Lecture notes on topological insulators

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CONTENTS

I. INTRODUCTION TO TOPOLOGY

The topological property of an object is the property that is invariant under *continuous* deformation of that object. For example, on a desk there is a rubber band and a nailed thumbtack. The thumbtack can be inside or outside of the rubber band. Without lifting the rubber band from the desk or breaking it, it is impossible to deform one state to the other. So these two states have different topologies.

If a physical system has a topological property. Then it would be robust against small changes in physical condition. The most famous example is the quantum Hall system (more details in a later chapter). Because of its inherent topology, its quantum Hall conductance can be universal, independent of the samples of choice, the laboratories or the times of measurement.

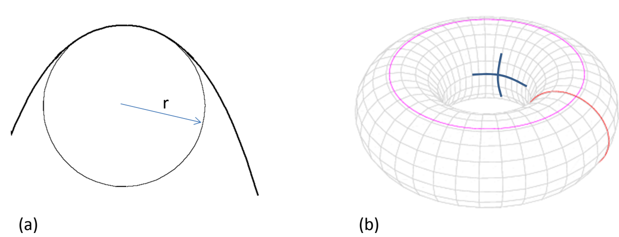

FIG. 1 (a) An osculating circle for a particular point on a curve. Similar concept can be extended to a 2D curved surface: a patch of the surface near a particular point can be approximated by a quadratic surface. (b) The inner side of a torus is similar to the surface of a saddle and has negative curvature. The outer side of the torus has positive curvature.

A. Intrinsic curvature and extrinsic curvature

The discussion here is brief, and only a pedagogical picture is provided. For more of the technical details, one can read, for example, Schutz, 1982, or Kreyszig, 1968.

Consider a smooth two-dimensional (2D) surface. Near a point p, the surface can be approximated by a quadratic surface. The closet quadratic surface being the one that matches this patch of surface to the second-order Taylor expansion in a given coordinate (see Fig. 1(a)). It is known that a quadratic surface can be one of three types: an ellipsoid, a paraboloid, or a hyperboloid. These three types of surface all have two principal directions with maximum or minimum curvature. Along these two directions, we have two **principle curvatures** $k_1 = 1/r_1, k_2 = 1/r_2$ (up to a sign), where r_1, r_2 are radii of osculating circles along the principle directions.

The **Gaussian curvature** at p is defined to be the product of k_1, k_2,

$$G = \frac{1}{r_1}\frac{1}{r_2}. \tag{1.1}$$

For example, on the inner side of a torus, the two principal directions curve toward opposite directions (like the surface of a saddle, see Fig. 1(b)). Therefore, the Gaussian curvature G is negative. On the outer side of the torus, G is positive.

An important property of the Gaussian curvature is that it cannot be changed by bending the surface. It can be changed only by stretching or squeezing. Bending cannot alter the shortest distance between two points on the surface, while stretching/squeezing can. Also, for a creature living on the 2D surface, it is possible for it to determine the Gaussian curvature by measuring the distances and angles (see next subsection) on the surface. Therefore, the Gaussian curvature is an **intrinsic curvature**.

In addition, the **mean curvature** H is defined to be the sum of k_1, k_2,

$$H = \frac{1}{r_1} + \frac{1}{r_2}. \tag{1.2}$$

For a cylinder, the radius r_1 for a straight line along a ridge is infinite, so $G = 0$, but $H \neq 0$. Note that a

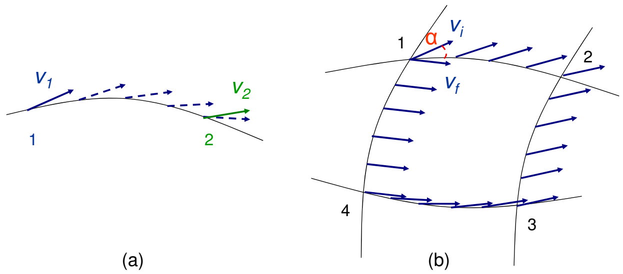

FIG. 2 (a) Parallel transport of a vector from 1 to 2. It offers a way to compare $\mathbf{v}_1$ and $\mathbf{v}_2$ on a curved surface. (b) A vector is parallel transported around a closed path. The final vector could point to a different direction from the initial vector.

flat sheet of paper has mean curvature $H = 0$. But if you bend the paper, then H is no longer zero. That is, the mean curvature can be changed simply by the act of bending. Therefore it is an **extrinsic curvature**. A cylindrical surface looks curved from a creature outside the surface in 3D space, but not to a creature living on the surface.

B. Parallel transport and anholonomy angle

We now try to define the Gaussian curvature without leaving the surface. This can be achieved with the anholonomy angle, which we now explain.

On a flat plane, one can easily compare two vectors lying on the plane to see if they are parallel to each other. For a curved surface, however, the comparison is not so evident. The tangent vectors at point p belong to the tangent plane T_p of point p. These tangent planes have different orientations at different points and the concept of parallelism seems to be lost for a 2D creature living on the surface.

Nonetheless, one can still have parallelism by following the rule of **parallel transport** (Levi-Civita, 1917): To compare the vectors at point 1 and point 2, first draw a **geodesic** (the shortest curve) between two points. Starting from point 1, carry the vector in such a way that it makes a fixed angle with the tangent vector along the a geodesic (see Fig. 2(a)) till it reaches point 2. You can then compare it with the vector already at point 2 to see if they are "parallel".

For a *closed loop* on a curved surface, after parallel-transporting a vector around the loop, the final vector $\mathbf{v}_f$ is usually different from the initial vector $\mathbf{v}_i$ (see Fig. 2(b)). The angle between these two vectors is called the **anholonomy angle** (or **defect angle**) α_A. One can define the Gaussian curvature at point p as the ratio between α_A and the area A of an infinitesimal triangle around p,

$$G = \lim_{A \to 0} \frac{\alpha_A}{A}. \tag{1.3}$$

For example, consider a sphere with radius r. It is known that a spherical triangle on its surface has an area $A = r^2(\theta_1 + \theta_2 + \theta_3 - \pi)$ (Girard theorem). On the other hand, if you carry a vector around the spherical triangle, the defect angle would be $\alpha_A = \theta_1 + \theta_2 + \theta_3 - \pi$, where θ_i are the angles of the triangle. This can be understood in the following way: Suppose the sphere is the earth, and you carry a pendulum walking around the triangular loop. At the starting point, the pendulum swings along the tangent vector of the path at this point. Since the sides of the spherical triangle are geodesics, the vector would be parallel transported, expect at the corners of the triangle. Each corner contributes an angle $\theta_i - \pi$ between two tangent vectors at the corner. Therefore, when the pendulum comes back, the anholonomy angle

$$\begin{aligned} \alpha_A &= (\theta_1 - \pi) + (\theta_2 - \pi) + (\theta_3 - \pi) \bmod 2\pi & (1.4) \\ &= \theta_1 + \theta_2 + \theta_3 - \pi. & (1.5) \end{aligned}$$

To calculate the Gaussian curvature at a point p on the surface, combine these two results above and get

$$G = \lim_{A \to 0} \frac{\alpha_A}{A} = \frac{1}{r^2}. \tag{1.6}$$

The larger the sphere, the smaller the curvature. Also, obviously, it is the same everywhere on the surface.

You can apply the same procedure to find out the Gaussian curvature of a cylinder. The result would be zero, as expected.

So far we have mentioned two definitions of the Gaussian curvature in Eq. (1.1) and Eq. (1.3). The third one is defined below, and they can all be shown to be equivalent. The third definition is as follows (Huang, 1978): Suppose there is a small area A covering a point p on the surface. The *unit normal vector* of A draws out another area G_A on the surface of a unit sphere S. This mapping is known as the **Gauss map**. The Gaussian curvature can be defined as,

$$G = \lim_{A \to 0} \frac{G_A}{A}. \tag{1.7}$$

From this definition, it is not difficult to see that the *total curvature* of a simple closed surface (with no holes) is equal to the total solid angle of a unit sphere, which is 4π. The total Gaussian curvature is closely related to the topology of a surface. Before explaining this, let's introduce a topological number of a surface called the Euler characteristic.

C. Euler characteristic

Consider a two-dimensional surface M (with or without boundary). Divide it into a patchwork of cells as in Fig. 3(a). Assume there are β_0 vertices (aka 0-simplexes), β_1 edges (1-simplexes), and β_2 faces (2-simplexes). Then

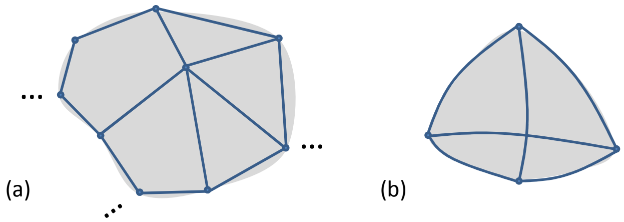

FIG. 3 (a) A surface is divided into cells. (b) Division of a sphere-like surface.

the **Euler characteristic** of this patchwork is defined as,

$$\chi(M) = \beta_0 - \beta_1 + \beta_2. \tag{1.8}$$

For example, for the sphere-like surface shown in Fig. 3(b), we have

$$\chi(S^2) = 4 - 6 + 4 = 2. \tag{1.9}$$

This number does not depend on how the surface is being divided. Also, if M' is **homeomorphic** to (topologically the same as) M, then $\chi(M') = \chi(M)$ (**Poincaré-Alexander theorem**). Thus, $\chi(M)$ is a number that characterizes the topology of M.

As an exercise, verify that the Euler characteristics of a disk and a torus are 1 and 0 respectively. For a general closed 2D surface M,

$$\chi(M) = 2(1-g), \tag{1.10}$$

where g is the number of holes in the surface. For example, for a torus, $g = 1$ and $\chi = 0$; for a sphere, $g = 0$ and $\chi = 2$.

In general, for a surface M with dimension D, we can divide it into a patchwork of cells, and define

$$\chi(M) = \sum_{k=0}^{D} (-1)^k \beta_k, \tag{1.11}$$

where β_k is the number of k-**simplexes**.

D. Gauss-Bonnet theorem

There is a deep connection between the curvature, which is a local property of a surface, and the topology, which is a global property: **Gauss-Bonnet theorem** tells us that the total Gaussian curvature of a *closed* 2D surface M is $2\pi\chi$, or

$$\frac{1}{2\pi}\int_M d^2a\ G = \chi(M). \tag{1.12}$$

That is, the total Gaussian curvature is a topological invariant.

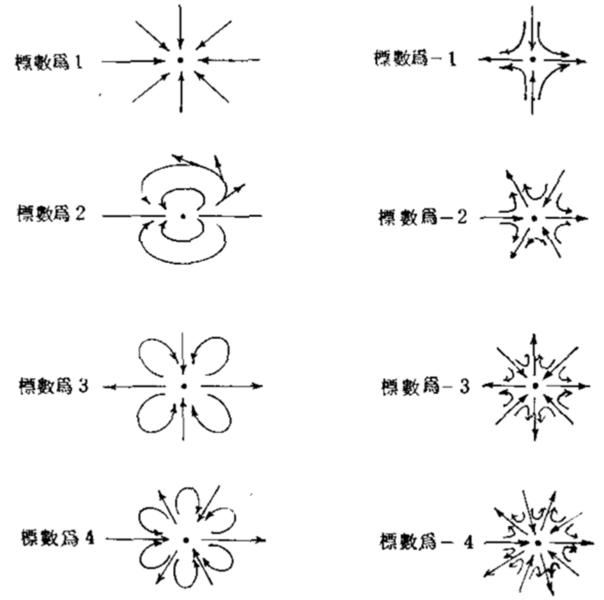

FIG. 4 Several examples of the index of a singularity in a 2D vector field. Note that if the arrows in the top-left figure are reversed, the index is still +1 (this is so only in even dimension). The figure is from Huang, 1978.

As we have mentioned, if the radius of a sphere is r, then $G = 1/r^2$. In this case, the Gauss-Bonnet theorem is trivially satisfied,

$$\frac{1}{2\pi}\int_M d^2a\ G = \frac{1}{2\pi}\int_M r^2 d\Omega\ \frac{1}{r^2} = \frac{4\pi}{2\pi} = 2. \tag{1.13}$$

What is amazing is that no matter how you squeeze and stretch the sphere to redistribute the G's, the total curvature is always 4π.

For reference, for a 2D surface M with a boundary ∂M that is sectionally smooth (with corners), the Gauss-Bonnet theorem is generalized as,

$$\int_M d^2a\ G + \int_{\partial M} d\ell\ \kappa_g + \sum_i (\pi - \theta_i) = 2\pi\chi(M), \tag{1.14}$$

in which κ_g is the **geodesic curvature** (see App. **??**) that measures the deviation from a geodesic curve, and the third term is a sum of *exterior angles* at the corners. The generalization of the Gauss-Bonnet theorem to higher dimension can be found in App. **??**.

E. Hopf-Poincaré theorem

Given a **zero** (such as a **source** or a **drain**) of a vector field, one can define an index according to the pattern of the surrounding flow. First consider a flow on a 2D surface: For a creature walking clockwise around the zero once, if the vectors of the flow on the creature's path

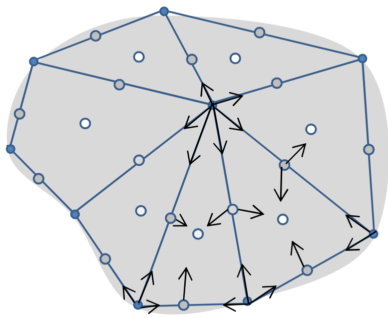

FIG. 5 Assign a flow to the surface divided by cells. Dark, grey, and white dots are sources, saddle points, and sinks.

Whenever $\chi(M) = 0$, the surface M is *parallelizable* (i.e., the hair, or vectors, on the surface could be "combed"). Therefore, S^1 and S^3 are parallelizable.

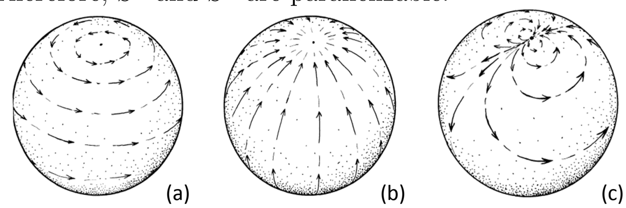

FIG. 6 Three examples of possible flows on S^2 surface. According to the Hopf-Poincaré theorem, the total index of the vector field must be 2.

rotate clockwise n-times, then the index is n. If they rotate counter-clockwise n-times, then the index is $-n$. For example, for the top-left figure in Fig. 4, the index is 1; for the top-right figure in Fig. 4, the index is -1.

On a D-dimensional manifold, the circular path surrounding the zero is replaced by a sphere S^{D-1}. The vectors on the sphere would trace out another sphere S'^{D-1}, and the index is given by the winding number of S'^{D-1} over S^{D-1}.

Hopf-Poincaré theorem (1927) states that, for a flow distributed on a *closed* manifold M, the total index of the vector field is equal to the Euler characteristic of M. That is,

$$\sum_i \mathrm{ind}(\mathbf{v}_i) = \chi(M). \tag{1.15}$$

For a 2D surface, this theorem can be explained as follows. This proof is based on Lect 12 of T. Tadashi's youtube lecture on *Topology and Geometry*:
First, divide the surface into cells with β_0 vertices, β_1 edges, and β_2 faces (see Fig. 5). Second, place a source at each vertex, a saddle point at the middle of each edge, and a sink in the middle of each face. We then have a continuous flow filling the surface. Since the indices of a source, a saddle point, and a sink are $+1, -1, +1$ respectively. We have

$$\sum_i \mathrm{ind}(\mathbf{v}_i) = (+1)\beta_0 + (-1)\beta_1 + (+1)\beta_2, \tag{1.16}$$

which is exactly the Euler characteristic defined in Eq. (1.8). End of proof.

For example, the total index of a vector field on the surface of a S^2 is 2, according to this theorem. In Fig. 6, we show 3 possible flow patterns, and their total indices are indeed all equal to 2. You may try to see if it's possible to find a vector field that breaks this rule. In general,

$$\chi(S^n) = 1 + (-1)^n. \tag{1.17}$$

The Hopf-Poincaré theorem on S^2 is sometimes called the **hairy ball theorem**: it's impossible to have a hairy ball free of any vortex (assuming the hair lies on the surface, of course). An alternative scenario is that, the flow of wind on the surface of the earth must have at least one location that has no wind at all.

Since the Euler characteristic of T^2 is zero, it's possible to have a smooth flow on T^2 without any vortex. For example, a flow with all vectors point to the azimuthal direction. On the other hand, if there is a vortex with index 1 somewhere on the surface of a torus, then there must be another vortex with index -1. In general, for any dimension n,

$$\chi(T^n) = 0. \tag{1.18}$$

Such a fact is related to the **Nielsen-Ninomiya theorem**, aka **fermion-doubling theorem**: massless lattice fermions always have to come up in pairs. This is valid in any *odd* spatial dimension, and neither TR nor SI symmetry needs be presumed. The Weyl point of a fermion is a source or a drain of the Berry flux. It is the zero of the vector field of Berry connection. The Berry index (topological charge) of this nodal point can be identified as the index of the zero. (for more details, see Lect **??**)

Finally, based on the Hopf-Poincaré theorem, one can deduce that the Euler characteristic $\chi(M)$ of any (closed) odd-dimensional manifold M is zero: After reversing the direction of the vector field, $\mathbf{v} \to -\mathbf{v}$, one has $\mathrm{ind}(\mathbf{v}_i) \to -\mathrm{ind}(\mathbf{v}_i)$ for each i (in *odd* dimension!). Thus $\sum_i \mathrm{ind}(\mathbf{v}_i) = 0$ (see p.39 of Milnor, 1965).

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