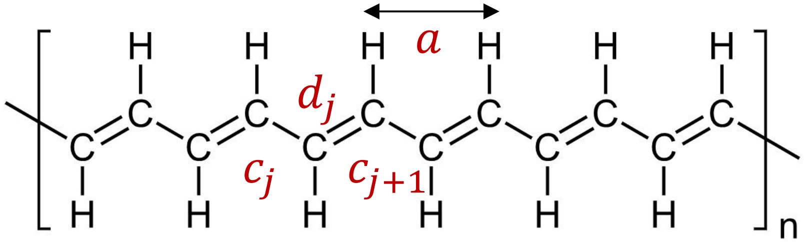
Su-Schrieffer-Heeger model of polyacetylene



$$H = \sum_{j=1}^{N} t_- \left(c_j^\dagger d_j + h.c. \right) + $$
$$+ \sum_{j=1}^{N} t_+ \left(c_{j+1}^\dagger d_j + h.c. \right)$$

$$\begin{cases} c_j = \dfrac{1}{\sqrt{N}} \displaystyle\sum_k e^{ijak} c_k \\ d_j = \dfrac{1}{\sqrt{N}} \displaystyle\sum_k e^{ijak} d_k \end{cases}$$

(Phys Rev Lett 1979)

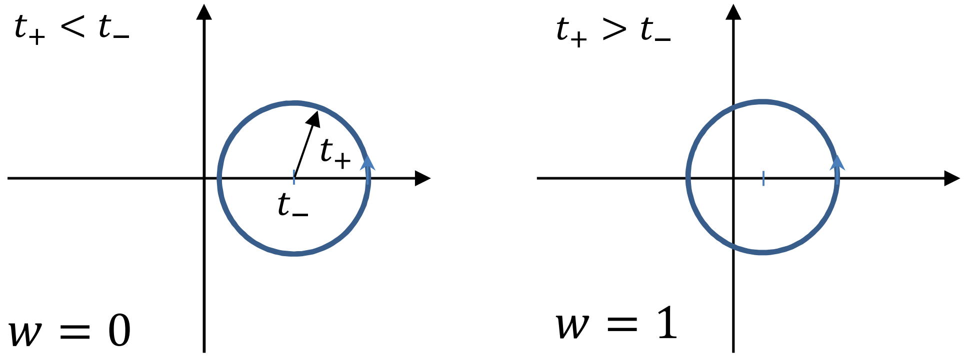
Wu Pei Su Bob Schrieffer Alan Heeger

$$H = \sum_k (c_k^\dagger, d_k^\dagger) \begin{pmatrix} 0 & t_- + t_+ e^{-iak} \\ t_- + t_+ e^{+iak} & 0 \end{pmatrix} \begin{pmatrix} c_k \\ d_k \end{pmatrix}$$
$$= \sum_k (c_k^\dagger, d_k^\dagger) \mathsf{H}(k) \begin{pmatrix} c_k \\ d_k \end{pmatrix}.$$

$$\mathsf{H}(k) = \mathbf{h}(k)\cdot\boldsymbol{\sigma} \qquad \mathbf{h}(k) = (t_- + t_+\cos ka, t_+\sin ka,\ 0)$$

➡ $\varepsilon_\pm = \pm\sqrt{\{(t_- + t_+\cos ka)^2 + (t_+\sin ka)^2\}}$ It's gapped when $t_+ \neq t_-$

- Winding number



Note: $$w = \frac{1}{2\pi}\int_{BZ} dk\ \frac{1}{h^2}\hat{\mathbf{z}}\cdot\mathbf{h}\times\frac{d\mathbf{h}}{dk}$$

- 1D Berry phase, aka Zak phase (Phys Rev Lett 1989)

$$\gamma = \int_{-\pi}^{\pi} dk\ A(k) \qquad A(k) = i\langle u(k)|\frac{d}{dk}|u(k)\rangle$$

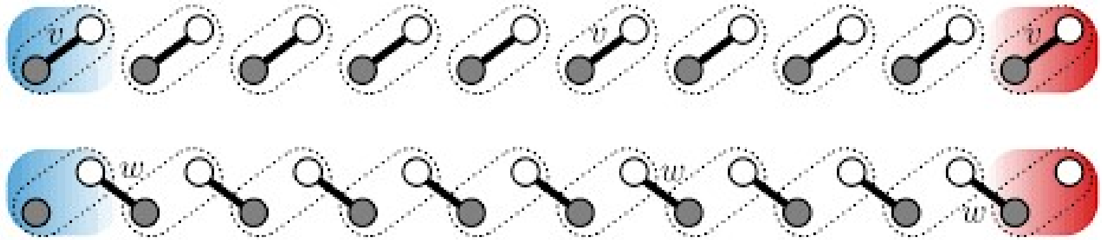
HW 1

$$\begin{cases} \gamma = 0 \ \ \text{if} \ \ t_+ < t_- \\ \gamma = \pi \ \ \text{if} \ \ t_+ > t_- \end{cases}$$

Bulk-edge correspondence

$t_+ < t_-$: $w = 0, \gamma = 0$

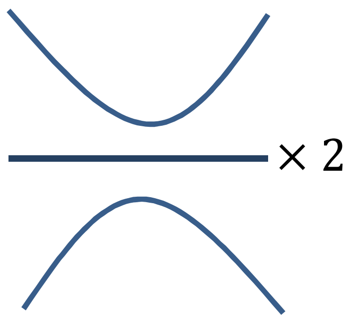
Edge state



Edge state

$t_+ > t_-$: $w = 1, \gamma = \pi$

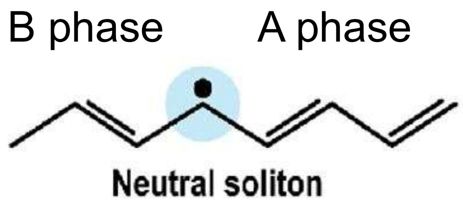
1. Edge states are located at zero energy



2. There is one electron per site

When an electron fills the right end, it has charge q/2, while the left end is -q/2 (w.r.t. to the trivial case)

Localized state at the domain boundary



Low energy continuum theory :

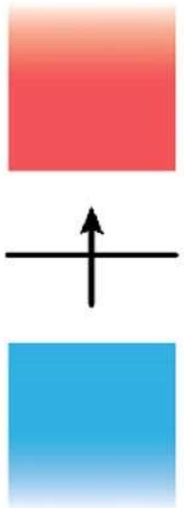
For small δt focus on low energy states with $k\sim\pi/a$

$$k \to \frac{\pi}{a} + q \;\; ; \;\; q \to -i\partial_x$$

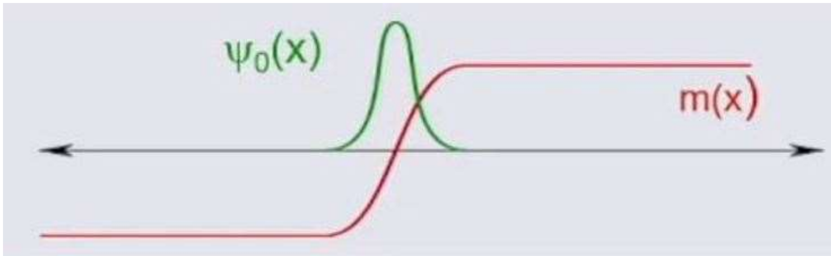
$$H = -v_F \frac{\partial}{i\partial x}\sigma_y - m(x)\sigma_x$$

$$v_F = at_+, m(x) = 2\delta(x)$$

aka Jackiw-Rebbi model (1976)

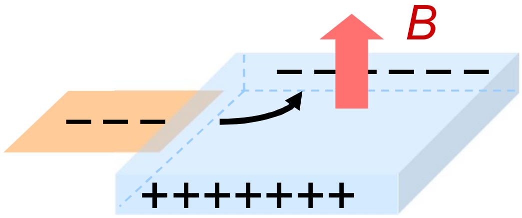


Neutral soliton
Charge : zero
Spin :1/2



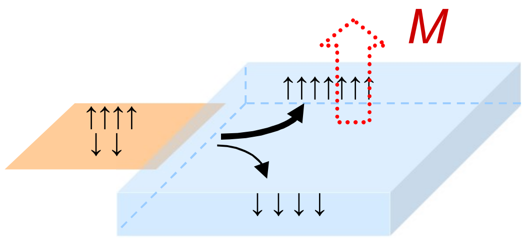
$$\psi_0(x) = e^{-\int_0^x m(x')dx'/v_F} \begin{pmatrix} a \\ b \end{pmatrix}$$

• classical Hall effect (E. Hall 1879)



✓ Lorentz force

• anomalous Hall effect (E. Hall, 1881)



Ingredients:

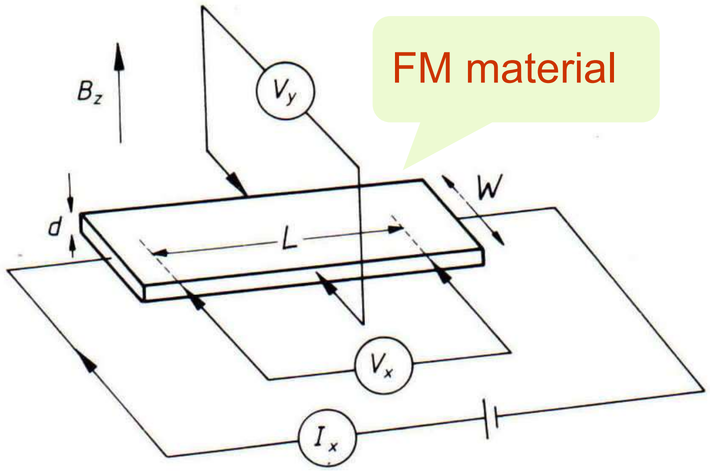
• magnetization (majority spin)

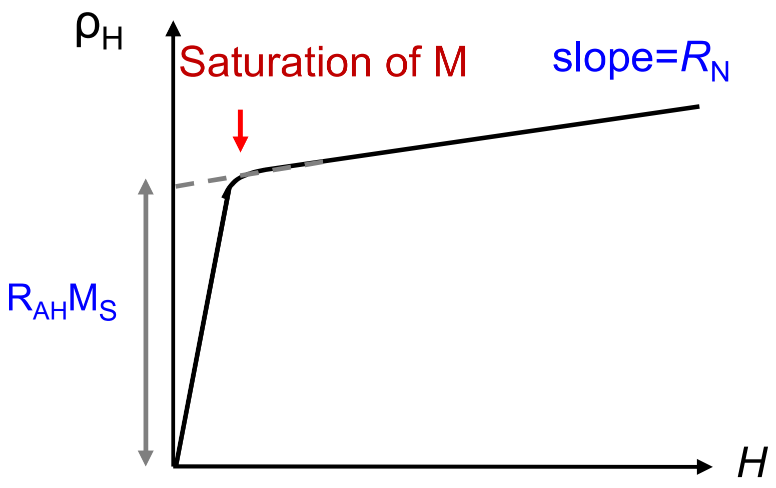
• spin-orbit coupling

(to couple the *majority-spin* direction to transverse motion)

Note: An example that requires no magnetization is provided by Haldane's graphene model

Experimental observation





Recall that ρ_H=B/ne for free electron gas

The usual term

$$\rho_H = R_N H + \rho_{AH}(H),$$

Anomalous term

$$\rho_{AH}(H) \equiv R_{AH} M(H)$$

Theory: Intrinsic mechanism (ideal lattice without impurity)

PHYSICAL REVIEW VOLUME 95, NUMBER 5 SEPTEMBER 1, 1954

Hall Effect in Ferromagnetics*

ROBERT KARPLUS,† *Department of Physics, University of California, Berkeley, California*

AND

J. M. LUTTINGER, *Department of Physics, University of Michigan, Ann Arbor, Michigan*

(Received May 21, 1954)

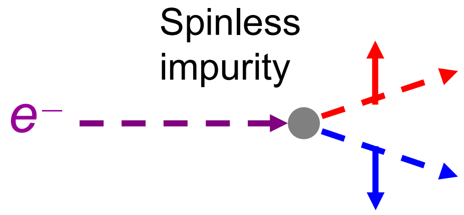
Both the unusually large magnitude and strong temperature dependence of the extraordinary Hall effect in ferromagnetic materials can be understood as effects of the spin-orbit interaction of polarized conduction electrons. It is shown that the interband matrix elements of the applied electric potential energy combine with the spin-orbit perturbation to give a current perpendicular to both the field and the magnetization. Since the net effect of the spin-orbit interaction is proportional to the extent to which the electron spins are aligned, this current is proportional to the magnetization. The magnitude of the Hall constant is equal to the square of the ordinary resistivity multiplied by functions that are not very sensitive to temperature and impurity content. The experimental results behave in such a way also.

- Linear response theory, with
  Spin-orbit coupling
  magnetization

- They find a transverse electron velocity (aka anomalous velocity) that depends only on band structure
- Gives correct order of magnitude of ρ_H for Fe, also explains $\rho_{AH} \propto \rho_L^2$ that's observed in some data

Theory: Extrinsic mechanisms (with impurities)

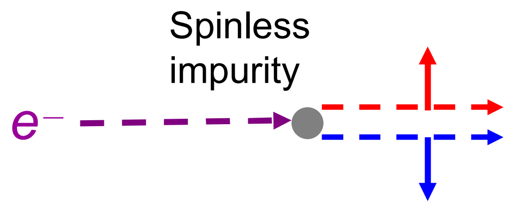
- Skew scattering (Smit,1955) $\sim$ Mott scattering



$$\rho_{AH} \propto \rho_L$$

Smit: K-L mechanism should be annihilated by the skew scatterings from impurities.

- Side jump (Berger, 1970)



$$\rho_{AH} \propto \rho_L^2$$

Anomalous velocity due to electric field of impurity $\sim$ anomalous velocity in K-L

In reality, it's not always clear which one is dominant.

- Hurd, *The Hall Effect in Metals and Alloys* (1972)

  "The difference of opinion between Luttinger and Smit seems never to have been entirely resolved."

30 years later:

- Crepieux and Bruno, PRB 2001

"It is now accepted that two mechanisms are responsible for the AHE: the skew scattering… and the side-jump…"

However, entering the 21th century

Science 2001

Spin Chirality, Berry Phase, and Anomalous Hall Effect in a Frustrated Ferromagnet

Y. Taguchi,[1] Y. Oohara,[2] H. Yoshizawa,[2] N. Nagaosa,[1,3] Y. Tokura[1,3]

An electron hopping on non-coplanar spin sites with spin chirality obtains a complex phase factor (Berry phase) in its quantum mechanical amplitude that acts as an internal magnetic field, and is predicted to manifest itself in the Hall effect when it is r
surement, neutro
that the gigantic
ferromagnet with
the spin chirality
tilting.

Science 2003

The Anomalous Hall Effect and Magnetic Monopoles in Momentum Space

Zhong Fang,[1,2*] Naoto Nagaosa,[1,3,4] Kei S. Takahashi,[5] Atsushi Asamitsu,[1,6] Roland Mathieu,[1] Takeshi Ogasawara,[3] Hiroyuki Yamada,[3] Masashi Kawasaki,[3,7] Yoshinori Tokura,[1,3,4] Kiyoyuki Terakura[8]

VOLUME 88, NUMBER 20 PHYSICAL REVIEW LETTERS 20 MAY 2002

Anomalous Hall Effect in Ferromagnetic Semiconductors

T. Jungwirth,[1,2] Qian Niu,[1] and A. H. MacDonald[1]

[1]*Department of Physics, The University of Texas, Austin, Texas 78712*

[2]*Institute of Physics ASCR, Cukrovarnická 10, 162 53 Praha 6, Czech Republic*

(Received 3 October 2001; published 6 May 2002)

We present a theory of the anomalous Hall effect in ferromagnetic (III, Mn)V semiconductors. Our theory relates the anomalous Hall conductance of a homogeneous ferromagnet to the Berry phase acquired by a quasiparticle wave function upon traversing closed paths on the spin-split Fermi surface. The quantitative agreement between our theory and experimental data in both (In, Mn)As and (Ga, Mn)As systems suggests that this disorder independent contribution to the anomalous Hall conductivity dominates in diluted magnetic semiconductors. The success of this model for (III, Mn)V materials is unprecedented in the longstanding effort to understand origins of the anomalous Hall effect in itinerant ferromagnets.

… and more

Old wine in new bottle

Karplus-Luttinger theory (1954)

= Berry curvature theory (2001)

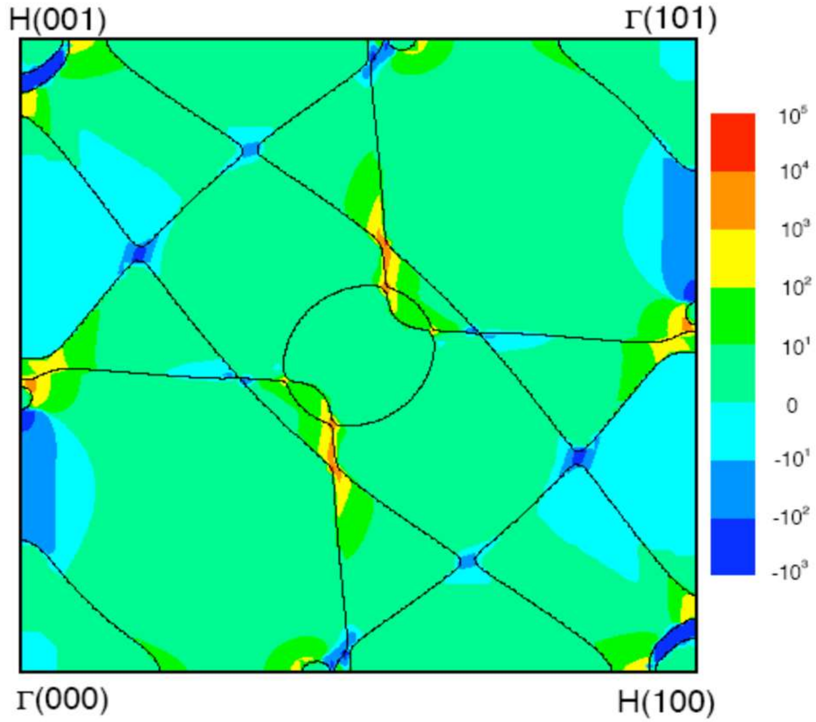
Anomalous velocity is essentially this term

$$\mathbf{v}_n(\mathbf{k}) = \frac{1}{\hbar}\frac{\partial \varepsilon_{n\mathbf{k}}}{\partial \mathbf{k}} + \frac{e}{\hbar}\mathbf{E}\times\mathbf{F}_n(\mathbf{k})$$

$\rightarrow$ AHE

$$\sigma_{xy} = \frac{e^2}{\hbar}\int_{filled}\frac{d^3k}{(2\pi)^3}\ F_z(\vec{k})$$

Berry curvature of fcc Fe
(Yao et al, PRL 2004)



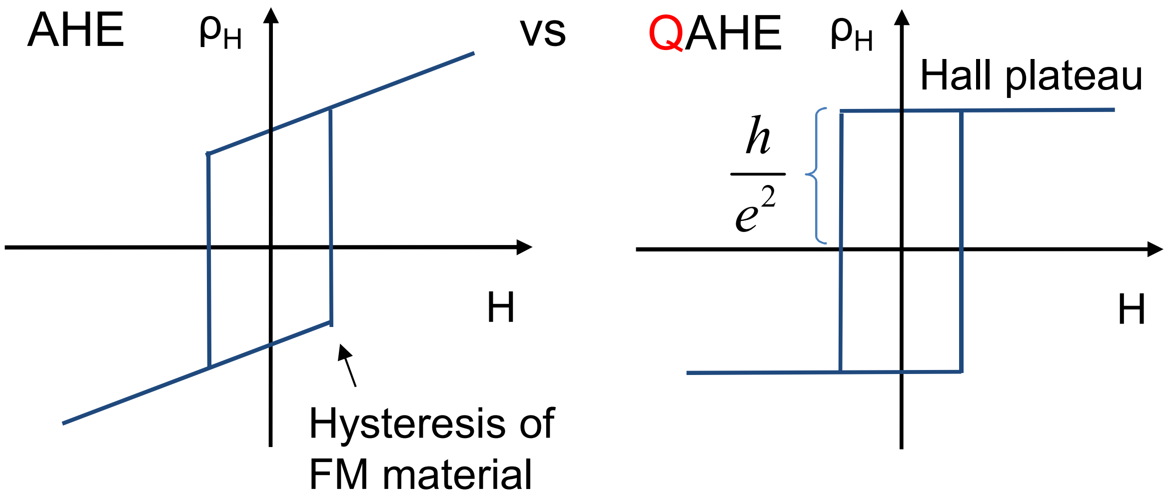
Ideal lattice without impurity

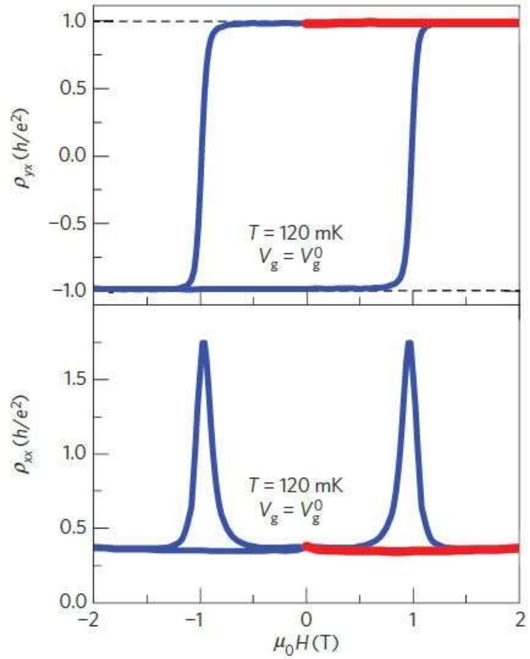
Karplus-Luttinger mechanism: Mired in controversy from the start, it simmered for a long time as an unsolved problem, but has now re-emerged as a topic with modern appeal. – Ong @ Princeton

Experimental Observation of the Quantum Anomalous Hall Effect in a Magnetic Topological Insulator

Cui-Zu Chang,[1,2*] Jinsong Zhang,[1*] Xiao Feng,[1,2*] Jie Shen,[2*] Zuocheng Zhang,[1] Minghua Guo,[1] Kang Li,[2] Yunbo Ou,[2] Pang Wei,[2] Li-Li Wang,[2] Zhong-Qing Ji,[2] Yang Feng,[1] Shuaihua Ji,[1] Xi Chen,[1] Jinfeng Jia,[1] Xi Dai,[2] Zhong Fang,[2] Shou-Cheng Zhang,[3] Ke He,[2†] Yayu Wang,[1†] Li Lu,[2] Xu-Cun Ma,[2] Qi-Kun Xue[1†]

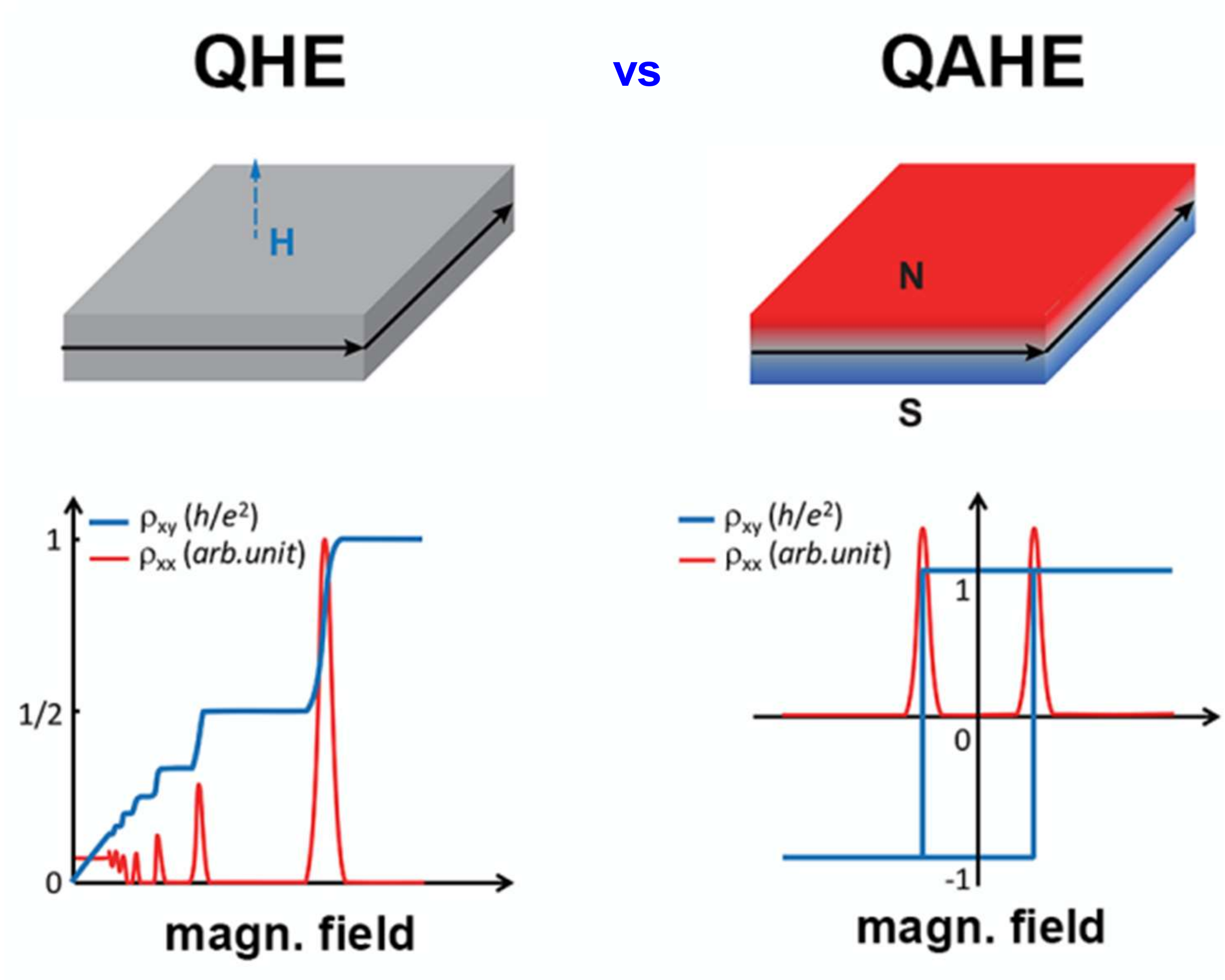
Science, 2013





C.Z. Chang et al, Nat Phys 2015

(accurate to 10^{-4})


QHE
vs
QAHE
H
N
S
ρxy (h/e²)
ρxx (arb.unit)
1
1/2
0
magn. field
ρxy (h/e²)
ρxx (arb.unit)
1
0
-1
magn. field

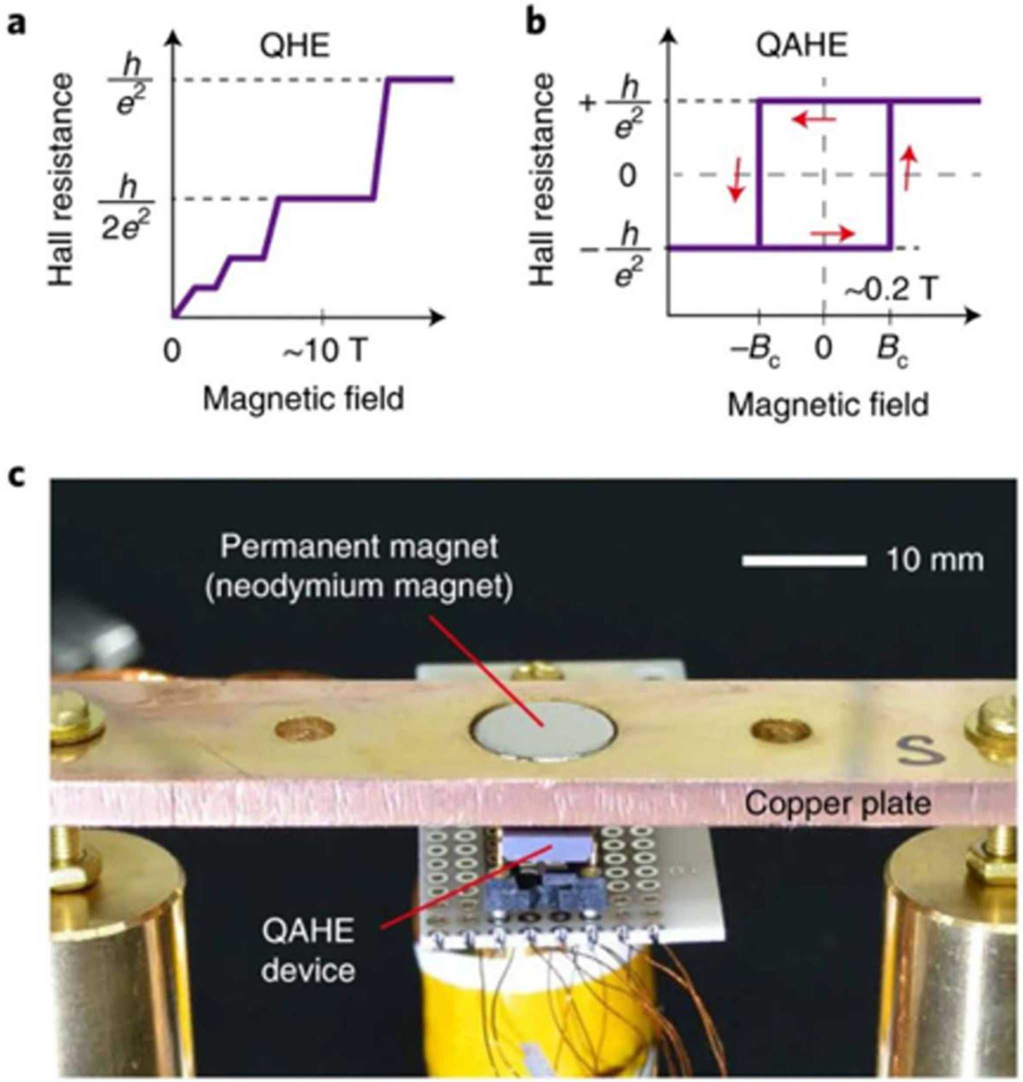
Quantum anomalous Hall effect

- Bi_2Te_3 theory (Yu et al, Science 2010)
- Bi_2Te_3 experiment (Chang et al, Nat Material 2015)
- manganese bismuth telluride ($MnBi_2Te_4$) (Deng et al, Science 2020)
- Twisted bilayer graphene (Serlin et al, Science 2020) **Orbital magnetism**
- $MoTe_2/WSe_2$ heterobilayers (Li et al, Nature 2021)
- $Cr_{1-x}(Bi_{1-y}Sb_y)_{2-x}Te_3$ (Okazaki et al, Nat Phys 2022)
- Twisted Bilayer $MoTe_2$ (Cai et al, Nature 2023)

…

Nature Phys, Okazaki et al 2022

QAHE with a permanent magnet defines a quantum resistance standard



a precision of 10 parts per billion (mK)

I. Quantum anomalous Hall effect

A. Qi-Wu-Zhang model

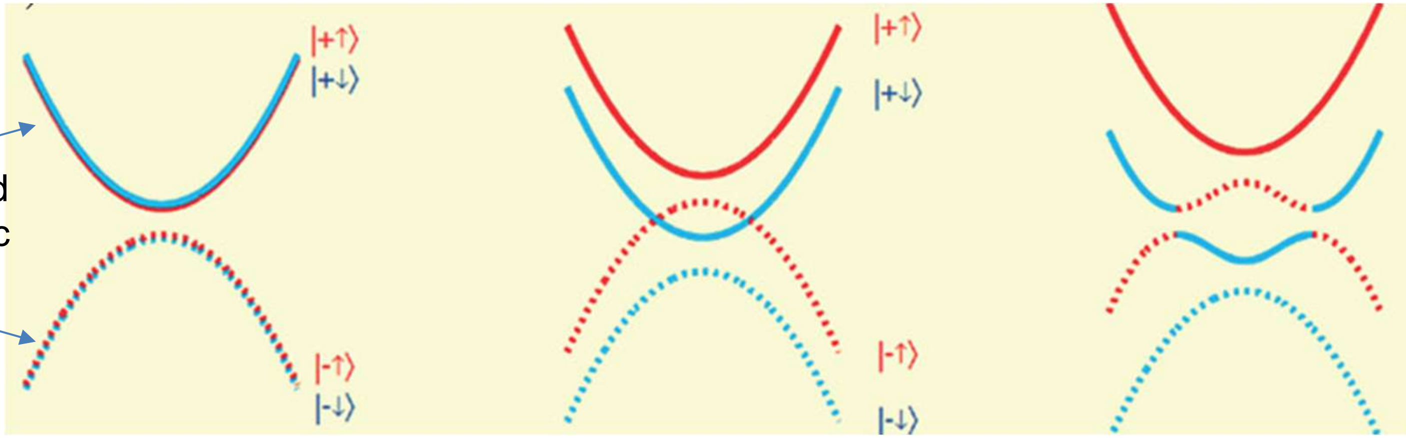
B. Edge state in Qi-Wu-Zhang model

- Engineering a topological band by level crossing

(Qi-Wu-Zhang model, 2006; Yu et al, Science 2010)

Using 2D surface states of magnetic topological insulator

symmetric and
anti-symmetric
combinations



w/o exchange field　　w/ exchange field　　w/ exchange and SOC

A. Qi-Wu-Zhang model - a toy model of QAHE (2006)

$$\mathsf{H}(\mathbf{k}) = \mathsf{H}_0 + \mathsf{H}_m + \mathsf{H}_{so}, \tag{1.1}$$

$$\mathsf{H}_0 = \varepsilon_0(\mathbf{k}) +$$

$$t \begin{pmatrix} 2-\cos k_x a - \cos k_y a & 0 \\ 0 & -(2-\cos k_x a - \cos k_y a) \end{pmatrix},$$

$$\mathsf{H}_m = m \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathsf{H}_{so} = \lambda \begin{pmatrix} 0 & \sin k_x a - i \sin k_y a \\ \sin k_x a + i \sin k_y a & 0 \end{pmatrix}.$$

Can be realized using ultracold fermions, see arXiv:2109.08885,

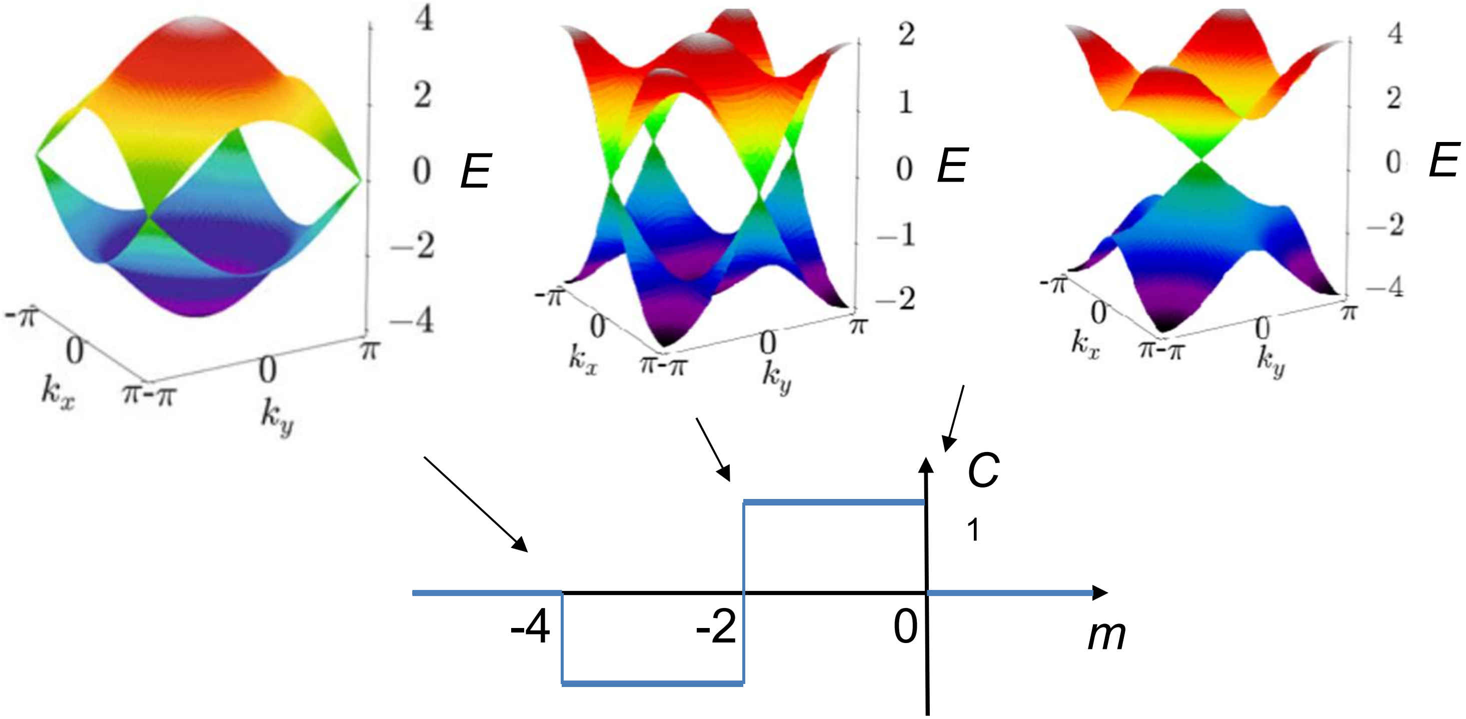
$$\mathsf{H}(\mathbf{k}) = \varepsilon_0(\mathbf{k}) + \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma}, \tag{1.2}$$

where

$$\mathbf{h}(\mathbf{k}) = \left(\lambda \sin k_x a, \lambda \sin k_y a, m + t \sum_{j=1}^{2} (1 - \cos k_j a) \right).$$

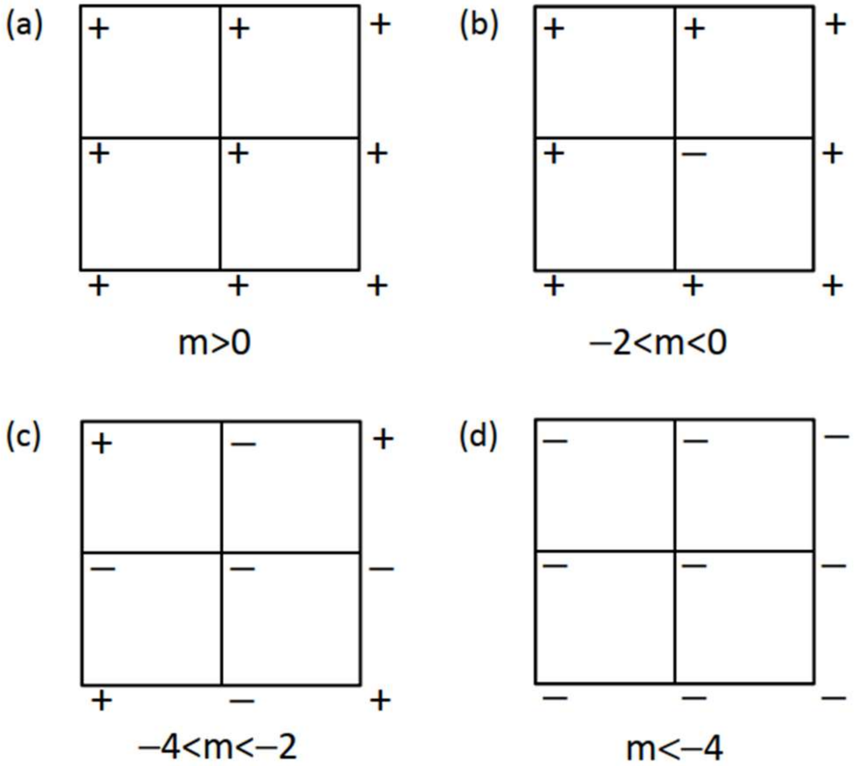
$$\varepsilon_{\pm}(\mathbf{k}) = \varepsilon_0(\mathbf{k}) \pm |\mathbf{h}(\mathbf{k})|$$

$$\begin{aligned}
\mathbf{k}_0 &= 0 \rightarrow \varepsilon_{\pm}(\mathbf{k}_0) = \varepsilon_0 \pm m, \\
\mathbf{k}_0 &= (\pi, 0), (0, \pi) \rightarrow \varepsilon_{\pm}(\mathbf{k}_0) = \varepsilon_0 \pm |m+2|, \\
\mathbf{k}_0 &= (\pi, \pi) \rightarrow \varepsilon_{\pm}(\mathbf{k}_0) = \varepsilon_0 \pm |m+4|.
\end{aligned}$$

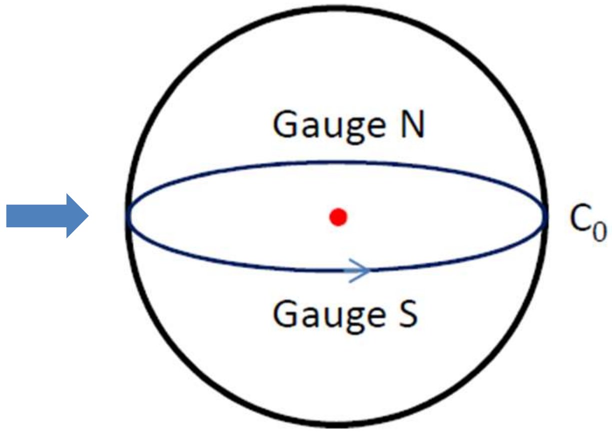


Figs. From Asboth et al, A short course on TI

1) $m > 0 : h_z(\mathbf{k}) > 0$ over the whole BZ.
2) $-2 < m < 0 : h_z(\mathbf{k}) < 0$ near $\mathbf{k} = 0$.
3) $-4 < m < -2 : h_z(\mathbf{k}) > 0$ near $\mathbf{k} = (\pi, \pi)$ (and its equivalent points).
4) $m < -4 : H_z(\mathbf{k}) < 0$ over the whole BZ.



$$\mathrm{H}(\mathbf{k}) = \varepsilon_0(\mathbf{k}) + \mathbf{h}(\mathbf{k}) \cdot \boldsymbol{\sigma},$$

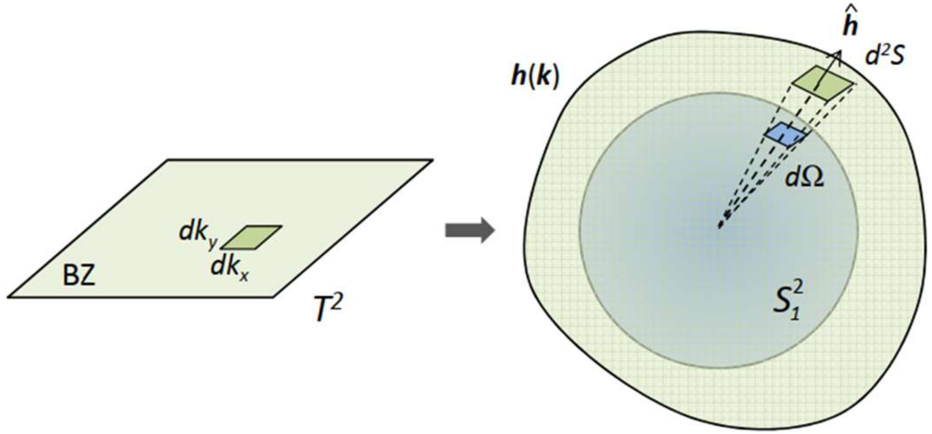


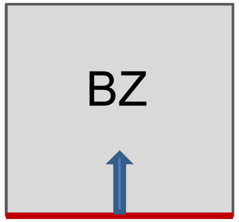
$$\boxed{F_z^{\pm}(\mathbf{k}) = \mp\frac{1}{2h^3}\mathbf{h}\cdot\frac{\partial \mathbf{h}}{\partial k_x}\times\frac{\partial \mathbf{h}}{\partial k_y}.}$$

Pf:
$$\begin{aligned} A_\ell^{\pm}(\mathbf{k}) &= i\langle \mathbf{h},\pm|\frac{\partial}{\partial k_\ell}|\mathbf{h},\pm\rangle \\ &= \frac{\partial h_\alpha}{\partial k_\ell} i\langle \mathbf{h},\pm|\frac{\partial}{\partial h_\alpha}|\mathbf{h},\pm\rangle \\ &= \frac{\partial h_\alpha}{\partial k_\ell} a_\alpha^{\pm}(\mathbf{h}), \end{aligned}$$

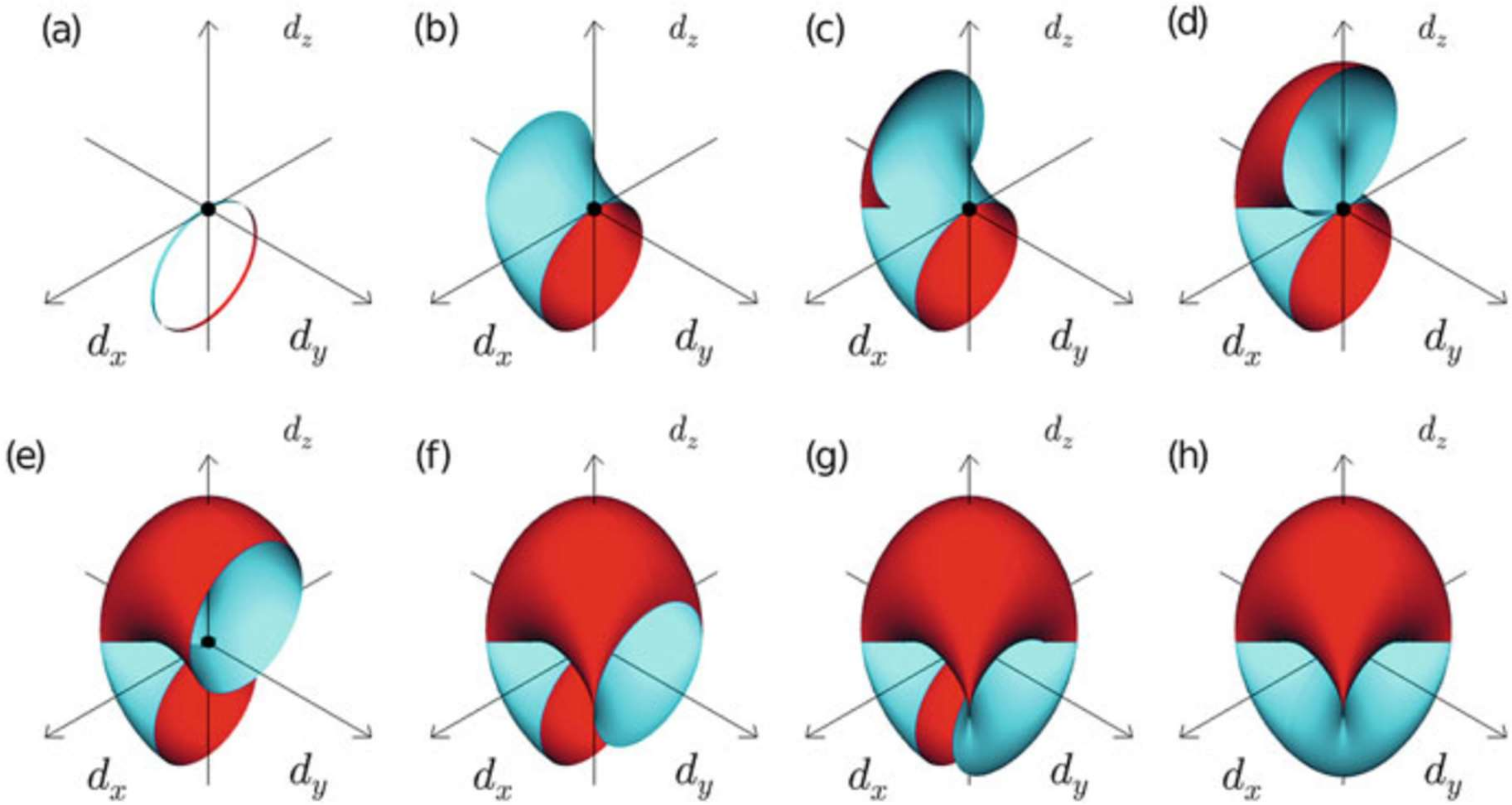
$\Rightarrow$

$$\begin{aligned} F_z^{\pm}(\mathbf{k}) &= \frac{\partial A_y^{\pm}}{\partial k_x} - \frac{\partial A_x^{\pm}}{\partial k_y} \\ &= \frac{\partial}{\partial k_x}\left(\frac{\partial h_\beta}{\partial k_y}a_\beta^{\pm}\right) - \frac{\partial}{\partial k_y}\left(\frac{\partial h_\alpha}{\partial k_x}a_\alpha^{\pm}\right) \\ &= \frac{\partial h_\alpha}{\partial k_x}\frac{\partial h_\beta}{\partial k_y}\left(\frac{\partial a_\beta^{\pm}}{\partial h_\alpha} - \frac{\partial a_\alpha^{\pm}}{\partial h_\beta}\right) \\ &= \frac{\partial h_\alpha}{\partial k_x}\frac{\partial h_\beta}{\partial k_y}\varepsilon_{\alpha\beta\gamma}f_\gamma^{\pm} \\ &= \mp\frac{1}{2h^3}\mathbf{h}\cdot\frac{\partial \mathbf{h}}{\partial k_x}\times\frac{\partial \mathbf{h}}{\partial k_y}, \end{aligned}$$

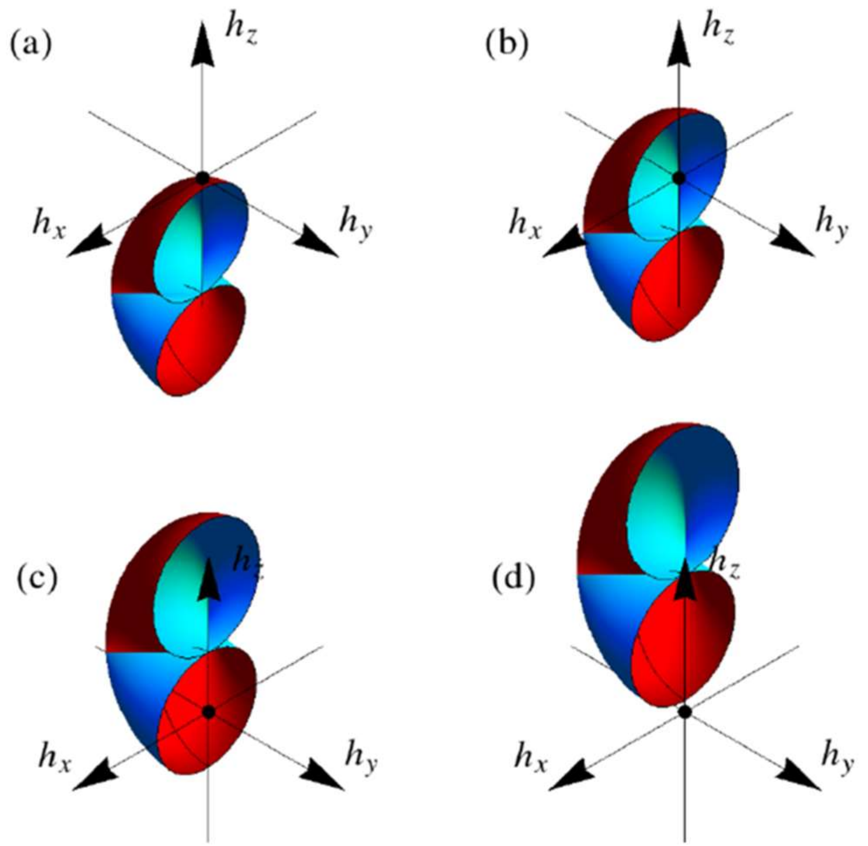




m = –2



Figs. From Asboth et al, A short course on TI



m > 0

-2 < m < 0

-4 < m < -2

m < -4

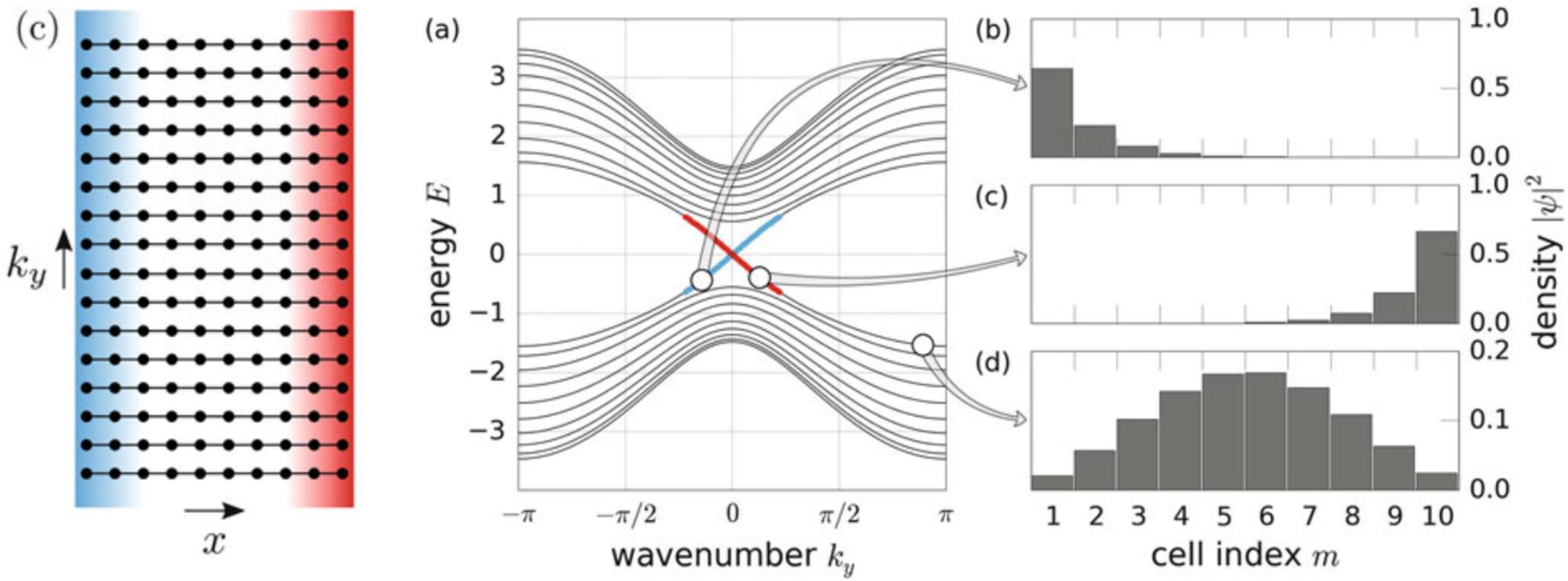
Figs. From Asboth et al, A short course on TI

$$
\begin{aligned}
\sigma_H &= \frac{e^2}{h}\frac{1}{2\pi}\int_{BZ} d^2k F_z^-(\mathbf{k}) \\
&= \frac{e^2}{h}\frac{1}{4\pi}\int_{BZ} d^2k \frac{1}{h^3}\mathbf{h}\cdot\frac{\partial \mathbf{h}}{\partial k_x}\times\frac{\partial \mathbf{h}}{\partial k_y}
\end{aligned}
$$

$$\Rightarrow \quad \sigma_H = w\frac{e^2}{h}, w \in Z.$$

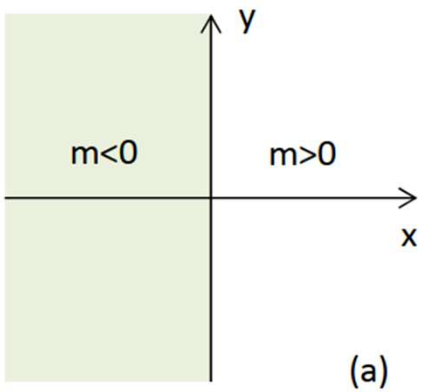
B. Edge state in Qi-Wu-Zhang model

Numerical calculation based on lattice QWZ model



Figs. From Asboth et al,
A short course on TI

$$
\begin{aligned}
\mathsf{H}(\mathbf{k}) &= \mathsf{H}_0 + \mathsf{H}_m + \mathsf{H}_{so}, \qquad (1.1)\\
\mathsf{H}_0 &= \varepsilon_0(\mathbf{k}) + \\
& t\begin{pmatrix} 2-\cos k_x a - \cos k_y a & 0 \\ 0 & -(2-\cos k_x a - \cos k_y a) \end{pmatrix},\\
\mathsf{H}_m &= m\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\\
\mathsf{H}_{so} &= \lambda\begin{pmatrix} 0 & \sin k_x a - i\sin k_y a \\ \sin k_x a + i \sin k_y a & 0 \end{pmatrix}.
\end{aligned}
$$



$$
m(x)\begin{cases} >0 \text{ for } x>0 \\ <0 \text{ for } x<0 \end{cases}
$$

Small k limit:

$$
\mathsf{H}(\mathbf{k}) = \varepsilon_0 + \begin{pmatrix} m & \lambda(k_x - ik_y) \\ \lambda(k_x + ik_y) & -m \end{pmatrix} + O(k^2).
$$

Re-quantize,

$$
\Rightarrow \quad \mathsf{H}(\mathbf{p}) = \varepsilon_0 + \begin{pmatrix} m(x) & \lambda\left(\frac{1}{i}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) \\ \lambda\left(\frac{1}{i}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) & -m(x) \end{pmatrix}
$$

$$\mathsf{H}(\mathbf{p})\psi(x,y) = \varepsilon\psi(x,y)$$

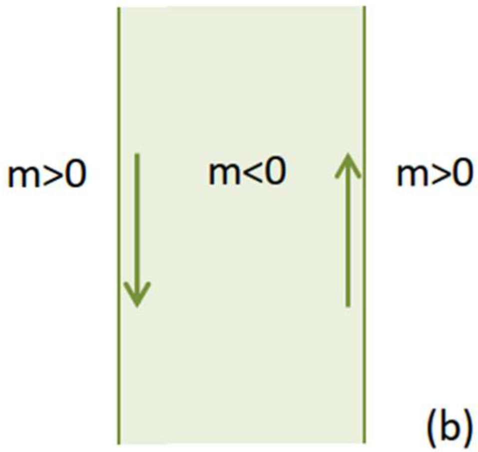
$$\psi(x,y) = \phi_1(x)\phi_2(y) \qquad \phi_2(y) = e^{ik_y y}$$

$$\begin{pmatrix} m(x) & \frac{\lambda}{i}\left(\frac{\partial}{\partial x} + k_y\right) \\ \frac{\lambda}{i}\left(\frac{\partial}{\partial x} - k_y\right) & -m(x) \end{pmatrix} \phi_1(x) = \varepsilon_e(k_y)\phi_1(x)$$

$$\Rightarrow \quad \phi_1(x) = e^{-\frac{1}{\lambda}\int_0^x dx' m(x')} \begin{pmatrix} a \\ b \end{pmatrix}$$

It can be verified as an eigenstate with eigenvalue $\varepsilon_e(k_y) = \lambda k_y$ if $(a,b) = (1,i)$.

$$\Rightarrow \quad \phi_1(x) = e^{-\frac{1}{\lambda}\int_0^x dx' m(x')} \begin{pmatrix} 1 \\ i \end{pmatrix}$$



(b)

Or, $$m(x) \begin{cases} > 0 \text{ for } x < 0 \\ < 0 \text{ for } x > 0 \end{cases}$$

$$\phi_1(x) = e^{\frac{1}{\lambda}\int_0^x dx' m(x')} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$