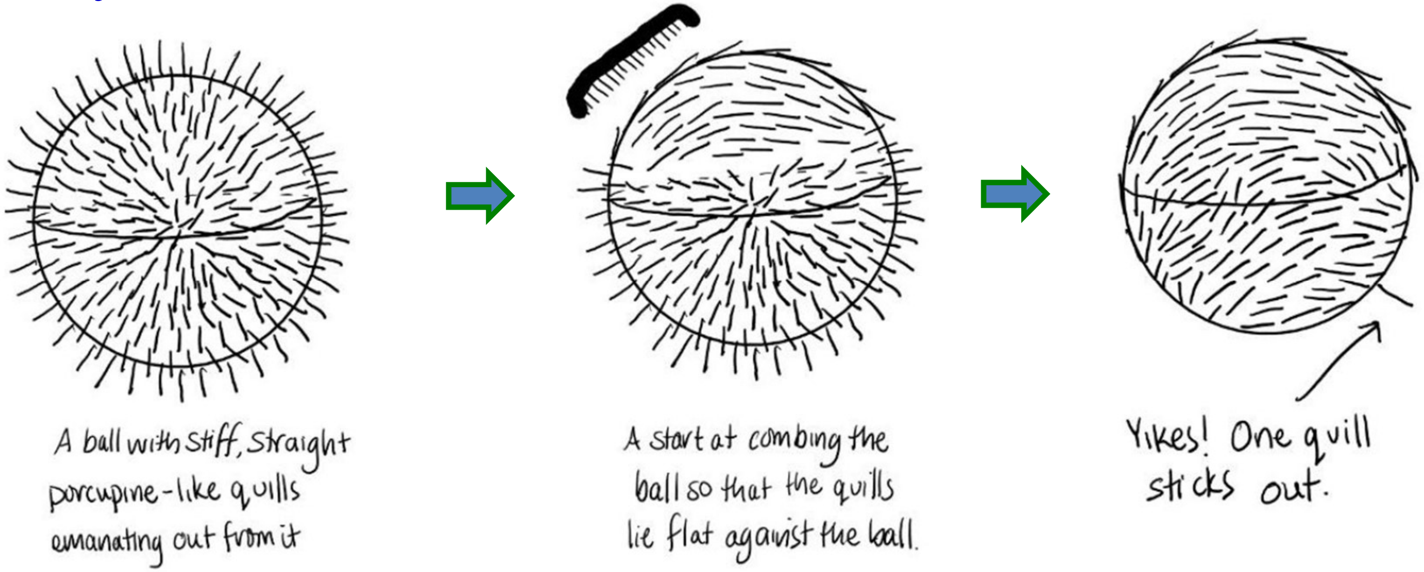


On a sphere

$$\sum_i ind(v_i) = 2$$



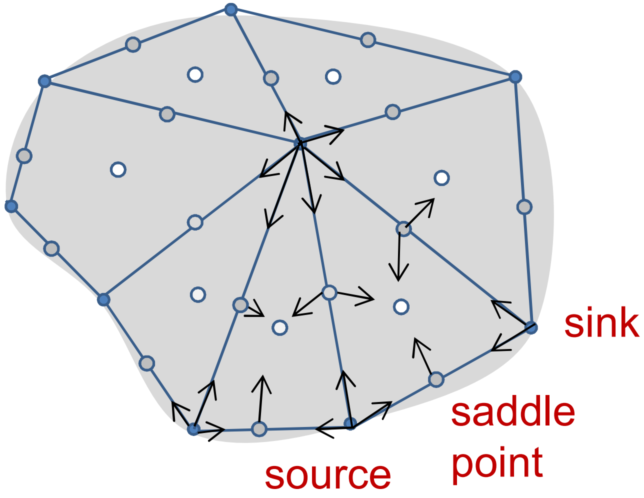
## Hairy ball theorem



# A “proof” of Hopf-Poincare theorem

Youtube course: *Topology & Geometry*, by Tadashi Tokieda
時枝正

Put a source on a vertex, a saddle point on an edge,
and a sink on a face

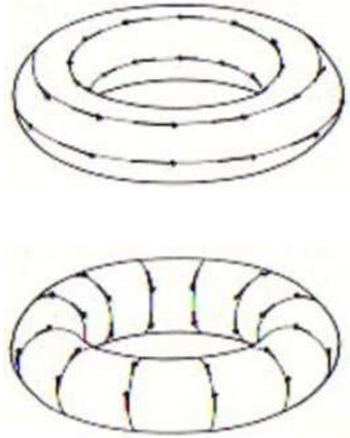



$$\sum_i \mathrm{ind}(\mathbf{v}_i) = (+1)\beta_0 + (-1)\beta_1 + (+1)\beta_2$$
$$= \chi(M)$$

$$\chi(M) = \sum_{k=0}^{D}(-1)^k \beta_k$$

# Vector field on a torus

$$\sum_i ind(v_i) = \chi(T^2) = 0$$



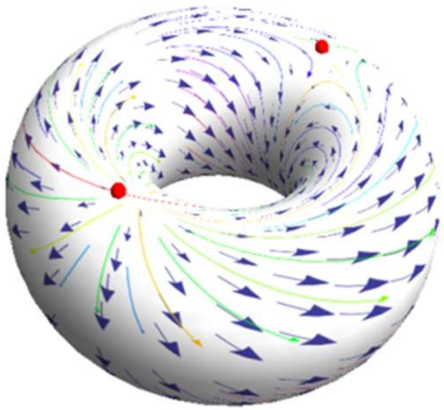
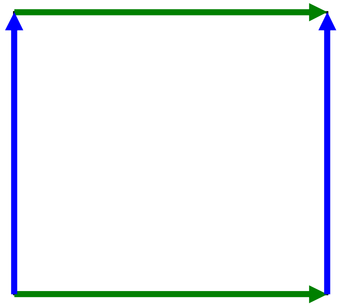
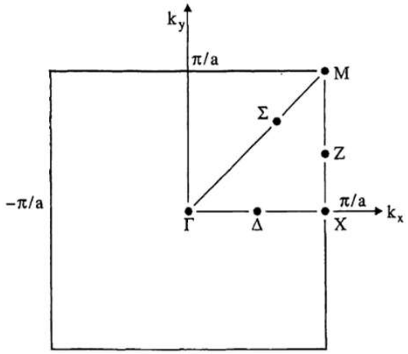


Fig from arxiv.org/pdf/1201.1162.pdf

Application: Brillouin zone as a torus (1D, 2D, 3D)






Berry connection **A**(***k***) as a vector field in BZ