Lectures at NDHU's 3rd P/F Summer School, July 05-10, 2004

Supersymmetry and Beyond

Feng-Li Lin

Physics Division, National Center for Theoretical Sciences

and

Physics Department, National Taiwan Normal Univ.

Outline

1. Supersymmetry and Superspace
2. Soft Susy breaking and MSSM
3. N=2 SYM and Seiberg-Witten Duality

Motivations

- To unite the internal symmetry and Poincare symmetry in a nontrivial way, however, there is a NO-GO theorem of Coleman and Mandula:

  The only symmetries that commute with the S-matrix are the Ponicare symmetry and the ordinary internal symmetries.

  This no-go theorem is by-passed by allowing anit-commuting generators and graded algebra.

  This leads to susy— a fermionic symmetry!

  Rem: With susy, the states with the same quantum number but different spins are grouped in the same multiplet.

  Rem: In non-relativistic case, different spin states can be grouped in the same multiplet of a bosonic symmetry, e.g. $SU(6)$ flavor-spin symmetry for heavy hadrons.

- Solve the hierarchy problem:

  In Standard Model, the 1-loop mass of the Higgs is plagued by quadratic divergence as following:

$$\delta m_H^2 = -\frac{g^2}{16\pi^2}\Lambda^2 \ .$$

  where Λ is a huge fundamental scale, it needs fine tuning to make δm_H^2 to be $\mathcal{O}(Tev)^2$.

  Susy ensures the quadratic divergences of the bosonic loops are exactly cancelled by the fermionic ones. The loop correction yields only harmless log divergence. This is also true for soft-broken susy.

  This solves the hierarchy problem!

  Exercise: Calculate the 1-loop diagrams and check the above statement.

- Extended susy gauge theories can be exactly solvable non-perturbatively.

  The beta function of the gauge coupling g is

$$\beta(g) = -\frac{g^3}{16\pi^2} b_0 \,, \qquad b_0 \equiv \frac{11}{3} C_2(G) - \sum_r \left(\frac{2N_f(r) + N_s(r)}{3}\right) C_2(r)$$

  where $C_2(r)$ is the quadratic Casimir of representation r; for $r = G$ denotes the adjoint repres. $N_f(r)(N_s(r))$ is the number of species of fermions(scalars) in that repres. r.

  Rem: for $N = 1$ $SU(N_c)$ Susy QCD(SQCD) with N_f flavors, $b_0 = 3C_2(G) - \frac{1}{2}N_f$.

  Rem: for $N = 4$ Susy Yang-Mills(SYM), $N_f = 4, N_s = 3$ so that $b_0 = 0$. It is shown that this theory is UV finite.

  Rem: Montonen-Olive conjectured that N=4 SYM has the S-duality, that is the theory of photon with coupling g^2 to light electron is equivalent to the one of "dual" photon with coupling $4\pi/g^2$ to light monopole.

  Rem: Seiberg-Witten(1994) showed that the $N = 2$ SQCD is exactly solvabe.

Historical Remarks

Susy was first constructed by P. Ramond in an attempt to obtain Dirac fermion in string theory in 1971 though it is the 2-d. worldsheet susy. At the same yearGol'fand and Likhtman wrote down the 4-d. susy algebras, also Gol'fand lost his job after publishing their paper.

Three years later, Wess and Zumino successfully constructed the 4-d. N=1 susy scalar and gauge field theories, the scalar one is now known as the Wess-Zumino model. At the same year, Salam and Strathdee introduce the concept of the superspace.

Ref: hep-ph/0101209 by Likhtman.

Spinor Convention

Use the Irreps of $SL(2,C) \simeq SO(1,3)$:

$$\psi_\alpha \quad : \quad (\frac{1}{2},0) = \text{left-handed 2-component Weyl spinor}$$

$$\bar{\psi}^{\dot{\alpha}} = (\psi_\alpha)^* \quad : \quad (0,\frac{1}{2}) = \text{right-handed 2-component Weyl spinor}$$

Also

$$\bar{\psi}_{\dot{\alpha}} \equiv (\psi_\alpha)^\dagger \ , \qquad \psi^\alpha = (\bar{\psi}_{\dot{\alpha}})^* \ .$$

Rising and lowering the spinor indices by Levi-Civita symbols

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \qquad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ .$$

Also we define

$$\sigma^m = (I, \vec{\sigma}) = \bar{\sigma}_m \ , \qquad \bar{s}^m = (I, -\vec{\sigma}) = \sigma_m$$

and

$$\bar{\sigma}^{m\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}} e^{\beta\delta} \sigma^m_{\delta\dot{\gamma}} \ , \qquad \sigma^m_{\alpha\dot{\beta}} = \epsilon_{\dot{\delta}\dot{\beta}} \epsilon_{\gamma\alpha} \bar{\sigma}^{m\dot{\delta}\gamma} \ .$$

<u>Spinor summing convention:</u>

$$\psi\chi = \psi^\alpha \chi_\alpha = -\psi_\alpha \chi_\alpha = \chi^\alpha \psi_\alpha = \chi\psi \ ,$$
$$\bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} \ .$$

Also,

$$(\chi\psi)^\dagger = \bar{\chi}\bar{\psi} \ , \qquad (\chi\sigma^m\psi)^\dagger = \psi\sigma^m\chi \ .$$

<u>Useful relation:</u>

$$\psi^\alpha \psi^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\psi\psi \ , \qquad \psi_\alpha \psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\psi\psi \ ,$$
$$\bar{\psi}^{\dot{\alpha}} \psi^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{a}\dot{\beta}}\bar{\psi}\bar{\psi} \ , \qquad \bar{\psi}_{\dot{\alpha}} \bar{\psi}_{\dot{\beta}} = -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}\bar{\psi} \ .$$

Susy Algebras

The susy transformations are generated by the susy charge operator Q^A_α, $A = 1, ..., N$, together with Poincare algebra they form a graded susy algebra as following:

$$
\begin{aligned}
&\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m \delta^A_B \ , \\
&[Q^A_\alpha, P_m] = [\bar{Q}^A_{\dot{\alpha}}, P_m] = 0 \ , \\
&[Q^A_\alpha, M_{mn}] = \sigma^{\ \beta}_{mn\alpha} Q^A_\beta \ , \quad [\bar{Q}^A_{\dot{\alpha}}, M_{mn}] = \bar{\sigma}^{\ \dot{\beta}}_{mn\dot{\alpha}} \bar{Q}^A_{\dot{\beta}} \\
&\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB} \ , \quad \{\bar{Q}^A_{\dot{\alpha}}, \bar{Q}^B_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{AB})^* \ .
\end{aligned}
$$

where $\bar{Q}^A_{\dot{\alpha}} \equiv (Q^A_\alpha)^\dagger$, and Z^{AB} are called the central charges which commutes with everyone else.

Rem: In the above, we have omitted the part for the $SU(N)$ R-symmetry which rotates N different susy charges. Therefore, R-symmetry does not commute with susy charges but commute with the Hamiltonian.

Irreps. of Susy

Susy Casimir: For $N = 1$, they are $P^2 = P_m P^m$ and $C^2 = C_{mn} C^{mn}$ with

$$C_{mn} = W_m P_n - W_n P_m \ , \quad W_m = \frac{1}{2}\epsilon_{mnpq} P^n M^{pq} - \frac{1}{4}\bar{Q}_{\dot{\alpha}} \bar{\sigma}_m^{\dot{\alpha}\dot{\beta}} Q_\beta$$

where W_m is the generalization of the Pauli-Lubanski vector in Poincare algebra.

Exercise: Show they are indeed Casimir invariants of $N = 1$ susy algebra.

Massive states: In the rest frame $P_n = (m, 0)$, and $C^2 = 2m^4 J_i J^i = 2m^4 j(j+1)$ with $[J_i, J_j] = i\epsilon_{ijk} J_k$. So, an irrep. state is labelled by $|m, j\rangle$. Moreover,

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Longleftrightarrow \{a_{1,2}, a^\dagger_{1,2}\} = 1$$

a la' $Q_{1,2} \equiv \sqrt{2m}\, a_{1,2}$, $\bar{Q}_{\dot{1},\dot{2}} = \sqrt{2m}\, a^\dagger_{1,2}$.

The Clifford vacuum state is defined by $|\Omega\rangle = Q_1 Q_2 |m,j\rangle$ so that $Q_1|\Omega\rangle = Q_2|\Omega\rangle = 0$, and the full massive susy irreps. is:

$$|\Omega\rangle\ ,\ a^{\dagger}_{1,2}|\Omega\rangle\ ,\ a^{\dagger}_{1}{a_2}^{\dagger}|\Omega\rangle\ .$$

There are $4(2j+1)$ states in the massive irreps of spin $j_3, j_3-\frac{1}{2}, j+\frac{1}{2}, j_3$ since $J_i = S_i$(Spin) w.r.t. $|\Omega\rangle$ and $[S_3, a^{\dagger}_1] = -a^{\dagger}_1$, $[S_3, a^{\dagger}_2] = -a^{\dagger}_2$.

Massless states: In the lightcone frame $P_m = (E,0,0,E), C^2 = 0$ and

$$\{Q_1, \bar{Q}_{\dot{1}}\} = 4E\ , \qquad \{Q_2, \bar{Q}_{\dot{2}}\} = 0.$$

Therefore, we can set $Q_2 = \bar{Q}_{\dot{2}} = 0$ so that the full $N=1$ massless susy irreps. is:

$$|\Omega_\lambda\rangle \text{ of helicity } \lambda\ , \quad \bar{Q}_{\dot{1}}|\Omega_\lambda\rangle \text{ of helicity } \lambda + \frac{1}{2}\ ,$$

also, paired with its CPT conjugate(with $\lambda, \lambda - \frac{1}{2}$)to have 4 states in total.

$\underline{N>1\text{, the massless states:}}$ There are 2^N states in the susy irreps. which take the form:

$$a^{\dagger}_{A_1}\cdots a^{\dagger}_{A_n}|\Omega\rangle$$

with C^N_n degeneracy. The helicity in the irreps. are λ,$\lambda+\frac{1}{2}$,$c\ldots$,$\lambda+\frac{N}{2}$. This is not the CPT eigenstate except for $\lambda=-N/4$.

Exercise: Work out the $N=2$, $N=4$ and $N=8$ massless susy irreps.

e.g. $N=2$ susy vector multiplet: $\Omega_0\oplus\Omega_{-1}$ yields $N=2$ SYM.

e.g. $N=2$ susy hypermultiplet: $\Omega_{\frac{-1}{2}}$ yields susy quark multiplet.

e.g. $N=4$ vector multiplet: Ω_{-1} yields $N=4$ SYM with 1 gauge bosons, 3 complex scalars and 4 Dirac fermions.

e.g. $N=8$ gravity multiplet: Ω_{-2} yields N=8 gravity multiplet.

Rem: The maximal number of susy is $N=8$ to avoid the spin higher than 3 massless CPT eigenstate.

Susy in higher spacetime dimensions

The dimension of Dirac spinor in d spacetime dimensions is

$$d_\gamma = \begin{cases} 2^{d/2} & d \text{ even,} \\ 2^{(d-1)/2} & d \text{ odd.} \end{cases}$$

This is the minimum number of susy, let's refer this to $N = (1)_{d_\gamma}$. More generally, $N = (p)_{p*\delta_\gamma}$.

In $4k+2$ dimension, $(\gamma_5)^2 = I$ so that the CPT conjugate spinors have the same chirality and we can have independent chiral and anti-chiral susy generators. Moreover, in $8k+2$, the Weyl spinor can be Majorana(real) so that the minimum number of susy is half of d_γ. Therefore,

- $d = 6$, $N = (p,q)_{(p+q)*d_\gamma}$.
- $d = 2, 10$, $N = (p,q)_{(p+q)*d_\gamma/2}$. E.g. for $d = 10$, $N = (1,1)_{32}$ for IIA superstring, $N = (2,0)_{32}$ for IIB superstring.

11-d. SUGRA:

The dimension of the maximal susy is 32, this implies that the maximal spacetime dimension for maximal susy is $d = 11$, that is, $N = (1)_{d_\gamma=32}$. The only susy massless irreps. is the gravity multiplet which contains the component fields

- e_M^A, the veilbein, 44 of $SO(9)$.
- $\psi_M^a, a = 1, \cdots, 32$, the gravitino, 128 of $SO(9)$.
- A_{MNP}, 3rd rank anti-symmetric tensor, 84 of $SO(9)$.

The lagrangian is very simple

$$-\frac{1}{\kappa^2} eR - \frac{1}{2} e\bar{\psi}_M e_A^M e_B^N e_C^P \gamma^{[A}\gamma^B\gamma^{C]} D_N \psi_P - \frac{1}{48} e\partial_{[M} A_{NPQ]} \partial^{[M} A^{NPQ]}$$

This is the low energy limit of the ultimate M-theory, from which superstring theories can be obtained by Kaluza-Klein reduction, e.g.

The IIA SUGRA(low energy limit of IIA string) can be obtained by KK reduction of 11-d. SUGRA: $g_{11,11}$ is the 10-d. dilaton and $A_{MN,11}$ give 10-d. B_{MN}. The other components of A_{MNP} give the RR-fields.

Type I SUGRA: The pure $N = (1,0)_{16}$ SUGRA is gravitationally anomalous, however, one can couple to $N = (1,0)$ SYM to cancel the anomaly with the particular choice of gauge group:

$$E_8 \times E_8, SO(32), [U(1)]^{248}, [U(1)]^{496}.$$

The first two choices are the field theory limit of the $E_8 \times E_8$ and the $SO(32)$ heterotic strings.

Rem: The lower dim. maximal SYM can be obtained from KK-reduction of the 10-d. $N = (1,0)_{16}$ SYM, and the isometry of the compactified space becomes the R-symmetry of the lower dim. SYM. e.g. $N = (4)_{16}$ in 4-d. with $SU(4) \simeq SO(6)$ R-symmetry.

Rem: The $E_8 \times E_8$ SUGRA-SYM can be obtained by KK-reduction of 11-d. SUGRA on the 1-d. orbifold. Each E_8 SYM lives on the domain wall at the orbifold point to cancel the gravitational anomaly. This is known as the Horava-Witten model—a prototype of the brane world scenario.

With Central Charges

For simplicity, we only consider N =even cases. We first bring the antisymmetric matrix Z^{AB} into block-diagonal form with each block has the form $Z_L \epsilon^{ab}$, with $\epsilon^{ab} = i\sigma^2$ and the eigenvalues $Z_L, L = 1, ..., \frac{N}{2}$ so that

$$\{Q_\alpha^{aL}, Q_\beta^{bM}\} = \epsilon_{\alpha\beta}\epsilon^{ab}\delta^{LM} Z_M$$

Then we define N pairs of annihilation operators

$$a_\alpha^L = Q_\alpha^{1L} + \epsilon_{\alpha\beta}\bar{Q}_{\dot{\gamma}2L}\bar{\sigma}^{0\dot{\gamma}\beta} \ , \quad b_\alpha^L = Q_\alpha^{1L} - \epsilon_{\alpha\beta}\bar{Q}_{\dot{\gamma}2L}\bar{\sigma}^{0\dot{\gamma}\beta}$$

and their corresponding creation operators by taking the hermitian conjugate..

Exercise: they satisfy the following anti-commutation relation:

$$\{a_\alpha^L, (a_\beta^M)^\dagger\} = 2(2m + Z_M)\sigma^0_{\alpha\dot{\beta}}\delta^L_M \ ,$$

$$\{b_\alpha^L, (b_\beta^M)^\dagger\} = 2(2m - Z_M)\sigma^0_{\alpha\dot{\beta}}\delta^L_M \ ,$$

The positive definiteness of the anti-commutation relation yields the BPS condition:

$$Z_M \leq 2m.$$

Especially, the BPS-saturated susy mutiplet($Z = 2m$) have only half the number of the usual massive multiplet, we call it short multiplet in contrast to the long multiplet for $Z < 2m$.

FACT: If disregarding the mass, for $N = 2$ SQCD, one has

$$\Omega_0^{short} = \Omega_{\frac{-1}{2}}(\text{massless hyper})\ , \qquad \Omega_{\frac{1}{2}}^{short} = \Omega_0 \oplus \Omega_{-1}(\text{massless vector})\ .$$

Rem: The BPS states are solitons(e.g. 't Hoof-Polyakov's monopoles or dyons in the N=2 case) whose mases are inversely proportional to the gauge coupling.

In the strong coupling limit, these BPS states become light and dynamical. Combining with the above FACT implies the existence of the electromagnetic duality similar to the one proposed by Montonen-Olive, or more precisely, the Seiberg-Witten duality.

Superspace

The generalized coordinate on superspace is $z^M \equiv (x^m, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$, where θ and $\bar{\theta}$ are Grassmann Lorentz spinor, i.e. they are anti-commuting to each others.

The general group element generated by the susy algebra can be denoted as

$$G(y, \xi, \bar{\xi}) = e^{i[-y^m P_m + \xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}]} \ ,$$

which generates a super-translation on an arbitrary coset element $\Omega(x, \theta, \bar{\theta})$:

$$\begin{aligned} G(y, \xi, \bar{\xi})\Omega(x, \theta, \bar{\theta}) &= e^{i[-y^m P_m + \xi Q + \bar{\xi}\bar{Q}]} e^{i[-x^m P_m + \theta Q + \bar{\theta}\bar{Q}]} \\ &= \Omega(x^m + y^m - i\xi\sigma^m\bar{\theta} + i\theta\sigma^m\bar{\xi}, \theta + \xi, \bar{\theta} + \bar{\xi}) \ . \end{aligned}$$

Therefore we can identify P_m, Q and $\bar{Q}$ as superspace differential operators:

$$P_m = i\partial_m \ , \quad Q_\alpha = \partial_\alpha - i\sigma^m_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_m \ , \quad \bar{Q}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta\sigma^m_{\beta\dot{\alpha}}\partial_m \ .$$

Superspace Covariant Derivatives

The supervielbein E_M^A is given by the the Maurer-Cartan form

$$\Omega^{-1} d\Omega = i dz^M (E_M^m P_m + E_M^\alpha Q_\alpha + E_M^{\dot\alpha} \bar{Q}_{\dot\alpha}) \ .$$

The supervielbein is non-trivial with nonzero torsion. Moreover, $det E_M^A = 1$.

The superspace covariant derivatives are given by

$$D_\alpha = \partial_\alpha + i\sigma^m_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_m \ , \quad \bar{D}_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i\theta^\beta\sigma^m_{\beta\dot\alpha}\partial_m \ .$$

By construction, the susy covariant derivatives commutes with the susy generators, and

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i\sigma^m_{\alpha\dot\alpha}\partial_m \ , \quad \{D_\alpha, D_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0$$

signal the nonzero torsion.

Rem: $\bar{D}_{\dot\alpha}\theta_\alpha = \bar{D}_{\dot\alpha} y^m = 0$ where $y^m = x^m + i\theta\sigma^m\bar\theta$.

Integration over Grassmann number

For a single Grassmann nmber θ,

$$\int d\theta\theta = 1 \ , \qquad \int d\theta = 0 \ , \qquad \delta(\theta) \equiv \theta$$

For superspace coordinates,

$$d^2\theta = -\frac{1}{4} d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \ , \qquad d^2\bar{\theta} = -\frac{1}{4} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} \ , \qquad d^4\theta \equiv d^2\theta d^2\bar{\theta} \ .$$

Therefore,

$$\int d^2\theta \ \theta\theta = 1 \ , \int d^2\bar{\theta} \ \bar{\theta}\bar{\theta} = 1 \ ,$$

otherwise are zero.

Superfields

A superfield living on the superspace can be expanding in terms of θ and $\bar\theta$ as following:

$\mathcal{F}(x,\theta,\bar\theta) = f(x)+\theta\psi(x)+\bar\theta\bar\chi(x)+\theta\theta m(x)+\bar\theta\bar\theta n(x)+\theta\sigma^m\bar\theta v_m(x)+\theta\theta\bar\theta\bar\lambda+\bar\theta\bar\theta\theta\psi(x)+\theta\theta\bar\theta\bar\theta d(x)$.

From $\delta_\xi\mathcal{F}(x,\theta,\bar\theta) = (\xi Q + \bar\xi\bar Q)\mathcal{F}(x,\theta,\bar\theta)$ one can read the susy transformations on the component fields.

Rem: $\delta_\xi d = \frac{i}{2}\partial_m[\psi\sigma^m\bar\xi + \xi\sigma^m\bar\lambda]$, a total derivative. This is in general not the case for other component fields.

Chiral superfields: it is subjected to the constraint

$$\bar D_{\dot\alpha}\Phi = 0 \; .$$

Solving this, we get

$$\begin{aligned}\Phi(y,\theta) &= A(y) + \sqrt{2}\theta\chi(y) + \theta\theta F(y) \; , \\ &= A(x) + i\theta\sigma^m\bar\theta\partial_m A(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\Box A(x) + \sqrt{2}\theta\chi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_m\chi(x)\sigma^m\bar\theta + \theta\theta F(x) \; .\end{aligned}$$

The susy transformation is

$$
\begin{aligned}
&\delta_\xi A = \sqrt{2}\xi\chi \ , \\
&\delta_\xi \chi = i\sqrt{2}\sigma^m\bar{\xi}\partial_m A + \sqrt{2}\xi F \ , \\
&\delta_\xi F = i\sqrt{2}\bar{\xi}\bar{\sigma}^m\partial_m \chi \ , \ \text{a total derivative .}
\end{aligned}
\tag{1}
$$

Rem: $\Phi^\dagger$ is called an anti-chiral field and satisfies $D_\alpha \Phi^\dagger = 0$.

Rem: Products and sums of the chiral superfields are also chiral superfields.

Vector superfields: it is constrained by

$$V^\dagger = V \ .$$

In components: $f = f^*, \ \bar{\chi} = \phi^*, \ m = n^*, \ v_m = v_m^*, \ \bar{\lambda} = \psi^*, \ d = d^*$.

e.g. if Φ is a chiral field, then $\Phi+\Phi^{\dagger}$ is a vector superfield.

In components:

$$\begin{aligned}\Phi+\Phi^{\dagger} &= (A+A^*)+\sqrt{2}(\theta\chi+\bar{\theta}\bar{\chi})+\theta\theta F+\bar{\theta}\bar{\theta}F^*+i\theta\sigma^m\bar{\theta}\partial_m(A-A^*)\\ &+ \frac{i}{\sqrt{2}}\theta\theta\bar{\theta}\bar{\sigma}^m\partial_m\chi+\frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^m\partial_m\bar{\chi}+\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box(A+A^*)\ .\end{aligned}$$

Since the coefficient $i\partial_m(A-A*)$ of the $\theta\sigma^m\bar{\theta}$-component is a gradient, this motivates us to define a susy gauge transformation:

$$V\longrightarrow V+\Phi+\Phi^{\dagger}\ .$$

This is the generalization of the gauge transformation

$$v_m\longrightarrow v_m+\partial_m\Lambda\ ,\quad \Lambda=i(A-A^*)\ .$$

Note that the first 5 component fields of $\Phi+\Phi^{\dagger}$ are unconstrained, then we can choose the so called Wess-Zumino gauge such that

$$V(x,\theta,\bar{\theta})=-\theta\sigma^m\bar{\theta}v_m+i\theta\theta\bar{\theta}\bar{\lambda}-i\bar{\theta}\bar{\theta}\theta\lambda+\frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D\ .$$

Note that fixing WZ gauge does not fix the abelian gauge inv.

The susy field strength is given by W_α and its hermitian conjugate:

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \ , \quad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V \ .$$

They are (anti-)chiral superfields since $D^3 = \bar{D}^3 = 0$.

Exercise: Show (i) $\bar{D}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}} = D^\alpha W_\alpha$. (ii) W_α is a susy-gauge invariant. (iii) in WZ gauge,

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}(\sigma^m\bar{\sigma}^n\theta)_\alpha F_{mn}(y) + \theta\theta\sigma^m_{\alpha\dot{\alpha}}\partial_m\bar{\lambda}^{\dot{\alpha}}$$

where $F_{mn} \equiv \partial_m v_n - \partial_n v_m$.

Rem: <u>The nonabelian generalization:</u>

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V}D_\alpha e^V$$

transform covariantly as

$$W_\alpha \longrightarrow e^{-i\Lambda}W_\alpha e^{i\Lambda}$$

under the nonabelian susy gauge transformation

$$e^V \longrightarrow e^{-i\Lambda^\dagger}e^V e^{i\Lambda}$$

where $\Lambda = \Lambda_a T^a$ and Λ_a's are chiral superfields.

Supersmmetric Actions

A susy-invariant action can be written down from the fact that:

The susy transformation of the $\theta\theta$ component of a chiral superfield is total derivatives, so is the the the $\theta\theta\bar{\theta}\bar{\theta}$ component of a vector superfield.

Susy action for a chiral superfield: The action

$$S = \int dx^4 \int d^4\theta \Phi^\dagger \Phi + \int d^4x [\int d^2\theta P(\Phi) + h.c.]$$

The function P is called superpotential.

In components:

$$\Phi^\dagger \Phi \mid_{\theta\theta\bar{\theta}\bar{\theta}} = \frac{1}{4} A^* \Box A + \frac{1}{4} \Box A^* A - \frac{1}{2} \partial_m A^* \partial^m A + F^* F + \frac{i}{2} \partial_m \bar{\chi} \bar{\sigma}^m \chi - \frac{i}{2} \bar{\chi} \bar{\sigma}^m \partial_m \chi \, ,$$

$$P(\Phi) \mid_{\theta\theta} = \frac{\partial P(A)}{\partial A} F - \frac{\partial^2 P(A)}{\partial A^2} \chi\chi \, .$$

Integrating out the auxiliary field F, we get a scalar potential

$$V(A, A^*) = F^* F = |\frac{\partial P(A)}{\partial A}|^2 \geq 0 \ .$$

Rem: For $P(\Phi) = m\Phi^2 + g\Phi^3$, it is called the Wess-Zumino model which is the most general unitary, **renormalizable** 4-d. susy action for a single chiral superfield.

However, from the effective field theory point of view, the most general susy Lagrangian is

$$\mathcal{L} = \int d^4\theta K(\Phi^i, \Phi^{j^\dagger}) + [\int d^2\theta P(\Phi^i) + h.c.]$$

for arbitrary K and P.

The function K is called the Kahler potential because the above susy action gives a nonlinear sigma model with the kinetic term

$$g_{ij^*}\partial_m A^i \partial^m A^{*j} \ \text{ where } g_{ij^*} \equiv \frac{\delta^2 K(A^i, A^{*j})}{\delta A^i \delta A^{*j}}$$

which is invariant under the Kahler transformation

$$K(A^i, A^{*j}) \longrightarrow K(A^i, A^{*J}) + \Lambda(A^i) + \Lambda^\dagger(A^{*j}) \ .$$

$N=1$ Susy Yang-Mills: Recall W_α is a chiral superfield, the SYM action is then

$$\begin{aligned}
&\frac{1}{8\pi} Im[\tau \int d^4x \int d^2\theta \; \mathrm{tr} W^\alpha W_\alpha] \\
&= \frac{1}{g^2} \int d^4x \; \mathrm{tr}[-\frac{1}{4} F_{mn} F^{mn} - i\lambda\sigma^m \nabla_m \bar{\lambda} + \frac{1}{2} D^2] - \frac{\theta_{YM}}{32\pi^2} \int d^4x \; \mathrm{tr} F_{mn} \tilde{F}_{mn}
\end{aligned}$$

where the complex coupling τ:

$$\tau = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2} \; .$$

Exercise: Derive the above.

Rem: Both the gauge field v_m and the gaugino λ are in the adjoint repes.

$N=1$ Susy QCD: For the charged matter, its chiral superfield transforms under the gauge transformation as

$$\Phi \longrightarrow e^{-ie\Lambda} \Phi$$

so that the gauge invariant kinetic Lagrangian is

$$\int d^4\theta \; \Phi^\dagger e^{eV} \Phi \; .$$

Rem: Note that the fermionic component of a chiral superfield is a chiral fermion, one needs two chiral superfields to yield a Dirac fermion.

Exercise: Show that in components the above Lagrangian yields

$$-D_m A_i^* D^m A^i - i\bar{\chi}_i \bar{\sigma}^m D_m \chi^i - i\sqrt{2} e \bar{\lambda}^{(a)} \bar{\chi}_i T_j^{(a)i} A^j + i\sqrt{2} e A_j^* T_i^{(a)j} \chi^i \lambda^{(a)} - \frac{1}{2} e^2 D^{(a)2}$$

where the so called D-term potential is given by

$$\frac{1}{2} e^2 D^{(a)2} = \frac{1}{2} e^2 (A_j^* T_i^{(a)j} A^i)^2 \geq 0 \; .$$

In Summary: In the coupled SYM-WZ model, the scalar potentials given by

$$V(A^i, A^{i*}) = V_F + V_D = |F|^2 + \frac{1}{2} e^2 D^2 \geq 0.$$

Spontaneous Susy Breaking

Recall $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2\sigma^m_{\alpha\dot{\beta}} P_m$, taking the trace, we get the Hamiltonian

$$H = \frac{1}{4}(Q_1\bar{Q}_1 + \bar{Q}_1 Q_1 + Q_2\bar{Q}_2 + \bar{Q}_2 Q_2)$$

so that the susy is spontaneously broken, i.e. $Q|0\rangle \neq 0$ if only if

$$\langle 0|H|0\rangle = V \neq 0 \ .$$

This implies the existence of a massless Goldstone spinor (Goldstino), that is $Q|0\rangle$ $\because$ $[H, Q]|0\rangle = 0$.

This means that susy is spontaneouslt broken if the auxiliary fields either F or D obtain a nonzero vacuum expectation value(vev). This leads to that

$$\text{either} \ \ \langle 0|\delta_\xi \chi|0\rangle = \sqrt{2}\xi\langle 0|F|0\rangle \neq 0 \quad \text{or} \quad \langle 0|\delta_\xi \lambda|0\rangle = i\xi\langle 0|D|0\rangle \neq 0 \ .$$

F-term breaking:

A model by O'raifeartaigh with the following superpotential

$$P(\Phi_i) = \lambda \Phi_1 \Phi_2 + g \Phi_3 (\Phi_2^2 - m^2) \ .$$

Susy is spontaneously broken since there is no solution to

$$\frac{\partial P}{\partial A_1} = \frac{\partial P}{\partial A_2} = \frac{\partial P}{\partial A_3} = 0 \ .$$

Rem: This mechanism is not working for realistic model since it yields

$$S\mathrm{Tr} M^2 \equiv \sum_j (-1)^{2j} (2j+1) m_j^2 = 0$$

implying that one of the scalar masses is lower than the fermion mass while the other is higher. One does not observe a light scalar sparticle.

D-term breaking:

A model by Fayet-Iliopoulos. For $U(1)$ gauge group, the D-term $\kappa \int d^4\theta V$ is uncharged and thus gauge invariant. Adding this term to the SYM-WZ model with superpotential

$$P(\Phi_\pm) = m\Phi_+\Phi_-$$

where $\Phi_\pm$ carry $\pm e\, U(1)$ charge.

The auxiliary fields are solved by

$$D = \kappa + \frac{e}{2}(A_+^* A_+ - A_-^* A_-)\ , \quad F_+ = mA_-^*\ , \quad F_- = mA_+^*\ .$$

The susy is spontaneously broken since there is no solution of $A_\pm$ to $D = F_+ = F_- = 0$.

Rem: D-term breaking yields

$$S\mathrm{Tr}M^2 = -2\xi^a D^a TrY_a$$

where Y_a are the generators of the additional $U(1)$ in the gauge group. $TrY \neq 0$ iff the $U(1)$ has mixed anomaly(one gauge and two gravity vertices.)

This implies that the D-term breaking could work in the realistic model if there is an anomalous $U(1)$. However, this is not the case for Minimal Susy SM(MSSM).

Nonrenormalization of Superpotential

Wilsonian effective Lagranigian: Consider a renormalizable Lagrangian L and impose the momentum cutoff Λ in loop diagram, then for the Wilsonian effective Lagrangian $\mathcal{L}_W$ we have

$$\text{S-matrix for the process below } \Lambda \text{ from } \mathcal{L}(\Phi) = \text{S-matrix from } \mathcal{L}_W(\Phi_\Lambda)$$

$\mathcal{L}_W$ is local and usually contains infinite number of terms allowed by the symmetries of the theory.

Rem: The Wilsonian effective Lagrangian is the same as the the 1-PI effective Lagrangian if there is no interacting massless particles, otherwise, the latter will suffer the IR ambiguity but the Wilsonian one is well-defined.

Statement of nonrenormalization theorem: As long as the cutoff preserves the susy and gauge invariance, to all orders in perturbation theory, the superpotential is not renormalized in form and coefficients, and the gauge coupling is 1-loop exact.

Rem: This implies that susy cannot be broken perturbatively but could be broken dynamically by some strong gauge dynamics.

There are two ways to prove the nonrenormalization theorem:

- Using the Feynman rule for the perturbation theory in Super-graphs. We will not go to details. For further study on Super-graphs and their application, please check Wess-Bagger or Superspace:1001 lessons in susy(hep-th/0108200).

- The power of holomorphy by Seiberg:

  Consider the WZ model with tree-level Superpotential $P_{tree} = m\Phi^2 + \lambda\Phi^3$ which is holomorphic in Φ) and is invariant under $U(1) \times U(1)_R$ perturbatively if we have the following charge assignment: $\Phi(1,1)$, $m(-2,0)$ and $\lambda(-3,-1)$.

  The constants m and λ effectively can be vev of some background fields which are charged under $U(1) \times U(1)_R$.

  By requiring holomorphy and the symmetries of the effective superpotential, we have

$$P_{eff} = m\Phi^2 f(\frac{\lambda\Phi}{m}) = \sum_{n=0}^{\infty} a_n \frac{\lambda^n \Phi^{n+2}}{m^{n-1}}$$

  where the second equality holds if λ is small.

  Exercise: Check that for $n \geq 2$, the term can not be 1PI so that its contribution should not be included in effective action.

Minimal Susy Standard Model

In reality susy is broken since we does not observe any susy particle at low energy. However, before considering the realistic model including the susy breaking effect, we shall construct first the minimal suspersymmetric generalization of Standard Model by

1. Generalize all the particles in the SM to the chiral/vector superfields by including their super-partners. Let's call them:
   $V^a(8,1,0)$, $V^i(1,3,0)$, $V(1,1,0)$, $Q(3,2,1/6)$, $\bar{U}(\bar{3},1,-2/3)$, $\bar{D}(\bar{3},1,1/3)$, $L(1,2,-1/2)$, $\bar{E}(1,1,1)$ of $SU(3)\times SU(2)\times U(1)$.

2. We need additional Higgs superfield for the Higgsino anomalies to cancel among themselves. Let's call them:
   $H_U(1,2,1/2)$ and $H_D(1,2,-1/2)$.

3. The Yukawa couplings and scalar potentials are defined by the following most general gauge-invariant renormalizable superpotential

$$\begin{aligned} P &= \mu H_U H_D + \lambda_U Q\bar{U}H_U + \lambda_D Q\bar{D}H_D + \lambda_E L\bar{E}H_D \\ &+ \{LH_U + QL\bar{D} + \bar{U}\bar{D}\bar{D} + LL\bar{E}\} \end{aligned}$$

The terms in $\{\cdots\}$ will give dangerous dimension-four operators which violate the baryon and lepton number and lead to the fast proton decay.

This is in contrast to the SM where the proton decay is produced by dimension-six operators and thus is suppressed at low energy.

To remove these dangerous terms, we need to impose the R-parity conservation. The R-parity leaves the SM fields but flip the sign of their super-partners. This leads to

– Susy particles are pair produced.

– The light susy particle (LSP) is stable and can be the candidate for dark matters.

4. From the above R-parity preserving superpotential, one can derive the scalar potential. To preserve the $SU(3)$ gauge symmetry, we shall set

$$0 = \langle \tilde{Q} \rangle = \langle \tilde{\bar{u}} \rangle = \langle \tilde{\bar{d}} \rangle \ .$$

Exercise: Compute the minimum of the potential and show that both electromagnetism and electroweak symmetry are not broken, i.e. the minimum is at $0 = \langle \tilde{L} \rangle = \langle \tilde{\bar{e}} \rangle = \langle h_U \rangle = \langle h_D \rangle$.

Therefore this simplest version of MSSM cannot be realistic.

From the effective theory point of view we shall break the susy by adding some soft breaking terms, namely, the irrelevant operators with dimension larger than four.

These soft susy-breaking terms are assumed to be generated dynamically in some hidden sectors which do not coupled to MSSM directly, and then the susy-breaking effect are transmitted indirectly to MSSM by some messenger sectors. These messengers could be

– gravity—Susy breaking at planckian scale.

– gauge fields of strong dynamics outside MSSM—initiated by Seiberg et al (1994—).

Soft Susy-breaking extension

Treating the MSSM as an effective theory, and the soft susy-breaking terms shall preserve the gauge symmetry and breaks susy at a scale M_s.

It is then convenient to characterize the susy-breaking effect by a spurious chiral superfield $U = \theta\theta M_s^2$ which contribute to the effective Lagrangian by the nonrenormalizable terms suppressed by $1/M$, e.g.

$$\frac{1}{M^2}\int d^4\theta \Phi^\dagger \Phi U^\dagger U + \frac{1}{M}\int d^2\theta W^\alpha W_\alpha U + \frac{1}{M}\int d^2\theta [m\Phi^2 + g\Phi^3]U \; .$$

These terms introduce the following soft susy-breaking Lagrangian:

$$\begin{aligned}
\mathcal{L}_{sb} &= \tilde{Q}^* m_Q^2 \tilde{Q} + \tilde{\bar{u}}^* m_{\bar{u}}^2 \tilde{\bar{u}} + \tilde{\bar{d}}^* m_{\bar{d}}^2 \tilde{\bar{d}} + \tilde{L}^* m_L^2 \tilde{L} + \tilde{\bar{e}}^* m_{\bar{e}}^2 \tilde{\bar{e}} + m_{H_U}^2 |h_U|^2 + m_{H_D}^2 |h_D|^2 \\
&+ m_I \lambda^I \lambda^I + c.c. \\
&+ \mu B h_U h_D + c.c. \\
&+ h_U \tilde{Q} A_U \tilde{\bar{u}} + h_D \tilde{Q} A_D \tilde{\bar{d}} + h_D L A_L \tilde{\bar{e}} + c.c. \; .
\end{aligned}$$

Counting the number of soft-braking parameters:

- 45 parameters of hermitian sparticle mass matrices $m^2_{(Q,\bar{u},\bar{d},L,\bar{e})}$.
- 2 parameters of real Higgs mass-squared $m^2_{H_{(U,D)}}$.
- 6 parameters of complex gaugino masses m_I.
- 2 parameters of complex Higgs mixing B.
- 54 parameters of complex tri-linear scalar coupling matrices $A_{U,D,L}$.

After modulo some $U(1)$ symmetries like PQ, R and lepton numbers, we still have more than 100 additional parameters than the ones in SM. This make the theory far less predictive.

To make the model more predictive, we shall make some ansatz at high energy scale:

- All the scalar masses are the same, m_0^2 —"universality" of scalar masses.
- All the gaugino masses are the same, M_0— "GUT relation".
- All the tri-linear couplings are proportional to theirs corresponding Yukawa couplings— "proportionality", i.e.

$$\mathcal{L}_{tri} = A(h_U \tilde{Q} Y_U \tilde{\bar{u}} + h_D \tilde{Q} Y_D \tilde{\bar{d}} + h_D L Y_L \tilde{\bar{e}}) + c.c.$$

where $Y_{U,D,L}$ are the Yukawa coupling matrices.

Some Phenomenology of MSSM

Rem: There are now only five additional parameters (m_0^2, M_0, A, B, μ) under the above ansatz. Various phenomenology at low energy via RG running will constrain the phase space of MSSM because the RG running will spoil the above alignments.

The RG running also provide a natural explanation of the negative Higgs mass: If assuming universality at high scale, then the RG equation of the $m^2_{H_U}$, m^2_T and $m^2_{\tilde{t}}$ shows that $m^2_{H_U}$ could change to the negative value.

Electroweak Symmetry breaking: The above RG argument provide a necessary but not sufficient condition for inducing the EW SSB. Here we study more on the details of the Higgs potential.

The Higgs doublet is

$$h_U = \begin{pmatrix} h_U^+ \\ h_U^0 \end{pmatrix} , \qquad h_D = \begin{pmatrix} h_D^0 \\ h_D^- \end{pmatrix}$$

The Higgs potential is

$$
\begin{aligned}
V_{Higgs} &= \frac{1}{8}(g^2+g'^2)(|h_U^a|^2-|h_D^a|^2)^2+\frac{1}{2}g^2|h_u^a h_D^{a*}|^2 \\
&+ (\mu^2+m_{H_U}^2)|h_U^a|^2+(\mu^2+m_{H_D}^2)|h_D^a|^2+\mu B(\epsilon_{ab}h_U^a h_D^b+c.c.)
\end{aligned} \tag{2}
$$

Taking the Higgs vev to be

$$
\langle h_U \rangle = \begin{pmatrix} 0 \\ v_u \end{pmatrix} , \qquad \langle h_D \rangle = \begin{pmatrix} v_d \\ 0 \end{pmatrix}
$$

and around this vacuum, there are fluctuations corresponding to

- two CP-even neutral scalars: H and h,
- two CP-odd neutral scalars: A and a Goldstone boson absorbed by Higgs mechanism,
- a pair of CP-even charged scalars: $H^{\pm}$.
- a pair of CP-odd massless charged Goldstone bosons absorbed by Higgs mechanism.

Define the parameters:

$$\tan\beta \equiv v_u/v_d \ , \quad m_Z^2 = \frac{1}{2}(g^2 + g'^2)(v_1^2 + v_2^2) \ , \quad m_W^2 = \frac{1}{2}g^2(v_u^2 + v_d^2) \ ,$$

where m_Z and m_W are the masses of Z and W bosons respectively.

Exercise: Diagonalizing the Higgs mass matrix, you get

$$\begin{aligned} m_A^2 &= 2|\mu|^2 + m_{H_U}^2 + m_{H_D}^2, \\ m_H^2 &= \frac{1}{2}[m_A^2 + m_Z^2 + \sqrt{(m_A^2 - m_Z^2)^2 + 4m_A^2 m_Z^2 \sin^2 2\beta}\,] \\ m_h^2 &= \frac{1}{2}[m_A^2 + m_Z^2 - \sqrt{(m_A^2 - m_Z^2)^2 + 4m_A^2 m_Z^2 \sin^2 2\beta}\,] \\ m_{H^\pm}^2 &= m_W^2 + m_A^2 \ . \end{aligned}$$

From the above, we find that the light Higgs is lighter than the pseudoscalar A and the Z-boson. On the other hand, the heavy Higgs is heavier than A and Z-boson, also the charged Higgs scalar is heavier than A and W-boson.

μ problem: In the above we see that μ is about the same order of the Higgs particles which is expected to be less than the $\mathcal{O}(Tev)$. However, the $\mu H_U H_D$ term is supersymmetric, and thus if μ is fundamental, it is expected to be order of some high energy scale, like GUT scale. There is no natural explanation about the discrepancy, which is then known as the μ problem.

FCNC suppression: In SM, the FCNC contribution to the $K - \bar{K}$ mixing is suppressed by the GIM mechanism and the factor $\frac{(m_c - m_u)^2}{m_W^2}$.

In MSSM, there are additional contributions for $K - \bar{K}$ mixing come from the gaugino-squark loop. Either the universality is exact so that there is no flavor mixing of squarks, or the off-diagonal term of the flavor-mixing matrix $V_{di} V_{si}^*$ is very tiny, otherwise, this loop will contribute to a dangerous amount of order of $\frac{(m_{\tilde{c}} - m_{\tilde{u}})^2}{\tilde{m}^2}$ where $\tilde{m}$ is the typical s-particle mass.

Gauge Couplings's Unification: In GUT's model, the gauge couplings are expected to unify at the GUT scale, however, detailed RG running show the opposite. Instead, the RG runnings of the susy GUT do unify the gauge couplings at GUT scale. This is considered as a compelling reason for susy.

Formalism of the gravity-mediated susy breaking: For the chiral matters with Kahler potential $K(\Phi_i, \Phi_i^\dagger)$ and superpotential $W(\Phi_i)$ coupled to $N=1, d=4$ SUGRA, the scalar potential is known to be

$$V = e^{M_p^{-2}K}[g^{i\bar{j}} D_i W (D_j W)^* - 3M_p^{-2}|W|^2]$$

where M_p is the planck scale and

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial A_i \partial A_j^*} , \qquad D_i W = \frac{\partial W}{\partial A_i} + M_p^{-2} \frac{\partial K}{\partial A_i} W .$$

For detailed derivation, please study Wess-Bagger.

Now let's assume K and W take the forms:

$$K = \sum z_i^\dagger z_i + \sum y_a^\dagger y_a , \qquad W = W_z(z_i) + W_y(y_a)$$

where we have introduced 2 sets of fields, the hidden sector fields z_i's and the visible sector fields y_a's.

Then, assume susy is broken in the hidden sector such that

$$\langle F \rangle \simeq m_{3/2} M_p$$

and requires $\langle V \rangle = 0$ (vanishing cosmological constant) which leads to

$$\langle W \rangle \simeq \mathcal{O}(1)\; m_{3/2} M_p^2 \; .$$

where $m_{3/2}$ is the mass of the gravitino after susy breaking by the susy version of the Higgs effect by which the Goldstino is absorbed as the longitudinal component of the massive gravitino.

If $W_y = 0$ and $K = z_i z_i + y_a y_a$, then the effective potential for y is

$$V(y) = e^{M_p^{-2}\langle K \rangle} M_p^{-4} |\langle W \rangle|^2 \sum |y_a|^2 \simeq m_{3/2}^2 \sum |y_a|^2 \; .$$

Obviously, the susy-breaking effect is transmitted from the hidden sector to the visible sector by giving an universal soft mass as large as $m_{3/2}$ to all the s-particles in the hidden sector.

Exercise: Show that a nontrivial W_y will give the B-term and the universal A-terms.

Rem: The SUGRA is not a renmormalizable theory, the need in determining K and W from first principle invokes more fundamental theory such as the string theory. In this respect, the gauge-mediating susy breaking effect is more straightforward, however, we will not go into the details in this lecture.

Some basics about N=2 SYM

The $N=2$ SYM vector multiplet includes one vector superfield and one chiral superfield in the adjoint repres. in the $N=1$ language, and the renormalizable Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{8\pi}\mathrm{Im}\,\mathrm{Tr}[\tau\int d^2\theta W_\alpha W^\alpha + 2\int d^4\theta \Phi^\dagger e^{-2V}\Phi] \\ &= \frac{1}{g^2}\mathrm{Tr}[\frac{-1}{4}F_{mn}F^{mn} + g^2\frac{\theta}{32\pi^2}F_{mn}\tilde{F}^{mn} + (D_m A)^\dagger D^m A - i\lambda\sigma^m D_m\bar{\lambda} \\ &\quad -i\bar{\chi}\bar{\sigma}^m D_m\chi - i\sqrt{2}[\lambda,\chi]A^\dagger - i\sqrt{2}[\bar{\lambda},\bar{\chi}]] - V\end{aligned}$$

where the form of the scalar potential is constrained by the $N=2$ susy to be

$$V = \frac{1}{2g^2}\mathrm{Tr}[A^\dagger, A]^2 \ .$$

Some of the basics of the theory are the following:

- Moduli space: Unlike the $N=1$ SYM, the classical $N=2$ SYM has a continuous family of vacuum states by solving $[A^\dagger, A]=0$ which implies that A takes value in the Cartan subalgebra of the gauge group G. The vacua is called moduli space.

For example, if $G = SU(2)$, we can take $A = \frac{1}{2}a\sigma^3$ and a a complex parameter labelling the vacua, and the gauge inv. quantity parameterizing the vacua is $u = \frac{1}{2}a^2 = \mathrm{Tr}A^2$.

The above vev of A breaks the $SU(2)$ to $U(1)$, this theory then belongs to the Coulomb phase of the residual $U(1)$, and has 't Hooft-Polyakov monopoles/dyons as in the Georgi-Glashow model.

The mass of the dyon of magnetic and electric quantum number (n_m, n_e) satisfies the Bogomolnyi-Prasad-Sommerfeld(BPS) bound:

$$M \geq \sqrt{2}|Z| \;\; \text{with} \;\; Z = a\left(n_e + i\frac{4\pi}{g^2}n_m\right) .$$

The mass formula has a symmetry under $n_e \leftrightarrow n_m$, $\frac{4\pi}{g^2} \leftrightarrow \frac{g^2}{4\pi}$, $a \leftrightarrow \frac{4\pi a}{g^2}$.

Is this strong/weak electromagnetic symmetry(or duality) also true quantum mechanically?

Olive and Montonen conjectured this is true for $N = 4$ SYM since it is an UV finite theory. However, this can not be true for $N = 2$ SYM because the monopoles and electrons are in the supermultiplet of different spins.

Central charge from susy transformations:

Exercise: The $N=2$ SYM Lagrangian is invariant under the susy transformations which include the usual susy transformation for the $N=1$ vector multiplet and the chiral multiplet with the modification:

$$\delta F = i\sqrt{2}\bar{\xi}\bar{\sigma}^m D_m \chi - 2iT^a A\bar{\xi}\bar{\lambda}^2 \ .$$

Since $N=2$ means that the Lagrangian shall be invariant under another fermionic transformation, this can be obtained by the replacement $\lambda \longrightarrow \chi$, $\chi \longrightarrow -\lambda$.

Exercise: Using the susy transformation to construct the supercurrents and then the supercharges and their commutators, you will find that

$$\{Q_{1\alpha}, Q_{2\beta}\} = \epsilon_{\alpha\beta}\, a(n_e + (4\pi/g^2)n_m)$$

where $n_{e/m}$ are dyon's electric/magnetic charges given by

$$Q_e = gn_e = -\frac{1}{ag}\int d^3x \partial_i (F^{a0i}A^a)\ , \qquad Q_m = \frac{4\pi}{g}n_m = -\frac{1}{ag}\int d^3x \partial_i (\tilde{F}^{a0i}A^a)\ .$$

Taking into the Witten's effect for non-zero θ parameter, the central charge changes to

$$Z = a\,(n_e + \tau_{cl} n_m)\ , \qquad \tau_{cl} \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}\ .$$

Wilsonian effective action for $N = 2$ SYM:

One can start from a $N = 2$ superspace formulation by adding additional fermionic coordinates to the $N = 1$ superspace, and construct the most general $N = 2$ SYM Lagrangian, in the $N = 1$ language it is

$$\frac{1}{4\pi}\mathrm{Im}\,[\int d^4\theta \frac{\partial \mathcal{F}}{\partial \Phi_i}\Phi_i^\dagger + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial \Phi_i \partial \Phi_j} W^{\alpha i} W_\alpha^j]\ .$$

$\mathcal{F}$ is called the prepotential, which relates the Kahler potential and the effective gauge coupling.

For $G = SU(2)$ with the large vev u, the Wilsonian effective Lagrangian for the single $N = 2$ vector multiplet of the residual unbroken $U(1)$ takes the form

$$\mathcal{L}_{eff} = \frac{1}{4\pi}\mathrm{Im}[\int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A}\bar{A} + \int d^2\theta \frac{1}{2}\frac{\partial^2 \mathcal{F}(A)}{\partial A^2} W^\alpha W_\alpha]$$

where A is the $N = 1$ chiral multiplet in the $N = 2$ vector multiplet.

1. There is no $N = 2$ invariant superpotential can be added to the above Lagrangian so that the vacuum degeneracy is not lifted by the quantum correction. However, the quantum correction does change the geometry of the moduli space described by

$$ds^2 = \mathrm{Im}\,\tau\; da d\bar{a}\;, \qquad \tau = \frac{\partial^2 \mathcal{F}}{\partial a^2}$$

2. In the classical theory $\mathcal{F}(A) = \frac{1}{2}\tau_{cl}A^2$. Quantum mechanically we have

$$\mathcal{F} = i\frac{1}{2\pi}A^2 \log\frac{A^2}{\Lambda^2} + \sum_{k=1}^{\infty}\mathcal{F}_k\,(\frac{\Lambda}{A})^{4k}A^2\;.$$

   where the first term is the tree-level plus 1-loop result and the rest are from the non-perturbative instanton effect.

   The theory is completely solved if $\mathcal{F}$ can be determined completely. This was done surprisingly by Seiberg and Witten by exploiting the isometry and monodromy of the quantum moduli space.

3. The above form for the metric on the moduli space is not covariant, and is not globally well-defined since τ is a multivalued function of a. We shall use a more symmetric parametrization.

To extract the isometry of the metric, we define

$$a_D = \frac{\partial \mathcal{F}}{\partial a}$$

so that

$$ds^2 = \text{Im}\ da_D d\bar{a} = -\frac{i}{2}(\frac{da_D}{du}\frac{d\bar{a}}{du} - \frac{da}{du}\frac{d\bar{a}_D}{du})$$

where $u = \mathrm{Tr} A^2$ is a local coordinate on the moduli space $\mathcal{M}$. If we arrange (a_D, a) as a column vector v, then the isometry of $\mathcal{M}$ is

$$v \longrightarrow Mv + c$$

where M is a 2×2 matrix of $SL(2, R)$ and c is a constant vector.

4. Seiberg and Witten proposed that the BPS spectrum is modified by the quantum correction so that the central charge becomes

$$Z = a\, n_e + a_D\, n_m$$

on which the subgroup $SL(2, Z)$ of the $SL(2, R)$ isometry is manifest as the electromagnetic duality as expected.

Seiberg-Witten duality

The $SL(2,Z)$ group is generated by the generators

$$T_n = \begin{pmatrix} 1 & n \\ 0 & b \end{pmatrix}, \text{ and } S = \begin{pmatrix} 0 & 1 \\ - & 0 \end{pmatrix}$$

We shall ask what is the physics for these transformation.

- $T_n : a_D \longrightarrow a_D + n\ a$. This gives $\tau = \frac{\partial a_D}{\partial a} \longrightarrow \tau + n$ or $\theta_{eff} \longrightarrow \theta_{eff} + 2n\pi$.
- S generates the S-duality by $\tau \longleftrightarrow \frac{-1}{\tau}$, and $A \longleftrightarrow A_D$.

<u>This can be seen by</u>

a. First introducing a Lagrangian multiplier V_D^α to implement the Bianchi identity $Im\ D_\alpha W = 0$ and adding the following term to $\int d^4x \mathcal{L}_{eff}$: (Exercise)

$$\frac{1}{4\pi}\mathrm{Im}\int d^4x d^4\theta V_D D W = \frac{1}{4\pi}\mathrm{Re}\int d^4x d^4\theta i D V_D W = -\frac{1}{4\pi}\mathrm{Im}\int d^4x d^2\theta W_D W\ .$$

b. Performing the Gaussian integration over W, we get an equivalent Lagrangian

$$\frac{1}{8\pi}\mathrm{Im}\int d^2\theta \frac{-1}{\tau(A)} W_D^2 \ .$$

c. Now also transforming A to A_D by S transformation:

$$\begin{pmatrix} A_D \\ h_D(A_D) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ h(A) \end{pmatrix}$$

in the kinetic term

$$\mathrm{Im}\int d^4\theta \ h(A)\bar{A}$$

which is then transformed to

$$\mathrm{Im}\int d^4\theta \ h_D(A_D)\bar{A}_D.$$

Moreover, using $h'(A) = \tau(a)$, then

$$-\frac{1}{\tau(A)} = -\frac{1}{h'(A)} = h'(A_D) = \tau_D(A_D) \ .$$

Finally, the meaning of the constant vector c in the isometry $v \longrightarrow Mv + c$ can be classified as following:

In the BPS spectrum of low energy $N = 2$ SYM, the dyon carries the electric charges which comes from the massive charged $N = 2$ vector multiplets. Moreover, pairing the two $N = 1$ chiral multiplets M and $\tilde{M}$ we get a hypermultiplet which couples to the gauge field with a $N = 2$ susy-inv. superpotential

$$\sqrt{2}n_e A M \tilde{M} \ .$$

The dyon mass is then under duality transformation of $\sqrt{2}n_e a$ which CANNOT allow a shift by c for its symmetry origin, so we need to set $c = 0$.

However, if one has additional massive charged hypermultiplet, such a shift can arise and play important role.

Rem: Under $v \longrightarrow Mv + c$, we need to transform the vector $w = (n_m, n_e)$ by wM^{-1} in order for the BPS spectrum to be $SL(2, Z)$ invariant for $c = 0$. If $c \neq 0$, there is no way to make the spectrum invariant from any compensating mtransformation of n_e or n_m.

Monodromies of the Moduli Space

1. Singularity at Infinity: For large u, the theory is weakly coupled so that

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \simeq \frac{2ia}{\pi} ln(a/\Lambda) + \frac{ia}{\pi}$$

and under a circuit of u-plane at large u, one has $\ln u \longrightarrow \ln u + 2\pi i$ and $\ln a \longrightarrow \ln a + \pi i$ so that

$$a_D \longrightarrow -a_D + 2a \ , \qquad a \longrightarrow -a \ .$$

Thus, there is a nontrivial monodromy at infinity at u-plane

$$M_\infty = PT_1^{-2} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \ , \quad P \text{ is the element -1 of} SL(2, Z)$$

2. Singularity at strong coupling: One is looking for the singularity at $u = u_0$ such that

$$a_D(u_0) = 0 \ .$$

This implies that the monopole becomes massless and dynamical such that the local symmetry is enhanced.

Exercise: More quantitatively, using the 1-loop beta function, near where $a_D = 0$, the (weak) magnetic coupling is

$$\tau_D \simeq -\frac{i}{\pi} \ln a_D$$

where the coefficient in front of logarithm is because monopole belongs to the spin $1/2$ susy multiplet.

Using $a_D \simeq c_0(u-u_0)$ and $\tau_D = dh_D/da_D$, then we have

$$a(\tau) = -h_D(a) \simeq -a_0 + \frac{i}{\pi} a_D \ln a_D \simeq a_0 + \frac{i}{\pi} c_0(u-u_0) \ln(c-c_0) \ .$$

Then we can read off the monodromy by circling u_0 and the result is

$$M_1 = S T_1^2 S^{-1} = \left(\begin{array}{cc} 1 & 0 \\ -2 & 1 \end{array} \right) \ .$$

Moreover, from the fact that anomalous $U(1)_R$ symmetry breaks to Z_8 symmetry which acts as a Z_2 symmetry on u, there should be another singularity at $u = -u_0$ with monodromy denoted by M_{-1}. Since the monodromies satisfy the relation: $M_1 M_{-1} = M_\infty$, we can determine

$$M_{-1} = (T_1 S) T_1^2 (T_1 S)^{-1} = \left(\begin{array}{cc} -1 & 2 \\ -2 & 3 \end{array} \right) \ .$$

Like M_1 arises from the vanishing mass of a monopole with $w = (1,0) \equiv w_1$ such that $w_1 M_1 = w_1$, one find that M_{-1} arises from the vanishing mass of a dyon with charge $(-1,1)$. In fact one can carry out the monodromy at infinity many times to get the dyons of different charges producing the singularities at $u = \pm u_0$.

Seiberg and Witten use the convention $u_0 = 1$ and use some nontrivial mathematics by knowing these monodromies, they get the solutions:

$$a(u) = \frac{\sqrt{2}}{\pi}\int_{-1}^{1} dx \sqrt{\frac{u-x}{1-x^2}}\,, \qquad a_D(u) = \frac{\sqrt{2}}{\pi}\int_{1}^{u} dx \sqrt{\frac{u-x}{1-x^2}}$$

which are the integral representation of the two hypergeometric functions by solving the Schrodinger equation with potential as the meromorphic function having poles at ± 1 and ∞, namely, the Picard-Fuchs equation.

In this sense, the quantum susy gauge theory is exactly solved.