Lecture notes on classical electrodynamics

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I. THE MAXWELL EQUATIONS

In this chapter, we outline the fundamental equations in electrodynamics.

A. Charge and current

1. Charge density

Consider a distribution of charge inside a volume $V$. If in a volume element $dv$ near point $r$, there is charge $dQ$, then the charge density at this location is

$$\rho(r) \equiv \frac{dQ}{dv}. \quad (1.1)$$

By the integration of $\rho(r)$, we can have the total charge $Q$ inside a volume $V$,

$$Q = \int_V dv \rho(r). \quad (1.2)$$

As we have mentioned in Chap 1, for a point charge $q$ at $r_1$, its charge density is,

$$\rho(r) = q\delta(r - r_1). \quad (1.3)$$

If there are point charges $q_1, q_2, \cdots, q_N$ at locations $r_1, r_2, \cdots, r_N$, then the charge density of this system is,

$$\rho(r) = \sum_{i=1}^{N} q_i\delta(r - r_i). \quad (1.4)$$

2. Current density

Electric current is defined as the amount of charge passing through a surface $S$ per unit time. If there is current $dI$ passing through a surface element $ds = d\hat{n}$, then (Fig. 1(a))

$$dI = J(r) \cdot ds = J \parallel (r) ds, \quad (1.9)$$

where $J(r)$ is the current density along the direction of charge motion, and $J \parallel = J \cdot \hat{n}$ is its component parallel to the surface normal $\hat{n}$.

After integration, we can find out the total current passing through surface $S$,

$$I = \int_S ds \cdot J(r). \quad (1.10)$$
If a small packet of charge $dQ$ is moving with velocity $v$, then within a time $dt$, the charges passing through $ds$ have spanned a volume $dv = (vd) \cdot ds$. Inside this volume,

$$dQ = \rho(vdt) \cdot ds,$$

which results in a current,

$$dI = \frac{dQ}{dt} = \rho v \cdot ds.$$  \hfill (1.12)

Compared with Eq. (1.9), one has

$$J(r) = \rho(r)v(r).$$  \hfill (1.13)

For point charges, with Eq. (1.4), one has

$$J(r) = \sum_{i=1}^{N} q_i v_i \delta(r - r_i),$$  \hfill (1.14)

where $v_i$ is the velocity of charge $i$.

Next, consider the current flowing on a surface. The surface has normal vector $\hat{n}$, and there is a line element $d\mathbf{r} \perp \hat{n}$ on the surface. The vector $\hat{n} \times d\mathbf{r}$ is tangent to the surface and perpendicular to $d\mathbf{r}$ (see Fig. 1(b)). The current $dI$ passes through $d\mathbf{r}$ is,

$$dI = K(r) \cdot \hat{n} \times d\mathbf{r},$$  \hfill (1.15)

where $K(r)$ is the surface current density along the direction of charge motion. Its dimension is [current]/[length].

After integration, we can find out the total current passing through a curve $C$ on the surface,

$$I = \int_C K(r) \cdot \hat{n} \times d\mathbf{r} = \int_C K(r) \times \hat{n} \cdot d\mathbf{r}.$$  \hfill (1.16)

3. Conservation of charge

Suppose the charge $Q$ inside a volume $V$ is leaking through its surface $S$ to the outside (Fig. 2). Due to charge conservation, the leaking current is,

$$I = -\frac{dQ}{dt}.$$  \hfill (1.17)

With Eqs. (1.2) and (1.10), we have

$$I = \int_S d\mathbf{r} \cdot J = \int_V dv \nabla \cdot \mathbf{J},$$  \hfill (1.18)

and

$$\frac{dQ}{dt} = \int_V dv \frac{\partial \rho(r,t)}{\partial t}.$$  \hfill (1.19)

Hence,

$$\int_V dv \nabla \cdot \mathbf{J} = - \int_V dv \frac{\partial \rho(r,t)}{\partial t}.$$  \hfill (1.20)

or

$$\int_V dv \left( \nabla \cdot \mathbf{J} + \frac{\partial \rho(r,t)}{\partial t} \right) = 0.$$  \hfill (1.21)

Since the charge should be conserved for any $dv$ in any location, so we can choose $V$ to be one of the $dv$, then

$$\int_V dv \left( \nabla \cdot \mathbf{J} + \frac{\partial \rho(r,t)}{\partial t} \right) = \int_V dv \left( \nabla \cdot \mathbf{J} + \frac{\partial \rho(r,t)}{\partial t} \right) \lim_{V \to \text{location}} = 0,$$  \hfill (1.22)

This is the equation of continuity written in differential form. It is valid if and only if charge is conserved.

B. Maxwell equations in vacuum

1. Electrostatics

According to Coulomb’s law, the electric force between two charges $q_i q_j$ at positions $r_i r_j$ is,

$$\mathbf{F} = \frac{qq}{4\pi \varepsilon_0} \frac{\mathbf{r} - r_1}{|\mathbf{r} - r_1|^3},$$  \hfill (1.24)

where the electric permittivity of free space $\varepsilon_0 = 8.8542 \times 10^{-12}$ C²/Nm².

If there are $N$ charges $q_1 q_2 \cdots q_N$ at positions $r_1 r_2 \cdots r_N$, then a test charge charge $q$ at $r$ feels a force,

$$\mathbf{F} = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{N} q q_i \frac{r - r_i}{|r - r_i|^3}.$$  \hfill (1.25)

The electric field $E$ from these $N$ charges is given as,

$$E(r) = \frac{\mathbf{F}}{q} = \frac{1}{4\pi \varepsilon_0} \sum_{i=1}^{N} q q_i \frac{r - r_i}{|r - r_i|^3}.$$  \hfill (1.26)

A continuous charge distribution can be divided into small packets with charges $\rho(r) dv'$ (Fig. 3). Identify $q_i$ with $\rho(r') dv'$ and replace the summation with an integral, one then has

$$\mathbf{E}(r) = \frac{1}{4\pi \varepsilon_0} \int dv' \rho(r') \frac{r - r'}{|r - r'|^3}.$$  \hfill (1.27)
Thus, \( \nabla \cdot \mathbf{E} = \frac{1}{4\pi \varepsilon_0} \int_{V} \rho(r') \, dv \). \[ \text{(1.44)} \]

where \( \rho \) is the charge density. This is the Poisson equation that has been referred to in Chap 1.

Furthermore, since the curl of divergence is zero, so

\[ \nabla \times \mathbf{E} = -\nabla \times \nabla \phi = 0. \quad \text{(1.37)} \]
so that
\[ \mathbf{B}(r) = \frac{\mu_0}{4\pi} \int_{V} dv' \mathbf{J}(r') \times \frac{r - r'}{|r - r'|^3}. \] (1.45)

This is the most general form of the Biot-Savart law that applies to all kinds of current distribution.

Again we can rewrite
\[ \frac{r - r'}{|r - r'|^3} = -\nabla \frac{1}{|r - r'|}. \] (1.46)

With the identity,
\[ \nabla \times (f \mathbf{v}) = \nabla f \times \mathbf{v} + f \nabla \times \mathbf{v}, \] (1.47)
we can write
\[ \mathbf{B}(r) = -\frac{\mu_0}{4\pi} \int_J dv' \mathbf{J}(r') \times \nabla \frac{1}{|r - r'|}. \] (1.48)
\[ = \nabla \times \mathbf{A}(r), \] (1.49)
with the vector potential,
\[ \mathbf{A}(r) = \frac{\mu_0}{4\pi} \int_J dv' \frac{\mathbf{J}(r')}{|r - r'|}. \] (1.50)

For a thin wire, it reduces to
\[ \mathbf{A}(r) = \frac{\mu_0 I}{4\pi} \oint_C \frac{dr'}{|r - r'|}. \] (1.51)

Since the divergence of curl is zero, so
\[ \nabla \cdot \mathbf{B}(r) = \nabla \cdot \nabla \times \mathbf{A}(r) = 0. \] (1.52)

This is Gauss’s law in magnetism. Also, if we take the curl of \( \mathbf{B} \), then
\[ \nabla \times \mathbf{B}(r) = \mu_0 \mathbf{J}(r). \] (1.53)

This is Ampère’s law.

Proof: First, we can show that for the steady case \( \nabla \cdot \mathbf{J} = 0 \), one has \( \nabla \cdot \mathbf{A} = 0 \). This is because
\[ \nabla \cdot \mathbf{A}(r) = \frac{\mu_0}{4\pi} \int_J dv' \mathbf{J}(r') \cdot \nabla' \frac{1}{|r - r'|}. \] (1.54)
\[ = -\frac{\mu_0}{4\pi} \int_J dv' \mathbf{J}(r') \cdot \nabla' \frac{1}{|r - r'|}. \] (1.55)
\[ = \frac{\mu_0}{4\pi} \int_J dv' \left( \frac{\nabla' \cdot \mathbf{J}(r')}{|r - r'|} \right) \bigg|_{r = 0} \] (1.56)
\[ = 0, \] (1.57)
where we have used the identity,
\[ \nabla \cdot (f \mathbf{v}) = \nabla f \cdot \mathbf{v} + f \nabla \cdot \mathbf{v}. \] (1.58)

Also, a surface term has been dropped.

Second,
\[ \nabla \times \mathbf{B}(r) = \nabla \times (\nabla \times \mathbf{A}) \] (1.59)
\[ = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \] (1.60)
\[ = -\nabla^2 \mathbf{A}(r) \cdot \mathbf{v} + \nabla \cdot \mathbf{A} = 0 \] (1.61)
\[ = -\frac{\mu_0}{4\pi} \int_J dv' \mathbf{J}(r') \cdot \nabla^2 \frac{1}{|r - r'|} \] (1.62)
\[ = \mu_0 \mathbf{J}(r). \quad QED \] (1.63)

When written in potential, we have
\[ \nabla^2 \mathbf{A}(r) = -\mu_0 \mathbf{J}(r). \] (1.64)

This is the vector Poisson equation in magnetostatics.

If we integrate Eq. (1.52) over a region \( V \) enclosed by surface \( S \), then
\[ \int_V dV \nabla \cdot \mathbf{B}(r) = \int_S ds \cdot \mathbf{B}(r) = 0, \] (1.65)

This shows that the magnetic flux through a closed surface is always zero. The existence of a magnetic monopole would contradict this result, but none has been found so far.

If we integrate Eq. (1.53) over a surface \( S \) with boundary \( C \), then
\[ \int_S ds \cdot \nabla \times \mathbf{B}(r) = \mu_0 \int_S ds \cdot \mathbf{J}(r), \] (1.66)
\[ or \quad \oint_C dr \cdot \mathbf{B}(r) = \mu_0 I, \] (1.67)

where \( I \) is the total current flowing through \( S \). This is the integral form of Ampère’s law.

3. Dynamic electromagnetic field

Eqs. (1.35), (1.37), (1.52), and (1.53) are the Maxwell equations for static electromagnetic field. For dynamics fields, they become
\[ \nabla \cdot \mathbf{E}(r, t) = \frac{\rho(r, t)}{\varepsilon_0}, \] (1.68)
\[ \nabla \cdot \mathbf{B}(r, t) = 0, \] (1.69)
\[ \nabla \times \mathbf{E}(r, t) = -\frac{\partial}{\partial t} \mathbf{B}(r, t), \] (1.70)
\[ \nabla \times \mathbf{B}(r, t) = \mu_0 \mathbf{J}(r, t) + \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E}(r, t). \] (1.71)

The charge density and the current density are related by the equation of continuity,
\[ \nabla \cdot \mathbf{J}(r, t) + \frac{\partial \rho(r, t)}{\partial t} = 0. \] (1.72)
The first change is that in Eq. (1.70), the right hand side (RHS) is no longer zero. This is Faraday’s law: a time-changing magnetic field produces an electric field.

The second change is that there is an extra term on the RHS of Eq. (1.71). This is the famous displacement current added by Maxwell: a time-changing electric field produces a magnetic field. The modified equation is called Ampère-Maxwell’s law.

When the fields are static, the Maxwell’s equations decouple into two sets of equations: two for electric field, and two for magnetic field. Thus, electrostatics and magnetostatics are independent of each other.

Integrating a divergence (e.g., $\nabla \cdot \mathbf{E}$) over a volume $V$ or a curl (e.g., $\nabla \times \mathbf{E}$) over a surface, and using the divergence theorem or the Stokes theorem, we have the integral form of the Maxwell equations (Fig. 5):

\[
\int_S ds \cdot \mathbf{E}(\mathbf{r}, t) = \frac{Q}{\varepsilon_0}, \quad (1.73)
\]
\[
\int_S ds \cdot \mathbf{B}(\mathbf{r}, t) = 0, \quad (1.74)
\]
\[
\int_C d\mathbf{r} \cdot \mathbf{E}(\mathbf{r}, t) = -\frac{d\Phi_B}{dt}, \quad (1.75)
\]
\[
\int_C d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}, t) = \mu_0 I + \frac{1}{c^2} \frac{d\Phi_E}{dt}, \quad (1.76)
\]

in which

\[
\Phi_B \equiv \int_S ds \cdot \mathbf{B}, \quad (1.77)
\]
\[
\Phi_E \equiv \int_S ds \cdot \mathbf{E}. \quad (1.78)
\]

They are the magnetic flux and the electric flux passing through surface $S$. Eq. (1.75) (Eq. (1.76)) tells us that a changing magnetic (electric) field through surface $S$ would induce electric (magnetic) circulation around the boundary $C$ of $S$.

C. Some history

In 1873, James C. Maxwell published "Treatise on electricity and magnetism" (Maxwell, 1891), in which he constructed a mathematical framework to describe all of the phenomena of electromagnetism. It has all the essence included but it’s hard to find “Maxwell equations” in the Treatise, since they are written as 20 equations in 20 variables scattered through the monograph. Some of the equations describe things like $\mathbf{D} = \varepsilon \mathbf{E}$, or $\mathbf{B} = \nabla \times \mathbf{A}$. It’s a pity that Maxwell died six years later at the age of 48, and was unable to pursue this subject further.

The four Maxwell equations we are familiar with nowadays are mainly the works of Oliver Heaviside and, independently, Heinrich R. Hertz (Fig. 6). It’s interesting to know that when the Treatise was just published, Heaviside (then 24) flipped through it in library and immediately saw the “prodigious possibilities in its power”. He then “determined to master the book”. (Mahon, 2017) Remember that at that time Maxwell is still not “Maxwell” and not many people trust his obscure, sometimes unintelligible theory on electromagnetism. Heaviside has no college education, and has forgotten most of the algebra and trigonometry learned in school. Thus, he quit his job that has a decent pay, stayed at home with his far-from-rich parents and started studying the Treatise. He remained “self-employed” ever since and never to get a job again. Heaviside has to learn all of the difficult mathematics of divergence, curl, and related theorems on his own, without friendly textbooks such as Arfken’s to ease the job.

In his later years, Heaviside recalls that “It took me several years before I could understand as much as I possibly could. Then I set Maxwell aside and followed my own course.”

The effort and sacrifice pay off. With his own formulation, Heaviside discovered things like electric inductance, contraction of the electric field of a moving charge (Heaviside ellipsoid), and magnetic-like field of gravity (gravito-magnetism).

In 1888, to the surprise of everybody, Hertz generated and detected electromagnetic wave in free space. This is the strongest boost to the status of Maxwell’s electromagnetic theory since at that time there was no other theory predicting the existence of EM wave. Afterwards, optics becomes a branch of electromagnetism.
More progress followed, such as the discovery of electron by J. J. Thomson in 1897, the theory of thermal radiation by Ludwig E. Boltzmann and others. The latter pursuit eventually leads to Max Planck’s important discovery of energy quantum.

Furthermore, in an attempt to resolve a paradox regarding motional electromotive force, Einstein discovered the theory of special relativity in 1905. As a result, Newton’s theory of mechanics needs to be revised. Nevertheless, Maxwell’s theory remains intact, since it is based on experimental evidences that have already included relativistic effects.

Problem:
1. The electric potential of an atom is given by
\[
\phi(r) = \frac{q}{4\pi\varepsilon_0} \frac{e^{-\alpha r}}{r},
\]
where \( q > 0 \), \( \alpha \) are constants.
(a) Find out the electron charge density \( \rho(r) \) outside the nucleus.
(b) Find out the total charge of this charge distribution. Hint: Poission equation.

References