Is Hagedorn string gas cosmology an alternative to inflation?

Outline:

1) String gas cosmology
2) Hagedorn phase
3) Cosmological perturbation in Hagedorn phase
4) Critics
   a) quasi-static phase?
   b) which frame?

String gas cosmology:

ingredients:

a) gravity

b) dilaton

c) string matters $\begin{cases} \text{KK modes} \\ \text{winding modes} \end{cases}$

d) T-duality

→ self-dual radius

→ limiting temperature

The action in string frame

$$S_0 = -\int d^{N+1}x \sqrt{-G}\, e^{-2\phi}\left[c + R + 4(\nabla\phi)^2\right]$$

$$S_m = \int dt \sqrt{-G_{00}}\, F(\lambda_i, \beta\sqrt{-G_{00}})$$

For simplicity, we assume

i) $c = 0$

ii) homogeneous universe

$$ds^2 = -dt^2 + e^{2\lambda(t)} dx_N^2$$

iii) Both KK & winding modes included in S_m

Define $\varphi \equiv 2\phi - N\lambda$

s.t. $\sqrt{-G}\, e^{-2\phi} = e^{-\varphi}$

$$\Rightarrow \quad S_0 = -\int dt\, e^{-\varphi} \sqrt{-G_{00}} \left[-G^{00} N \dot{\lambda}^2 + G^{00} \dot{\varphi}^2 \right]$$

<u>Note</u>: S_0 is invariant under T-duality

$$\lambda \to -\lambda, \quad \phi \to \phi - N\lambda, \quad \varphi \to \varphi$$

$\Rightarrow$ Exists the minimal size of the Universe, also implies

$$T(R) = T\left(\frac{1}{R}\right)$$

$\Rightarrow$ Hagedorn temperature T_H occurs at self-dual radius.

S_m is also invariant under T-duality since we have included both KK & winding modes.

for KK modes:

$$E \sim \frac{1}{R}$$

for winding modes:

$$E \sim R$$

$\Rightarrow$ KK modes dominate for large R, otherwise winding modes do.

Moreover, one can start with the Nambu–Goto action and derive the equation of states:

a) $P = +\frac{1}{d}\rho$ for KK modes

b) $P = -\frac{1}{d}\rho$ for winding modes

try to contract the Universe in contrast to KK modes.

$\Rightarrow$ At self-dual radius, KK & winding modes are equally important

$\Rightarrow$ i) $P = 0$ pressureless

ii) $F = E - T_H S = 0$

$\Rightarrow S = \beta_H E$

Hagedorn entropy

iii) The Universe will undergo

either a) oscillatory if $R = R_{SD} \simeq l_s$ is an equilibrium point

b) quasi-static if $R = R_{SD}$ is NOT an equilibrium point

Now, come to see the equations of motion:

$$\delta G_{00} \Rightarrow \quad -N\dot{\lambda}^2 + \dot{\varphi}^2 = e^{\varphi} E$$

$$\delta\lambda \Rightarrow \quad \ddot{\lambda} - \dot{\varphi}\dot{\lambda} = \frac{1}{2} e^{\varphi} P$$

$$\delta\varphi \Rightarrow \quad \ddot{\varphi} - N\dot{\lambda}^2 = \frac{1}{2} e^{\varphi} E$$

$$E \equiv -2\frac{\delta S_m}{\delta G_{00}} = F + \beta\frac{\partial F}{\partial \beta}$$

$$P \equiv -\frac{\delta S_m}{\delta \lambda} = -\frac{\partial F}{\partial \lambda}$$

These equations implies conservation law of entropy:

$$\dot{E} + N\dot{\lambda}P = 0 \iff \dot{S} = 0 \quad \left(S = \beta^2 \frac{\partial F}{\partial \beta}\right)$$

$\Rightarrow$ The matter evolves adiabatically

$$\Rightarrow \quad \beta(\lambda), \quad E(\lambda) = E(\lambda, \beta(\lambda))$$

$$P(\lambda) = -\frac{1}{N}\frac{\partial E}{\partial \lambda}$$

Near the self-dual radius, i.e.

$$\lambda \simeq 0$$

$$\Rightarrow \quad E \simeq T_H S = \text{const.}$$

$$\Rightarrow \quad P \simeq 0$$

particles

$S \sim E^{N/(N+1)} < E$

for large E

The Universe is quasi-static in the Hagedorn phase, and the Hubble horizon is ∞ since

$$H = \frac{\dot{a}}{a} = \dot{\lambda} \simeq 0$$

$\Rightarrow$ This solves the <u>horizon problem</u> if we assume Hagedorn phase as the initial condition for big-bang cosmology.

Then, solving the EOMs near such region, we find (Tseytlin-Vafa)

$$\varphi(t) = \varphi_0 + \log\left(\frac{1}{E}\frac{1}{t(t-t_0)}\right)$$

$$\lambda(t) = \mu_0 + \mu_1 \log\left[\frac{t}{t-t_0}\right]$$

$\delta G_{00} \Rightarrow t_0^2(1 - N\mu_1^2) = 0$

$\Rightarrow$ i) $t_0 = 0$ Static (but not $\varphi(t)$)

ii) $\mu_1 = \sqrt{N}$ or $\mu_1 = -\sqrt{N}$

$$\dot{\lambda} = -\mu_1 t_0 \frac{1}{t(t-t_0)}$$

Moreover, near $t \simeq t_0$

$$H^{-1} \simeq \sqrt{N}\,|t - t_0| \simeq 0$$

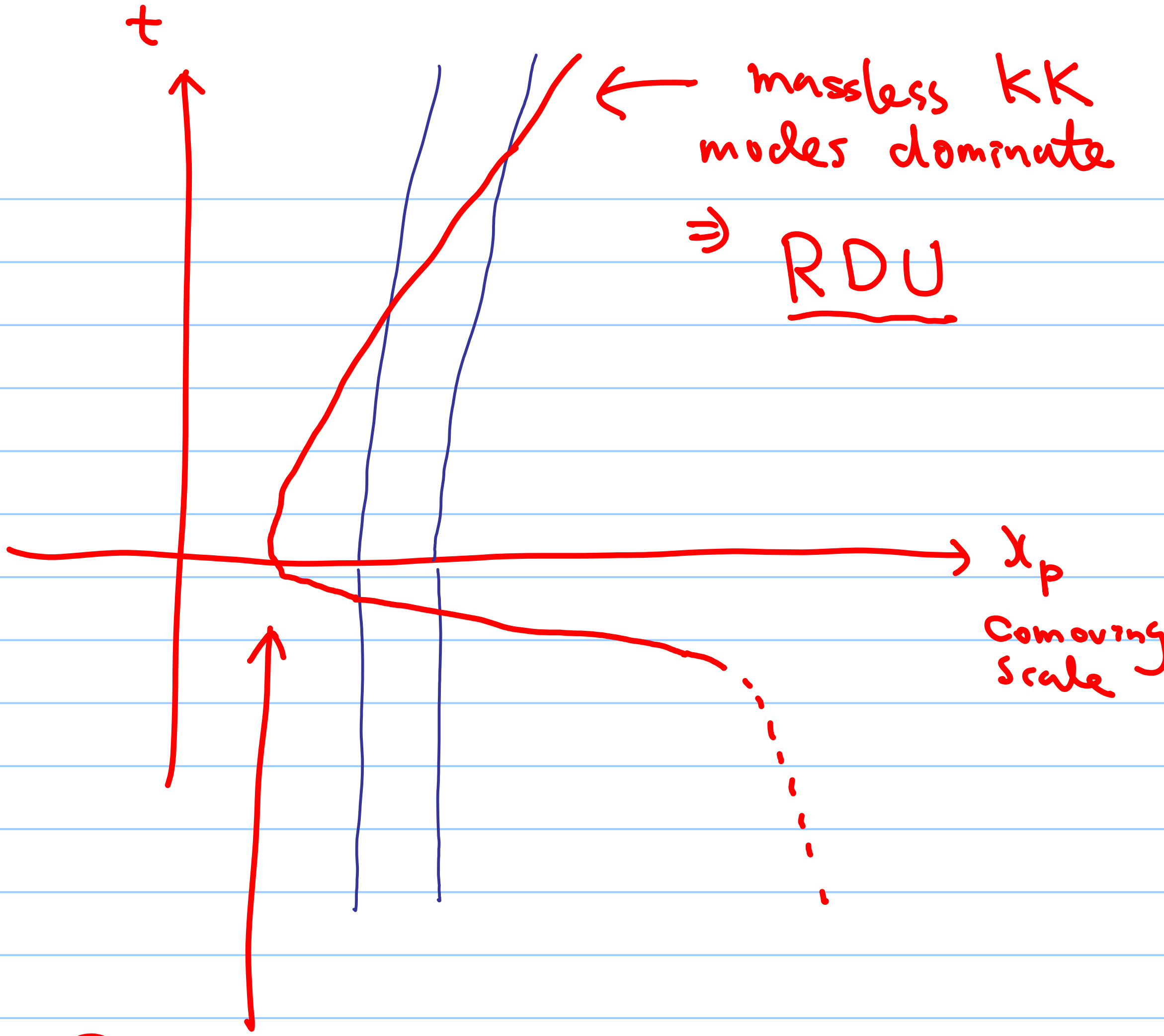
Two regimes: $\mu_1 = \sqrt{N}$

i) $t \gtrsim t_0 > 0 \longrightarrow t = \infty$

The Universe shrinks and come to a stop at $t = \infty$

ii) $t = -\infty \rightarrow t \lesssim t_0 < 0$

The Universe expands away from Hagedorn phase, but the Hubble horizon shrink from ∞ to minimal size, then transist to RDU.



? how to unwind the winding modes ?

⇒ Does Brandenberger - Vafa mechanism work efficiently well in this period ?

BV → Yes / NO ← Linde et al, Greene et al

From the above picture, we see that it is possible to generate the large scale structure w/o horizon problem through the thermal fluctuations during the transient period from Hagedorn phase to radiation dominated phase.

If so, is the power spectrum scale invariant ?

(Moreover, we will come back to the issue of quasi-static later.)

Hagedorn thermodynamics :

Ingredients :

1) Single string partition function

— Sum over all string states

$$Z(\beta) = \int_0^\infty d\varepsilon \; \omega(\varepsilon) \, e^{-\beta \varepsilon}$$

Two way to get the density of states $\omega(\varepsilon)$:

a) from spectrum of closed strings

$$\varepsilon^2 = \frac{\ell^2}{R^2} + \frac{w^2 R^2}{\ell_s^4} + \frac{2}{\ell_s^2}\left[-2 + \sum_{I=1}^{N-1} \sum_{m=1}^{\infty} m\left(N_m^I + \tilde{N}_m^I\right)\right]$$

See Green, Schwarz, Witten

b) from Random walk, the leading behavior is

$$\omega(\varepsilon) \sim V \cdot \frac{1}{\varepsilon} \frac{e^{\beta_H \varepsilon}}{(\sqrt{\varepsilon})^N} \sim V \frac{e^{\beta_H \varepsilon}}{\varepsilon^{1+N/2}}$$

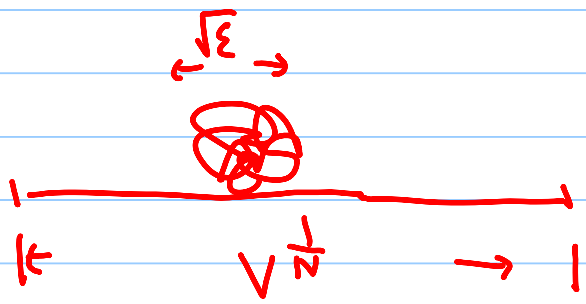
$\frac{1}{\varepsilon}$: starting point

$(\sqrt{\varepsilon})^N$: volume of walk

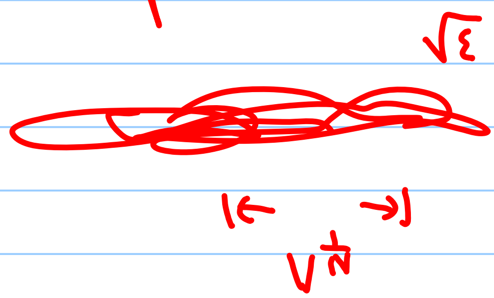
- length of the walk $\sim \varepsilon / \ell_s \sim \beta_H \varepsilon$
- # of possible walks $\sim e^{\beta_H \varepsilon}$

  (Boltzmann law $e^{n \ln 2}$)

- Cartoon



On the other hand, if all dim. are compact



$$\Rightarrow \quad V \sim \varepsilon^{N/2}$$

So $\omega(\varepsilon) \sim \dfrac{e^{\beta_H \varepsilon}}{\varepsilon}$

In either case, the density of states implies long string dominance.

2) Multi-string partition function
(String gas)

We assume long strings obey the classical statistics (Maxwell-Boltzmann)

$\Rightarrow$ The full partition function of the string gas

$$Z(\beta) \simeq e^{z(\beta)}$$

3) However, the canonical ensemble is not always well-defined. If so, one need to switch to the microcanonical ensemble

$$\Omega(E) \equiv \sum_i \delta(E - E_i)$$

$\Omega(E)$ is the density of states and the entropy

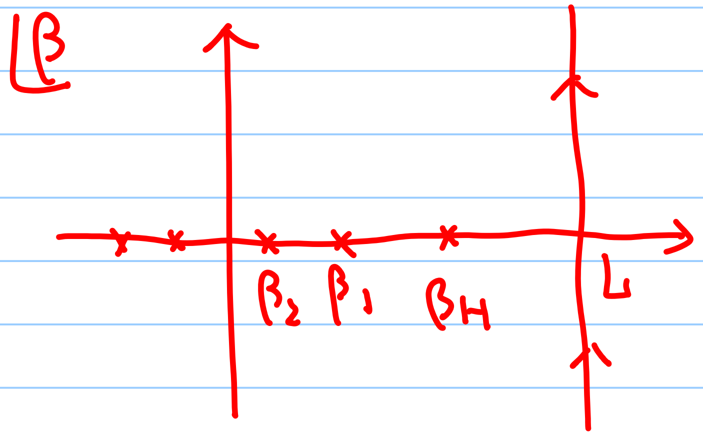
$$S = \ln \Omega(E)$$

Moreover, formully

$$Z(\beta) = \int_0^\infty dE\, \Omega(E)\, e^{-\beta E}$$

and

$$\Omega(E) = \int_{L-i\infty}^{L+i\infty} \frac{d\beta}{2\pi i}\, e^{\beta E}\, Z(\beta)$$



for heterotic string

$$\beta_n = l_s \left[\sqrt{1 - \frac{1}{2}\sum_{i=1}^{N} \left(\frac{n_i}{R_i} l_s\right)^2} + \sqrt{2 - \frac{1}{2}\sum_{i=1}^{N} \left(\frac{n_i}{R_i} l_s\right)^2} \right]$$

$$\beta_0 = \beta_H$$

Two approaches are equivalent if exist the saddle points in evaluating the integral of $Z(\beta)$, that is, the fluctuations in energy is small.

i.e. Canonical E fluctuates

microcanonical E is fixed

4) Two extremal cases

a) Small compact space

$$\omega(\varepsilon) \simeq \frac{e^{\beta_H \varepsilon}}{\varepsilon} \Rightarrow$$

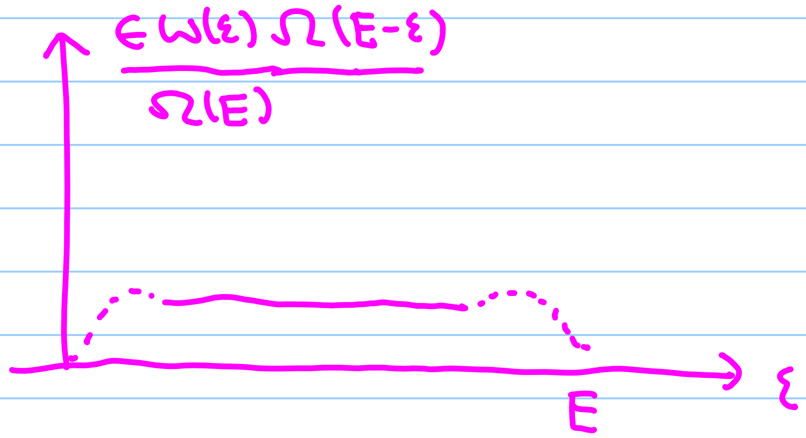
$$\Omega(E) \simeq \beta_H e^{\beta_H E + \lambda_H}$$

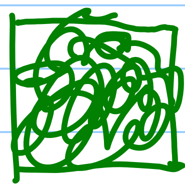
$$\Rightarrow S = \ln \Omega \simeq \beta_H E$$

$$\Rightarrow \quad C_V = \left(\frac{\partial E}{\partial T}\right)_V = -\beta^2 \left(\frac{\partial E}{\partial \beta}\right)_V = \infty$$

$\Rightarrow$ The Hagedorn temperature is limiting.

The energy distribution





String gas with uniform energy distribution for all sizes of strings

$\Rightarrow$ microcanonical = canonical

b) Non-compact space

$$\omega(\varepsilon) \simeq V \frac{e^{\beta_H \varepsilon}}{\varepsilon^{N/2+1}} \Rightarrow$$

$$\Omega(E) \simeq V \frac{e^{\beta_H E + \gamma_0 V}}{E^{N/2+1}}$$

$n_H \sim \mathcal{O}(l_s^{-N})$

$$\Rightarrow S = \ln \Omega(E) = \text{const.} + \beta_H E + n_H V - \left(\frac{N}{2}+1\right) \ln E$$

$$\Rightarrow \beta = \left(\frac{\partial S}{\partial E}\right)_V = \beta_H - \left(\frac{N}{2}+1\right) \frac{1}{E}$$

$$\Rightarrow E = \left(\frac{N}{2}+1\right) \frac{1}{\beta_H - \beta}$$

Approach the Hagedorn temperature from above.

$$\Rightarrow C_V = -\beta^2 \left(\frac{\partial E}{\partial \beta}\right)_V = -\left(\frac{N}{2}+1\right) \left(\frac{T_H}{T - T_H}\right)^2 < 0$$

The specific heat is finite except at $T = T_H$ but negative $\Rightarrow$

microcanonical $\neq$ canonical

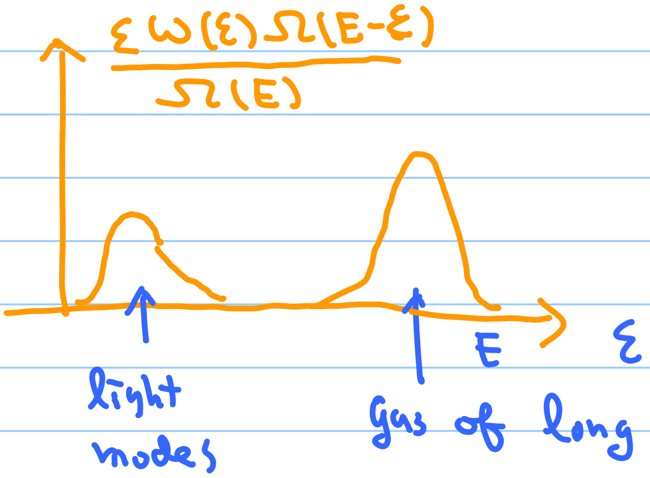
$\downarrow$

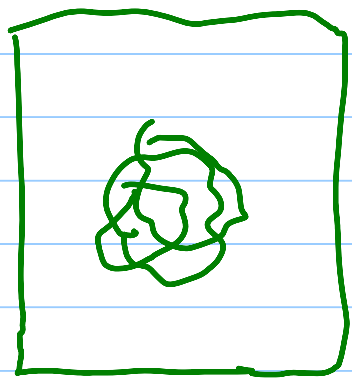
$$C_v = \beta^2 \left(\langle E^2 \rangle - \langle E \rangle^2 \right)$$

always positive

$\Rightarrow$ The Hagedorn temperature is non-limiting.

energy distribution





Some subtlety & maths :

a) Maxwell - Boltzmann

$$\log Z(\beta) = \sum_{r=1}^{\infty} \frac{1}{r} \mathfrak{z}(r\beta) \simeq \mathfrak{z}(\beta)$$

Single string dominance

☆ be more careful for N=0 case.

b) $\mathfrak{z}(\beta) = \int d\varepsilon \, \omega(\varepsilon) \, e^{-\beta\varepsilon}$

$$\text{leading term} = \begin{cases} \log\left[(-1)^{\frac{N+1}{2}} (\beta-\beta_H)^{N/2}\right] & N = \text{odd} \\ \log\left[(-1)^{N/2+1} (\beta-\beta_H)^{N} \log(\beta-\beta_H)\right], & N = \text{even} \end{cases}$$

Brower , Lowe , Tan

5) In the cosmological setting, we need to consider the large compact space by also including the sub-leading correction to resolve the Hagedorn singularity.

From full string spectrum, there are more singularities than the Hagedorn one in $z(\beta)$, i.e.

$$z(\beta) \simeq -\sum_n g_n \log(\beta - \beta_n) + \text{regular part}$$

$$\beta_0 = \beta_H \quad , \quad \beta_1 = \beta_H - \left| \Theta\left(\frac{l_s^3}{R^2} \right) \right| < \beta_H$$

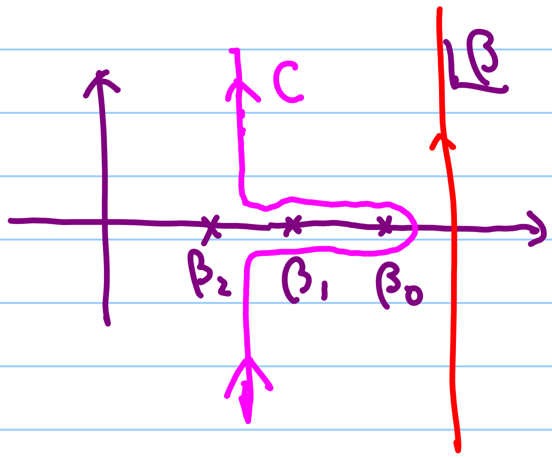
R = size of the compact space $\gg l_s$

$$\omega(\varepsilon) = \int_c \frac{d\beta}{2\pi i} e^{\beta \varepsilon} z(\beta)$$

$$= \sum_n g_n \frac{e^{\beta_n \varepsilon}}{\varepsilon}$$

$\because$ $g_n \log(\beta - \beta_n)$ introduce a discontinuity

$2\pi i\, e^{\beta\varepsilon} g_n\, \Theta(\beta_n - \beta)$



$$\Rightarrow \quad \omega(\varepsilon) = \begin{cases} V \dfrac{e^{\beta_H \varepsilon}}{\varepsilon^{N/2+1}} , & \varepsilon \ll \dfrac{R^2}{\ell_s^3} \\ \dfrac{e^{\beta_H \varepsilon}}{\varepsilon} , & \varepsilon \gg \dfrac{R^2}{\ell_s^3} \end{cases}$$

as expected

Including the correction to $\Omega(E)$ due to β_1 (the finite size effect), we get

$$\Omega(E,R) \simeq \Omega_0 + \Omega_1$$

$$\simeq \beta_H e^{\beta_H E + n_H V} \left[1 + \delta\Omega_1(E,R) \right]$$

$$\delta\Omega_{(1)}(E,R) = -\frac{(\beta_H E)^{2N-1}}{(2N-1)!} e^{-(\beta_H - \beta_1)(E - \rho_H V)}$$

$$\rho_H \sim \mathcal{O}(l_s^{-N-1})$$

$\Rightarrow$

$$S(E,R) \simeq \beta_H E + n_H V + \log(1 + \delta\Omega_1)$$

$$1/T(E,R) \simeq \left(\beta_H + \frac{\frac{\partial \Omega_1}{\partial E}}{1 + \delta\Omega_1}\right)$$

$$\simeq T_H^{-1}\left(1 - \frac{\beta_H - \beta_1}{\beta_H} \delta\Omega_1\right)$$

$\downarrow$ $R \gg l_s$, $\beta_H - \beta_1 \simeq l_s^3/R^2$

$$\simeq \frac{1}{T_H} + \frac{(\beta_H E)^{2N-1}}{(2N-1)!} \frac{l_s^2}{R^2} e^{-l_s^3 E/R^2}$$

Approach the Hagedorn temp. from below!

$\Rightarrow$

$$E \simeq \frac{R^2}{\ell_s^3} \log\left[\frac{\ell_s^3}{R^2} \frac{T}{(1-T/T_H)}\right] + \text{const.}$$

$\Rightarrow$

$$\boxed{C_V \simeq \frac{R^2/\ell_s^3}{T\left(1-\frac{T}{T_H}\right)} > 0}$$

The finite size effect helps to cure the pathology of non-compact space Hagedorn behavior.

Cosmological perturbation in Hagedorn phase

1) The CMB power spectrum is generated by the thermal fluctuation :

for any observable $\mathcal{O}^\alpha$ conjugate to the field Q^α, we have

$$\langle \mathcal{O}^\alpha \rangle = \frac{\partial \log Z}{\partial Q^\alpha}$$

$$\Rightarrow \langle \mathcal{O}^\alpha \mathcal{O}^\beta \rangle = \langle \mathcal{O}^\alpha \rangle \langle \mathcal{O}^\beta \rangle = \frac{\partial^2 \log Z}{\partial Q^\alpha Q^\beta}$$

$\Rightarrow$ mean square fluctuation

$$\langle \delta \mathcal{O}^\alpha \delta \mathcal{O}^\beta \rangle = \langle \mathcal{O}^\alpha \mathcal{O}^\beta \rangle - \langle \mathcal{O}^\alpha \mathcal{O}^\beta \rangle$$

$$= \frac{\partial^2 \log Z}{\partial Q^\alpha \partial Q^\beta}$$

Generalize this to stress tensor, we have

$$\langle \delta T^\mu{}_\nu \, \delta T^\sigma{}_\lambda \rangle = \langle T^\mu{}_\nu T^\sigma{}_\lambda \rangle - \langle T^\mu{}_\nu \rangle \langle T^\sigma{}_\lambda \rangle$$

$$= 2 \frac{G^{\mu\alpha}}{\sqrt{-G}} \frac{\partial}{\partial G^{\alpha\nu}} \left(\frac{G^{\sigma\delta}}{\sqrt{-G}} \frac{\partial \log Z}{\partial G^{\delta\lambda}} \right)$$

$$+ 2 \frac{G^{\sigma\alpha}}{\sqrt{-G}} \frac{\partial}{\partial G^{\alpha\lambda}} \left(\frac{G^{\mu\delta}}{\sqrt{-G}} \frac{\partial \log Z}{\partial G^{\delta\nu}} \right)$$

Especially

$$\langle \delta\rho^2 \rangle = \langle \delta T^0{}_0 \, \delta T^0{}_0 \rangle$$

$$= \langle \rho^2 \rangle - \langle \rho \rangle^2$$

$$= -\frac{1}{V^2} \frac{\partial}{\partial \beta} \left(F + \beta \frac{\partial F}{\partial \beta} \right)$$

$$= -\frac{1}{V^2} \frac{\partial E}{\partial \beta} = \frac{T^2}{V^2} C_V$$

need to use

$$\frac{\partial F}{\partial G_{00}} = \frac{-1}{2}\frac{\beta}{\sqrt{-G_{00}}}\frac{\partial F}{\partial(\beta\sqrt{-G_{00}})}$$

$$\frac{\partial F}{\partial G^{00}} = -\frac{1}{(G^{00})^2}\frac{\partial F}{\partial G_{00}}$$

Using the previous result for C_V we get

$$\langle \delta\rho^2 \rangle \simeq \frac{1}{V^2}\frac{R^2}{l_s^3}\frac{T}{1-T/T_H}$$

c.f.

open strings

$$\langle \delta\rho^2 \rangle \sim \frac{1}{V}\frac{1}{l_s^{N-1}}\frac{T^3}{(1-T/T_H)^3}$$

particles

$$\langle \delta\rho^2 \rangle \sim T^N/V$$

Cosmological perturbation theory in

Einstein frame : scalar pert. in longitudinal gauge

$$ds^2 = -(1+2\Phi)\,dt^2 + a^2(t)\,(1-2\Phi)\,d\vec{x}^2$$

perfect fluid is assumed.

$\Rightarrow$ The $\delta G^0{}_0$ yields the Poisson equation

$$\nabla^2\Phi = G_{N+1}\,\delta\rho$$

or

$$|\Phi_k|^2 = G_{N+1}^2\,k^{-4}\,\langle\delta\rho_k^2\rangle$$

$\Rightarrow$ The Power spectrum

$$P_\Phi(k) = k^N\,|\Phi_k|^2$$

$$= k^{N-4}\,G_{N+1}^2\,\langle\delta\rho_k^2\rangle$$

Recall $\langle \delta\rho^2 \rangle \sim \frac{1}{V^2} \frac{R^2}{\ell_s^3} \frac{T}{1-T/T_H}$

and

$$\langle \delta\rho_k^2 \rangle \sim V \langle \delta\rho^2 \rangle \Big|_{R=k^{-1}} \underset{V\sim R^N}{\sim} \frac{k^{N-2}}{\ell_s^3} \frac{T}{1-T/T_H}$$

$$\Rightarrow P_\Phi(k) = k^{2(N-3)} \frac{G_{N+1}^2}{\ell_s^3} \frac{T(t_e)}{1-T(t_e)/T_H}$$

$$\simeq k^{2(N-3)} \left(\frac{\ell_p}{\ell_s}\right)^4 \frac{1}{1-T(t_e)/T_H}$$

then

(i) $N=3 \Rightarrow$ scale-inv. spectrum

(ii) $\frac{1}{1-T(t_e)/T_H}$ yield small red tilt.

k 小, $1-T(t_e)/T_H$ 小 $\Rightarrow$ enhance

入 大

Similarly, one can evaluate the spectrum of tensor modes,

$$P_h(k) = G_{N+1}^2 \, k^{N-4} \langle \delta T^i_j(k) \, \delta T^i_j(k) \rangle$$

$$\Big\downarrow \; N=3$$

$$\sim \left(\frac{\ell_p}{\ell_s}\right)^4 \left(1 - T/T_H\right) \log^2\left[\frac{1}{\ell_s^2 k^2}\left(1 - T/T_H\right)\right]$$

nearly scale-inv.

Critics:

(A) Not clear how the transition from Hagedorn to radiation-dominated occur.

It needs efficient Brandenberger-Vafa mechanism to unwind the winding modes such that the Universe can expand.

Some study by Greene et al shows that this may not be the case.

(B) The mis-use of cosmological perturbation in Einstein frame for the background in String frame.

The Hubble horizons are related by

$$H_E^{-1} = e^{\frac{-2\phi}{N-1}} \left(H_s - \frac{2\dot{\phi}}{N-1} \right)^{-1}$$

Recall

$$\varphi_s(t) = \varphi_0 + \log\left(\frac{1}{E} \frac{1}{t(t-t_0)} \right)$$

$$\lambda_s(t) = \mu_0 + \mu_1 \log\left[\frac{t}{t-t_0} \right]$$

$$\phi = \frac{1}{2}\left(\varphi_s + N\lambda_s \right)$$

$$\Rightarrow \quad e^{\phi} = g_0 \frac{\left(\frac{t}{t-t_0} \right)^{N\mu_1/2}}{\sqrt{Et(t-t_0)}}$$

$$\dot{\phi} = - \frac{2t - (1-N\mu_1)t_0}{2t(t-t_0)}$$

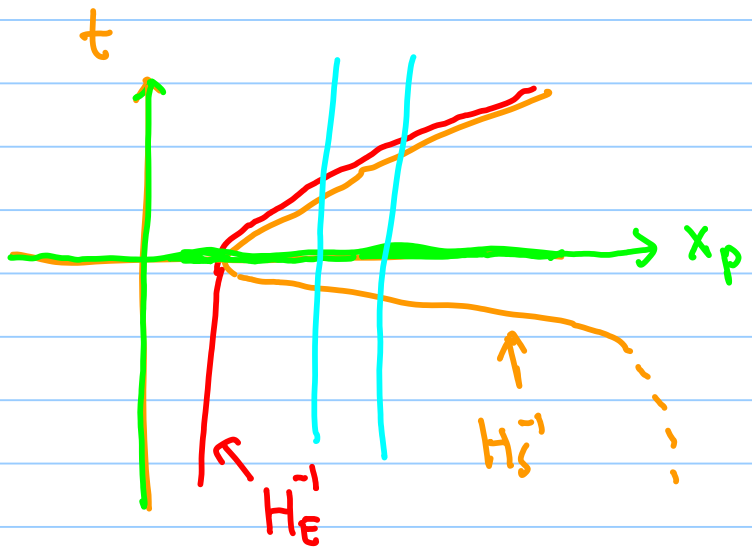
$$H_s - \frac{2\dot{\phi}}{N-1} = \frac{1}{N-1} \frac{2t + (\mu_1 - 1)t_0}{t(t-t_0)}$$

Near $t \sim t_0$

$$H_E^{-1} = \frac{N-1}{\mu_1+1} e^{\frac{-2\phi}{N-1}} (t-t_0)$$

$$\simeq (t - t_0)^{-\gamma}$$

some $\gamma > 0$



$\Rightarrow$ blue spectrum by using the perturbation with wavelength greater than the horizon H_E^{-1}:

$$2\Phi(k) = -\frac{\delta\rho(k)}{\rho}$$

$$\Rightarrow P_\Phi \sim k^N |\Phi(k)|^2 \sim k^N V k^{2N-2} \sim k^{2N-2}$$

(c) Which frame to be used ?

Choosing the frame in which the matter is minimally coupled.

Since we consider the stringy matter, string frame is the natural choice.

Consider the cosmological perturbation theory in string frame, it leads to

(i) if the dilaton velocity is negligible, then the power spectrum of scalar fluctuation is scale invariant.

(ii) if the dilaton velocity is NOT negligible, then the power spectrum is again blue: $P_{\Phi} \sim k^4$

(D) Since the dilaton velocity is not negligible in Hagedorn string gas cosmology, it makes this scenario not viable unless the dilaton is stablized by some means.

This opens the possibility to improve the scenario.