[30%] 1. A grounded conducting sphere of radius *a* carries charge *q*. The dielectric constant outside the sphere varies with the radial distance from the center of the sphere,

$$e(r) = e_0\left(1+\frac{b^2}{r^2}\right), \quad (r>a)\,.$$

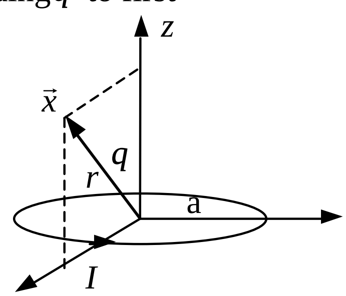
(a) Find the potential $f\,(r)$ in the region outside the sphere.

(b) What is the polarization surface charge density $s_p = \vec{P}\cdot\hat{n}$ on the dielectric surface at r=a?

[30%] 2. A circular ring with current *I* and radius *a* is located on the *x*-*y* plane (see Figure).

(a) Find the vector potential $\vec{A}(\vec{x}) = \frac{m_0}{4p}\int\frac{Id\vec{\ell}\,'}{|\vec{x}-\vec{x}\,'|}$ at any point in space..

(b) Using the result in (a), find the vector potential near the *z*-axis by expanding q to first order, then calculate the magnetic field $\vec{B}(z)$ on the positive *z*-axis.



[40%] 3. The electromagnetic field momentum is $\vec{p}_{field} = \frac{1}{c^2}\int dV\vec{E}\times\vec{H}$.

(a) A localized charge distribution produces $\vec{E} = -\nabla f$. This field coexists with a *H* field that is generated by a localized charge density *J* (time-independent). Show that if $f\vec{H}$ falls off rapidly at large distance, then $\vec{p}_{field} = \frac{1}{c^2}\int dV f\vec{J}$.

(b) Based on the result in (a), show that if the *E* field changes little over the localized charge distribution, then $\vec{p}_{field} = \frac{1}{c^2}\vec{E}(0)\times\vec{m}$, where 0 is the location within the current distribution, and $\vec{m}$ is the magnetic moment generated by the current.

$$\int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c; \quad \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial f}{\partial q}\hat{q}+\frac{1}{r\sin q}\frac{\partial f}{\partial f}\hat{f}\,; \quad \nabla\cdot\vec{g} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g_r\right)+\frac{1}{r\sin q}\left[\frac{\partial}{\partial q}\left(\sin q\, g_q\right)+\frac{\partial g_f}{\partial f}\right], \text{ (spherical coord.)}$$

$$\int_0^p\frac{\pm\cos f\,df}{(b\pm\cos f)^{1/2}} = \frac{4}{\sqrt{2a}}E(a)-b\sqrt{2a}\,K(a); \quad \int_0^p\frac{\pm\cos f\,df}{(b\pm\cos f)^{3/2}} = \sqrt{2a}\,K(a)-\frac{b}{b-1}\sqrt{2a}\,E(a), \quad a=\frac{2}{1+b}$$

$$E(a) = \frac{p}{2}\left(1-\frac{a}{4}-\frac{3}{64}a^2-\dots\right); \quad K(a) = \frac{p}{2}\left(1+\frac{a}{4}+\frac{9}{64}a^2+\dots\right)$$

$$B_r = \frac{1}{r\sin q}\frac{\partial}{\partial q}\left(\sin q\, A_f\right), \quad B_q = -\frac{1}{r}\frac{\partial}{\partial r}\left(rA_f\right), \quad B_f = 0.$$

$$x_i\nabla\cdot\vec{J} = \nabla\cdot(x_i\vec{J})-\vec{J}\cdot\nabla x_i; \quad x_i x_j\nabla\cdot\vec{J} = \nabla\cdot(x_i x_j\vec{J})-x_i J_j - x_j J_i$$