Lecture notes on topological insulators

Ming-Che Chang

Department of Physics, National Taiwan Normal University, Taipei, Taiwan

(Dated: December 29, 2018)

Contents

I. DIMENSIONAL REDUCTION

In Chap. **??**, we have shown that there is a connection between 1D charge pump and 2D quantum Hall effect. In this chapter we explain how to *deduce* the former from the latter using a procedure called dimensional reduction (Qi and Zhang, 2011). We then apply the same procedure to deduce topological insulator from 4D quantum Hall effect. This chapter could be skipped at first reading.

In this chapter, we write the Berry connections a_i in lowercase, and reserve the uppercase for electromagnetic potentials A_μ.

A. 1D charge pump and 2D quantum Hall effect

The quantization of Hall conductance in 2D electron systems can be understood from **Laughlin's gauge symmetry argument**. By imposing periodic boundary condition on the longitudinal direction, the 2D system becomes a cylinder in Fig. 1(a). In addition to a radial magnetic field perpendicular to the cylinder surface, Laughlin adds a fictitious, time-changing magnetic flux $\phi(t)$ through the cylinder. Its electromotive force $-d\phi/dt$ is circular and simulates the longitudinal electric field in real samples.

However, we know that the **Aharonov-Bohm phase** acquired by an electron circling the cylinder is $e^{2\pi i\phi/\phi_0}$, where $\phi_0 = hc/e$ ($c = 1$ from now on, in this chapter) is the magnetic flux quantum. Therefore, when ϕ increases by one ϕ_0, the system should come back to its original state. This gauge symmetry condition can be relaxed

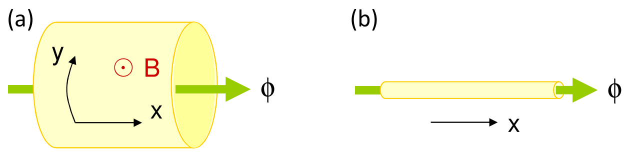

FIG. 1 (a) 2D quantum Hall system. (b) 1D charge pump.

a little bit by allowing an integer number of electrons to transport from one edge of the cylinder to the other. The transverse current density

$$\begin{aligned} j_x &= -\frac{1}{L_y}\frac{\partial E_\phi}{\partial \phi}, && (1.1)\\ &= -\frac{n(-e)}{\phi_0}\frac{V_y}{L_y} = n\frac{e^2}{h}E_y, && (1.2)\end{aligned}$$

in which we have used $\Delta E_\phi = n(-e)V_y, n \in Z$. Therefore, gauge symmetry demands precise quantization of the Hall conductance.

Suppose we shrink the dimension along the y-direction (see Fig. 1(b)), then because of the confinement, the longitudinal eigen-modes become widely separated in energy. As a result, the electron dynamics along the y-direction would be frozen in the lowest mode, and we have essentially a 1D system along the x-direction. The charge transport between two edges thus results in a 1D charge pump.

As we have explained earlier, there is a formal resemblance between the two systems. The first Chern number is,

$$C_1 = \frac{1}{2\pi}\int_{BZ} d^2k(\partial_{k_x}a_y - \partial_{k_y}a_x). \quad (1.3)$$

The change of electric polarization is

$$P(\lambda_2) - P(\lambda_1) = \frac{1}{2\pi}\int dk d\lambda(\partial_k a_\lambda - \partial_\lambda a_k). \quad (1.4)$$

Therefore, it should be possible to reach the 1D electric polarization from the formulation of the 2D quantum Hall system.

A quantum Hall system has the low-energy effective action (Zhang, 1992),

$$S_{CS}^{2+1} = \frac{\sigma_H}{2}\int dtd^2x\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda, (\mu,\nu,\lambda = 0,1,2), \quad (1.5)$$

in which

$$\sigma_H = \frac{1}{2\pi}\int_{BZ} d^2k\epsilon_{ij}\partial_{k_i}a_j \cdot \frac{e^2}{h}, (i,j = 1,2). \quad (1.6)$$

To proceed further, some knowledge of functional derivative is required.

If $F[f_i(x)]$ is a *functional* of $f_i(x), i = 1,2,\cdots$, then its functional derivative with respect to $f_i(x_0)$ is,

$$\frac{\delta F}{\delta f_i(x_0)} \equiv \lim_{\epsilon\to 0}\frac{F[f_i(x)+\epsilon\delta(x-x_0)] - F[f_i(x)]}{\epsilon}. \quad (1.7)$$

For example, if

$$F[f(x),g(x)] = \int f(x)g(x)dx, \quad (1.8)$$

then

$$\frac{\delta F}{\delta f(x)} = g(x). \quad (1.9)$$

Extending to multiple variables, and with more algebra, one can show that the current density,

$$j^\mu(\tilde{x}) = -\frac{\delta S^{2+1}_{CS}}{\delta A_\mu(\tilde{x})}, \ \tilde{x} = (t, \mathbf{x}) \quad (1.10)$$

$$= -\sigma_H \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda(x). \quad (1.11)$$

This gives ($j^\mu = (\rho, \mathbf{j})$),

$$\rho = \sigma_H B, \quad (1.12)$$

$$\mathbf{j} = \sigma_H \mathbf{E} \times \hat{\mathbf{z}}. \quad (1.13)$$

The first equation, or

$$\sigma_H = \partial\rho/\partial B, \quad (1.14)$$

is called the **Středa formula**, which shows that the Hall conductance is related to the response of charge density to a magnetic field. The second equation is the familiar Hall current.

The magnetic flux threading the tube results in a shift of momentum,

$$k_y L_y \to k_y L_y + 2\pi\frac{\phi}{\phi_0}. \quad (1.15)$$

When $L_y \to 0$, the system resides in the lowest k_y-mode and the first term can be ignored. We then make the following substitutions,

$$k_y \to \lambda = \frac{\phi}{L_y}\frac{e}{h}, \quad (1.16)$$

$$dk_y\frac{\partial}{\partial k_y} \to d\phi\frac{\partial}{\partial\phi}, \quad (1.17)$$

$$A_{0,1}(x,y,t) \to A_{0,1}(x,t), \quad (1.18)$$

$$\int_0^{L_y} dy A_2(x,y,t) \to \phi(x,t), \quad (1.19)$$

$$a_x(k_x,k_y) \to a_x(k_x,\phi), \quad (1.20)$$

$$a_y \to 0. \quad (1.21)$$

As a result, the effective action S^{2+1}_{CS} becomes,

$$\begin{aligned} S^{1+1} &= \frac{\bar{\sigma}_H}{2}\int dtdx\int_0^{L_y} dy[A_0(\partial_1 A_2 - \partial_2 A_1) \\ &+ A_2(\partial_0 A_1 - \partial_1 A_0) + A_1(\partial_2 A_0 - \partial_0 A_2)] \quad (1.22) \\ &= \frac{\bar{\sigma}_H}{2}\int dtdx[A_0\partial_1\phi - A_1\partial_0\phi \\ &\qquad +\phi(\partial_0 A_1 - \partial_1 A_0)] \quad (1.23) \\ &= \bar{\sigma}_H\int dtdx(A_0\partial_1\phi - A_1\partial_0\phi) \quad (1.24) \\ &= \bar{\sigma}_H\int dtdx\epsilon^{\mu\nu}A_\mu\partial_\nu\phi, (\mu = 0,1). \quad (1.25) \end{aligned}$$

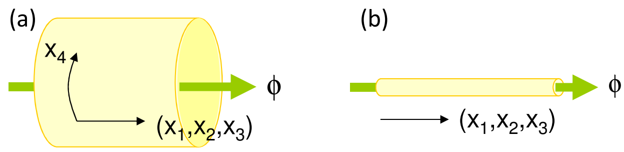

FIG. 2 (a) 4D quantum Hall system. (b) 3D topological insulator.

At the mean time, the Hall conductance σ_H reduces to

$$\begin{aligned} \bar{\sigma}_H &= \frac{1}{2\pi}\int dk_x dk_y(\partial_{k_x}a_y - \partial_{k_y}a_x)\cdot\frac{e^2}{h} \quad (1.26) \\ &= -\frac{1}{2\pi}\int dk_x d\phi\partial_\phi a_x\cdot\frac{e^2}{h} \quad (1.27) \\ &= \int d\phi\partial_\phi P(\phi)\cdot\frac{e}{h}, \quad (1.28) \end{aligned}$$

in which $P(\phi)$ is the electric polarization studied in Sec. **??**.

$$P(\phi) = -e\int\frac{dk_x}{2\pi}a_x. \quad (1.29)$$

Therefore,

$$\begin{aligned} S^{1+1} &= \frac{e}{h}\int d\phi\partial_\phi P\int dtdx\epsilon^{\mu\nu}A_\mu\partial_\nu\phi \quad (1.30) \\ &= \int\frac{d\phi}{\phi_0}S^{1+1}_\phi, \quad (1.31) \end{aligned}$$

where

$$S^{1+1}_\phi = \int dtdx\epsilon^{\mu\nu}A_\mu\partial_\nu P. \quad (1.32)$$

This is the effective action for the 1D charge pump.

From the effective action, we have the current density,

$$\begin{aligned} j^\mu(x,t) &= -\frac{\delta S^{1+1}_\phi}{\delta A_\mu(x,t)} \quad (1.33) \\ &= -\epsilon^{\mu\nu}\partial_\nu P(x,t), \quad (1.34) \end{aligned}$$

which gives ($j^\mu = (\rho, j)$)

$$\rho = -\frac{\partial P}{\partial x}, \ j = +\frac{\partial P}{\partial t}. \quad (1.35)$$

This is consistent with the polarization-induced charge and current in college electromagnetism.

B. 3D topological insulator and 4D quantum Hall effect

We now apply the same recipe to the 4D quantum Hall system. It turns out that after the dimensional reduction, we would reach the 3D topological insulator. Such a connection is first point out in Qi *et al.*, 2008. There is no 3D quantum Hall effect since the Chern numbers are nonzero only in even spatial dimensions.

The low-energy effective action for 4D quantum Hall system is (Qi *et al.*, 2008),

$$S_{CS}^{4+1} = -\frac{C_2}{6}\int dtd^4x\epsilon^{\mu\nu\rho\sigma\tau}A_\mu\partial_\nu A_\rho\partial_\sigma A_\tau\cdot\frac{e^3}{h^2}, \quad (1.36)$$

in which the subscripts run over $0, 1, 2, 3, 4$, and $x^\mu = (x^0, x^1, x^2, x^3, x^4)$. The second Chern number is defined as (see Sec. **??**),

$$C_2 = \frac{1}{32\pi^2}\int_{BZ} d^4k\epsilon_{ijkl}\mathrm{tr}\left(\mathsf{f}_{ij}\mathsf{f}_{kl}\right), \quad (1.37)$$

where $k_i = (k_1, k_2, k_3, k_4)$, and f_{ij} is the non-Abelian Berry curvature,

$$\begin{aligned} \mathsf{f}_{ij} &= \partial_i\mathsf{a}_j - \partial_j\mathsf{a}_i - i[\mathsf{a}_i, \mathsf{a}_j], && (1.38)\\ \text{and}\quad (\mathsf{a}_j)_{\alpha\beta} &= i\langle u_\alpha|\partial_{k_j}|u_\beta\rangle. && (1.39)\end{aligned}$$

The current density

$$\begin{aligned} j^\mu(x) &= -\frac{\delta S_{CS}^{4+1}}{\delta A_\mu(x)}, && (1.40)\\ &= \frac{C_2}{2}\epsilon^{\mu\nu\rho\sigma\tau}\partial_\nu A_\rho\partial_\sigma A_\tau\cdot\frac{e^3}{h^2}. && (1.41)\end{aligned}$$

Given $A^\mu = (A^0, A^1, A^2, A^3, A^4) = (0, 0, 0, E_zx^0, B_zx^2)$, the only zon-zero terms in Eq. (1.41) are $\partial_0A_3\partial_2A_4$ and $\partial_2A_4\partial_0A_3$. Therefore, one has

$$j^1 = C_2\frac{e^3}{h^2}B_zE_z. \quad (1.42)$$

Since the current density is proportional to the product B_zE_z, the C_2 coefficient can only be obtained via *non-linear* response theory.

We now compactify the dimension along x_4 (see Fig. 2), and adopt the following modifications,

$$\begin{aligned} k_4 &\to \lambda = \frac{\phi}{L_4}\frac{e}{\hbar}, && (1.43)\\ dk_4\frac{\partial}{\partial k_4} &\to d\phi\frac{\partial}{\partial\phi}, && (1.44)\\ A_{0,1,2,3}(x) &\to A_{0,1,2,3}(x^0, x^1, x^2, x^3), && (1.45)\\ \int_0^{L_4} dx_4A_4 &\to \phi(x^0, x^1, x^2, x^3), && (1.46)\\ a_{1,2,3}(k) &\to a_{1,2,3}(k_1, k_2, k_3, \phi), && (1.47)\\ a_4 &\to 0. && (1.48)\end{aligned}$$

The effective action S_{CS}^{4+1} becomes

$$\begin{aligned} S^{3+1} &= -\frac{\bar{C}_2}{6}\int dtd^3x\epsilon^{\mu\nu\rho\sigma}(\phi\partial_\mu A_\nu\partial_\rho A_\sigma && (1.49)\\ &\qquad +A_\mu\partial_\nu\phi\partial_\rho A_\sigma\\ &\qquad +A_\mu\partial_\nu A_\rho\partial_\sigma\phi)\cdot\frac{e^3}{h^2}\\ &= -\frac{\bar{C}_2}{2}\int dtd^3x\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\partial_\sigma\phi\cdot\frac{e^3}{h^2}. && (1.50)\end{aligned}$$

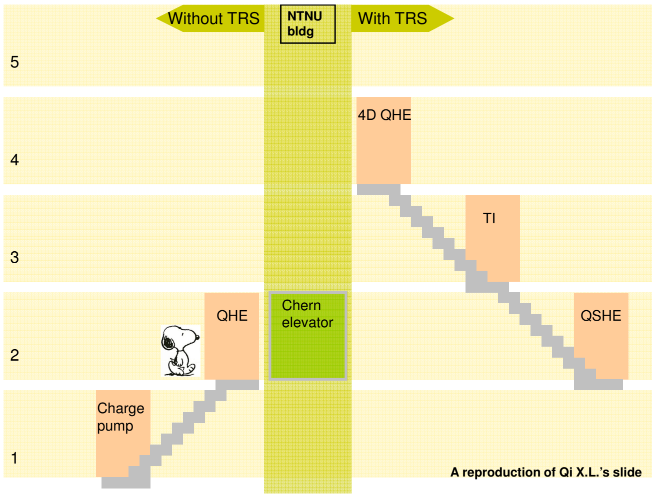

FIG. 3 Snoopy explores the building of topology.

The Chern number reduces to

$$\bar{C}_2 = \frac{1}{8\pi^2}\int d^3kd\phi\epsilon_{\phi jkl}\mathrm{tr}\left(\mathsf{f}_{\phi j}\mathsf{f}_{kl}\right). \quad (1.51)$$

With the help of the Chern-Simons identity (see Sec. **??**),

$$\frac{1}{4}\epsilon_{ijkl}\ \mathrm{tr}\left(\mathsf{f}_{ij}\mathsf{f}_{kl}\right) = \partial_i\left[\epsilon_{ijkl}\ \mathrm{tr}\left(\mathsf{a}_j\partial_k\mathsf{a}_l - \frac{2i}{3}\mathsf{a}_j\mathsf{a}_k\mathsf{a}_l\right)\right], \quad (1.52)$$

we have

$$\bar{C}_2\frac{e^3}{h^2} = \int d\phi\frac{\partial P_3}{\partial\phi}\cdot\frac{e}{h}, \quad (1.53)$$

in which the magneto-electric polarizability is

$$P_3(\phi) = \frac{1}{8\pi^2}\int_{BZ} d^3k\epsilon_{jkl}\mathrm{tr}\left(\mathsf{a}_j\partial_k\mathsf{a}_l - \frac{2i}{3}\mathsf{a}_j\mathsf{a}_k\mathsf{a}_l\right)\cdot\frac{e^2}{h}. \quad (1.54)$$

It is related to the axion angle in Chap. **??**. Comparing with Eq. (**??**), one has $P_3 = (\theta/2\pi)e^2/h$.

Now, we have

$$S^{3+1} = \int\frac{d\phi}{\phi_0}S_\phi^{3+1}, \quad (1.55)$$

where

$$\begin{aligned} S_\phi^{3+1} &= -\frac{1}{2}\int dtd^3x\epsilon^{\mu\nu\rho\sigma}A_\mu\partial_\nu A_\rho\partial_\sigma P_3 && (1.56)\\ &= -\frac{1}{2}\int dtd^3xP_3\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu\partial_\rho A_\sigma && (1.57)\\ &= \int dtd^3xP_3\mathbf{E}\cdot\mathbf{B}, && (1.58)\end{aligned}$$

which is the same as the action given by Eq. (**??**).

From the effective action, we have the current density,

$$\begin{aligned} j^\mu &= -\frac{\delta S_\phi^{3+1}}{\delta A_\mu(x)} && (1.59)\\ &= \frac{e^2}{\hbar}\frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu A_\rho\partial_\sigma\theta, && (1.60)\end{aligned}$$

which gives

$$\rho_\theta = -\frac{\alpha}{4\pi^2}\nabla\cdot(\theta\mathbf{B}), \tag{1.61}$$

$$\mathbf{j}_\theta = \frac{\alpha}{4\pi^2}\nabla\times(\theta\mathbf{E}) + \frac{\alpha}{4\pi^2}\frac{\partial}{\partial t}(\theta\mathbf{B}). \tag{1.62}$$

These are the same as Eq. (**??**) in axion electrodynamics.

Finally, Fig. 3 summaries what we have learned in this chapter.

References

Qi, X.-L., T. L. Hughes, and S.-C. Zhang, 2008, Phys. Rev. B **78**, 195424.

Qi, X.-L., and S.-C. Zhang, 2011, Rev. Mod. Phys. **83**, 1057.

Zhang, S.-C., 1992, International Journal of Modern Physics B **6**(1), 25.