# Lecture notes on topological insulators

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**Contents**



## I. REVIEW OF BLOCH THEORY

### A. Translation symmetry

For a perfect crystal with discrete translation symmetry, the Hamiltonian is

$$H = \frac{p^2}{2m} + V_L(\mathbf{r}), \text{ with } V_L(\mathbf{r}+\mathbf{R}) = V_L(\mathbf{r}), \tag{1.1}$$

in which $V_L(\mathbf{r})$ is the potential of the atomic lattice, and $\mathbf{R}$ is a **lattice translation vector**. Define a **lattice translation operator** $T_{\mathbf{R}}$ that acts on electronic states as follows,

$$T_{\mathbf{R}}\psi(\mathbf{r}) = \psi(\mathbf{r}+\mathbf{R}). \tag{1.2}$$

It can be shown that, because $H$ has the translation symmetry,

$$T_{\mathbf{R}}H(\mathbf{r})\psi(\mathbf{r}) = H(\mathbf{r})T_{\mathbf{R}}\psi(\mathbf{r}). \tag{1.3}$$

That is, $[T_{\mathbf{R}}, H] = 0$.

Because $T_{\mathbf{R}}$ commutes with $H(\mathbf{r})$, one can find their simultaneous eigenstates,

$$H\psi = \varepsilon\psi, \tag{1.4}$$
$$T_{\mathbf{R}}\psi = c_{\mathbf{R}}\psi, \tag{1.5}$$

where $\varepsilon$ and $c_{\mathbf{R}}$ are eigenvalues of $H$ and $T_{\mathbf{R}}$, and $|c_{\mathbf{R}}| = 1$. Furthermore, successive translations satisfy

$$T_{\mathbf{R}}T_{\mathbf{R}'} = T_{\mathbf{R}'}T_{\mathbf{R}} = T_{\mathbf{R}+\mathbf{R}'}. \tag{1.6}$$

This leads to

$$c_{\mathbf{R}}c_{\mathbf{R}'} = c_{\mathbf{R}'}c_{\mathbf{R}} = c_{\mathbf{R}+\mathbf{R}'}. \tag{1.7}$$

To satisfy these equations, $c_{\mathbf{R}}$ needs to be an exponential, $c_{\mathbf{R}} = e^{i\mathbf{k}\cdot\mathbf{R}}$. Therefore,

$$H\psi_{\varepsilon\mathbf{k}} = \varepsilon\psi_{\varepsilon\mathbf{k}}, \tag{1.8}$$
$$T_{\mathbf{R}}\psi_{\varepsilon\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{R}}\psi_{\varepsilon\mathbf{k}}. \tag{1.9}$$

The simultaneous eigenstate of $H$ and $T_{\mathbf{R}}$ is called the **Bloch state**.

If one writes the Bloch state in the following form,

$$\psi_{\varepsilon\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}u_{\varepsilon\mathbf{k}}(\mathbf{r}), \tag{1.10}$$

then Eq. (1.9) gives

$$u_{\varepsilon\mathbf{k}}(\mathbf{r}+\mathbf{R}) = u_{\varepsilon\mathbf{k}}(\mathbf{r}). \tag{1.11}$$

That is, a Bloch state is a plane wave times a **cell-periodic function** $u_{\varepsilon\mathbf{k}}(\mathbf{r})$. The latter contains, *in one unit cell*, all information of the Bloch state.

The Schrödinger equation for $u_{\varepsilon\mathbf{k}}$ is,

$$\tilde{H}_{\mathbf{k}}(\mathbf{r})u_{\varepsilon\mathbf{k}} = \varepsilon u_{\varepsilon\mathbf{k}}, \tag{1.12}$$

in which

$$\tilde{H}_{\mathbf{k}}(\mathbf{r}) \equiv e^{-i\mathbf{k}\cdot\mathbf{r}}H(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.13}$$
$$= \frac{1}{2m}(\mathbf{p}+\hbar\mathbf{k})^2 + V_L(\mathbf{r}). \tag{1.14}$$

Since $u_{\varepsilon\mathbf{k}}$ can be restricted to one unit cell (with periodic boundary condition), we expect it to have discrete energy eigenvalues $\varepsilon_n$ ($n \in Z^+$) for each $\mathbf{k}$, and write

$$\tilde{H}_{\mathbf{k}}(\mathbf{r})u_{n\mathbf{k}} = \varepsilon_{n\mathbf{k}}u_{n\mathbf{k}}. \tag{1.15}$$

The quantum numbers $n$ and $\mathbf{k}$ are called the **band index** and the **Bloch momentum**, and $\varepsilon_{n\mathbf{k}}$ are the energy dispersions of Bloch bands.

The Bloch state $\psi_{n\mathbf{k}}$ translates under $\mathbf{R}$ as (see Eq. (1.9)),

$$\psi_{n\mathbf{k}}(\mathbf{r}+\mathbf{R}) = e^{i\mathbf{k}\cdot\mathbf{R}}\psi_{n\mathbf{k}}(\mathbf{r}). \tag{1.16}$$

If one shifts the momentum $\mathbf{k}$ by a **reciprocal lattice vector G**, then since $e^{i\mathbf{G}\cdot\mathbf{R}} = 1$ (for any $\mathbf{R}$),

$$\psi_{n\mathbf{k}+\mathbf{G}}(\mathbf{r}+\mathbf{R}) = e^{i\mathbf{k}\cdot\mathbf{R}}\psi_{n\mathbf{k}+\mathbf{G}}(\mathbf{r}). \tag{1.17}$$

Since the two Bloch states $\psi_{n\mathbf{k}}$ and $\psi_{n\mathbf{k}+\mathbf{G}}$ satisfy the same Schrödinger equation (with $\varepsilon_{n\mathbf{k}} = \varepsilon_{n\mathbf{k}+\mathbf{G}}$) and the same boundary condition (Eqs. (1.16)and (1.17)), they can differ (for non-degenerate states) at most by a phase factor $\phi(\mathbf{k})$. For convenience, one can choose the **periodic gauge** with $\phi(\mathbf{k}) = 0$, $\psi_{n\mathbf{k}+\mathbf{G}} = \psi_{n\mathbf{k}}$. Note that for a quantum phase with non-trivial topology (such as the quantum Hall phase), one can no longer set $\phi(\mathbf{k}) = 0$ for *all* $\mathbf{k}$. This is called the **topological obstruction** (see Chap. **??**). In any case, $\psi_{n\mathbf{k}}$ (or $u_{n\mathbf{k}}$) in the first Brillouin zone should contain enough information of the electronic state.

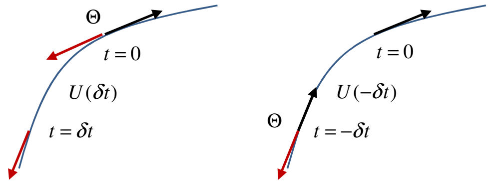


FIG. 1 Time reversal $\Theta$ followed by $U(\delta t)$ (left) and $U(-\delta t)$ followed by time reversal $\Theta$ (right) result in the same state.

## B. Time reversal symmetry

**Time reversal operator** $\Theta$ maps a state to its time-reversed state (or, motion-reversed state),

$$|\alpha\rangle \rightarrow |\tilde{\alpha}\rangle = \Theta|\alpha\rangle. \tag{1.18}$$

Naturally, if a dynamical system has time-reversal symmetry (TRS), then for a state $|\alpha\rangle$ evolving with $U(t) = e^{-iHt/\hbar}$, one expects (see Fig. 1)

$$U(t)\Theta|\alpha\rangle = \Theta U(-t)|\alpha\rangle. \tag{1.19}$$

For an infinitesimal evolution $U(\delta t) \simeq 1 - iH\delta t/\hbar$, Eq. (1.18) leads to $-iH\Theta = \Theta iH$. If $\Theta$ is a unitary operator, then we have $-H\Theta = \Theta H$. That is, if a state has energy $\varepsilon$, then its time-reversed state has energy $-\varepsilon$. This causes the eigen-energies to be bottomless, which is unreasonable (see Sakurai, 1985, p.272).

Time-reversal transformation, like translation or rotation, preserves the squared modulus $\langle\alpha|\alpha\rangle$ of a quantum state $|\alpha\rangle$ if the system under consideration has that symmetry. According to Wigner's study, an operator of transformation that preserves $|\langle\beta|\alpha\rangle|$ can only be either unitary or anti-unitary. Therefore, $\Theta$ must be an **anti-unitary operator**, which can be written as

$$\Theta = U_T K, \tag{1.20}$$

where $U_T$ is a unitary operator, and $K$ is a **complex conjugate operator**, $Ki = -iK$. As a result, if $H$ has TRS, then

$$H\Theta = \Theta H. \tag{1.21}$$

Note: even though $[H, \Theta] = 0$, there is no conserved quantity associated with TRS since $U(t)\Theta \neq \Theta U(t)$.

For states under TR, one has

$$\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\alpha|\beta\rangle, \quad \text{or} \quad \langle\beta|\alpha\rangle^*. \tag{1.22}$$

*Pf*:

$$\begin{aligned} \langle\tilde{\beta}|\tilde{\alpha}\rangle &= \langle UK\beta|UK\alpha\rangle &(1.23)\\ &= \langle K\beta|K\alpha\rangle^* &(1.24)\\ &= \langle\beta|\alpha\rangle^* = \langle\alpha|\beta\rangle. \qquad QED &(1.25)\end{aligned}$$

For the matrix elements of an operator $O$, one has

$$\langle\tilde{\beta}|O|\tilde{\alpha}\rangle = \langle\alpha|\Theta^{-1}O^\dagger\Theta|\beta\rangle. \tag{1.26}$$

*Pf*: We would try *not* to use the dual operation of $\Theta$ explicitly. That is, $\Theta$ is only allowed to act on ket states. First define $|\tilde{\gamma}\rangle = O^\dagger|\tilde{\beta}\rangle$, or

$$\Theta|\gamma\rangle = O^\dagger\Theta|\beta\rangle, \tag{1.27}$$

then $\langle\tilde{\gamma}| = \langle\tilde{\beta}|O$, and

$$\begin{aligned} \langle\tilde{\beta}|O|\tilde{\alpha}\rangle &= \langle\tilde{\gamma}|\tilde{\alpha}\rangle &(1.28)\\ &= \langle\alpha|\gamma\rangle &(1.29)\\ &= \langle\alpha|\Theta^{-1}O^\dagger\Theta|\beta\rangle. \qquad QED &(1.30)\end{aligned}$$

If an operator $O$ transforms under time reversal as,

$$\Theta^{-1}O^\dagger\Theta = \pm O, \tag{1.31}$$

then $\langle\Theta\alpha|O|\Theta\alpha\rangle = \pm\langle\alpha|O|\alpha\rangle$.

### 1. Spinless state

Given a time-reversed state $|\tilde{\psi}\rangle = \Theta|\psi\rangle$, one expects

$$\begin{aligned} \langle\psi|\Theta^{-1}\mathbf{r}\Theta|\psi\rangle &= \langle\psi|\mathbf{r}|\psi\rangle, &(1.32)\\ \langle\psi|\Theta^{-1}\mathbf{p}\Theta|\psi\rangle &= -\langle\psi|\mathbf{p}|\psi\rangle, &(1.33)\end{aligned}$$

Since this is valid for every time-reversed state, we demand

$$\begin{aligned} \Theta^{-1}\mathbf{r}\Theta &= \mathbf{r}, &(1.34)\\ \Theta^{-1}\mathbf{p}\Theta &= -\mathbf{p}. &(1.35)\end{aligned}$$

This also implies that the angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ changes sign under TR.

A spin-less state is described by a scalar function, and this two relations can be satisfied with $\Theta = K$ (i.e. $U = 1$). Hence

$$\psi(\mathbf{r}, t) \xrightarrow{TR} \Theta\psi(\mathbf{r}, t) = \psi^*(\mathbf{r}, t). \tag{1.36}$$

Furthermore, given the Schrödinger equation,

$$H\psi(\mathbf{r}, t) = i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r}, t), \tag{1.37}$$

its complex-conjugate counterpart is,

$$H\psi^*(\mathbf{r}, t) = i\hbar\frac{\partial}{\partial(-t)}\psi^*(\mathbf{r}, t). \tag{1.38}$$

That is $\psi^*(\mathbf{r}, t)$ evolves to $-t$ the same way as $\psi(\mathbf{r}, t)$ evolves to $t$. Hence, $\psi^*(\mathbf{r}, t)$ is indeed a time-reversed state.

For the Hamiltonian in Eq. (1.1), $K^{-1}HK = H$ (note that $K^{-1} = K$). However, for a crystal in a magnetic field,

$$H = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + V_L(\mathbf{r}), \tag{1.39}$$

where $q = -e$ is the charge of an electron and $\mathbf{A}$ is the vector potential, we have

$$K^{-1}HK = \frac{(p+q\mathbf{A})^2}{2m} + V_L(\mathbf{r}) \neq H. \quad (1.40)$$

That is, the magnetic field breaks the TRS, as expected.

For a Bloch state, one has

$$\psi_{n\mathbf{k}}(\mathbf{r}) \to \Theta\psi_{n\mathbf{k}}(\mathbf{r}) = \psi^*_{n\mathbf{k}}(\mathbf{r}). \quad (1.41)$$

Under a translation,

$$\begin{aligned} T_{\mathbf{R}}\psi^*_{n\mathbf{k}}(\mathbf{r}) &= \psi^*_{n\mathbf{k}}(\mathbf{r}+\mathbf{R}) && (1.42)\\ &= e^{-i\mathbf{k}\cdot\mathbf{R}}\psi^*_{n\mathbf{k}}(\mathbf{r}). && (1.43)\end{aligned}$$

If the state is not degenerate, then according to Eq. (1.9), $\psi^*_{n\mathbf{k}}(\mathbf{r})$ with the eigenvalue $e^{-i\mathbf{k}\cdot\mathbf{R}}$ could be identified as $\psi_{n-\mathbf{k}}(\mathbf{r})$. That is (see Sec. 16.3 of Dresselhaus *et al.*, 2008),

$$\psi^*_{n\mathbf{k}}(\mathbf{r}) = \psi_{n-\mathbf{k}}(\mathbf{r}). \quad (1.44)$$

### 2. Spin-1/2 state

For a state $\psi$ with spin, in addition to Eqs. (1.34), (1.35), we also require the spin operator under TR to satisfy

$$\Theta^{-1}\mathbf{s}\Theta = -\mathbf{s}. \quad (1.45)$$

A spin-1/2 state is a two-component **spinor**, and the spin operators are **Pauli matrices**, $\mathbf{s} = \frac{\hbar}{2}\boldsymbol{\sigma}$. Obviously, $\Theta = K$ cannot satisfy Eq. (1.45), and a unitary rotation $U$ (not operating on $\mathbf{r}$ and $\mathbf{p}$) in $\Theta = UK$ is required.

From Eq. (1.45), one has

$$\begin{aligned} s_x\Theta &= -\Theta s_x, && (1.46)\\ s_y\Theta &= -\Theta s_y, && (1.47)\\ s_z\Theta &= -\Theta s_z. && (1.48)\end{aligned}$$

Using the standard representation for Pauli matrices, where only $\sigma_y$ has *complex* matrix elements, we have

$$\begin{aligned} \sigma_x U &= -U\sigma_x, && (1.49)\\ \sigma_y U &= +U\sigma_y, && (1.50)\\ \sigma_z U &= -U\sigma_z. && (1.51)\end{aligned}$$

$U$ is a $2\times 2$ matrix (for spin-1/2 states), which can be expanded by Pauli matrices. Since $U$ anti-commutes with $\sigma_{x,z}$, but commutes with $\sigma_y$, the equations above can be satisfied with $U = \sigma_y$, or $U = e^{i\delta}\sigma_y$, where $\delta$ is an arbitrary phase. A popular choice of $e^{i\delta}$ is $-i$. Thus,

$$\Theta = -i\sigma_y K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} K. \quad (1.52)$$

Other choices of $e^{i\delta}$ are allowed, such as $+i$ or 1.

In general, for a state with spin $j$, which can be an integer or a half-integer,

$$\Theta = e^{-iJ_y\pi/\hbar}K, \quad (1.53)$$

in which $J_y$ is a spin operator (Sakurai, 1985). For spin 1/2,

$$\Theta = e^{-is_y\pi/\hbar}K = -i\sigma_y K. \quad (1.54)$$

A Bloch state with spin-1/2 transforms as

$$\begin{pmatrix} \varphi_{\mathbf{k}1} \\ \varphi_{\mathbf{k}2} \end{pmatrix} \xrightarrow{TR} \Theta \begin{pmatrix} \varphi_{\mathbf{k}1} \\ \varphi_{\mathbf{k}2} \end{pmatrix} = \begin{pmatrix} -\varphi^*_{\mathbf{k}2} \\ +\varphi^*_{\mathbf{k}1} \end{pmatrix}. \quad (1.55)$$

Applying the time-reversal transformation twice gives $\Theta^2 = -1$.

### 3. Kramer degeneracy

In general, if a particle has *integer spin*, then applying the TR transformation twice gives $\Theta^2 = 1$. However, if a particle has *half-integer spin*, then

$$\Theta^2 = -1. \quad (1.56)$$

This fact is crucial to the existence of the **Kramer degeneracy**: If a system has TRS and its spin is a *half-integer*, then eigenstates $\psi$ and $\Theta\psi$ are degenerate and orthogonal to each other.

*Pf*: Since $H\Theta = \Theta H$, so if $\psi$ is an eigenstate with energy $\varepsilon$, $H\psi = \varepsilon\psi$, then

$$H\Theta\psi = \Theta H\psi = \varepsilon\Theta\psi. \quad (1.57)$$

That is, $\Theta\psi$ is also an eigenstate with energy $\varepsilon$.

Furthermore, using the identity $\langle\beta|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\beta}\rangle$, one has

$$\begin{aligned} \langle\psi|\Theta\psi\rangle &= \langle\Theta(\Theta\psi)|\Theta(\psi)\rangle && (1.58)\\ &= -\langle\psi|\Theta\psi\rangle, && (1.59)\end{aligned}$$

in which $\Theta^2 = -1$ has been used to get the second equation. Therefore, $\langle\psi|\Theta\psi\rangle = 0$. QED.

For example, if a Bloch state $\psi_{n\mathbf{k}\uparrow}$ has energy $\varepsilon_{n\mathbf{k}\uparrow}$, then its time-reversed state $\Theta\psi_{n\mathbf{k}\uparrow} = -\psi_{n-\mathbf{k}\downarrow}$ (see Eq. (1.55)) has energy $\varepsilon_{n-\mathbf{k}\downarrow}$, and with time reversal symmetry $\varepsilon_{n\mathbf{k}\uparrow} = \varepsilon_{n-\mathbf{k}\downarrow}$ (Kramer degeneracy). For a solid with space inversion symmetry, one has $\varepsilon_{n-\mathbf{k}s} = \varepsilon_{n\mathbf{k}s}$ ($s =\uparrow$ or $\downarrow$). When the solid has *both* symmetries, there is a two-fold degeneracy at each $\mathbf{k}$-point,

$$\varepsilon_{n\mathbf{k}s} = \varepsilon_{n-\mathbf{k}-s} = \varepsilon_{n\mathbf{k}-s}. \quad (1.60)$$

An energy band thus has a *global* two-fold degeneracy over the whole Brillouin zone.

On the other hand, if there is TRS but no space inversion symmetry, so that $\varepsilon_{n-\mathbf{k}s} \neq \varepsilon_{n\mathbf{k}s}$, then the two-fold degeneracy at a $\mathbf{k}$-point is not guaranteed, except at the

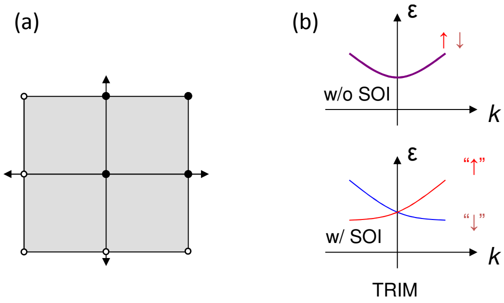


FIG. 2 (a) The TRIM are shown as black and white dots in the first Brillouin zone. Only four of them (black dots) are independent. (b) The Bloch energy levels of a system with time-reversal symmetry but without space-inversion symmetry.

$\mathbf{k}$-point that differs from $-\mathbf{k}$ by a reciprocal lattice vector $\mathbf{G}$,

$$\mathbf{k} = -\mathbf{k} + \mathbf{G}. \tag{1.61}$$

These $\mathbf{k}$-points are called **time-reversal-invariant momenta** (TRIM), see Fig. 2(a). At a TRIM,

$$\varepsilon_{n\mathbf{k}s} = \varepsilon_{n-\mathbf{k}-s} = \varepsilon_{n,-\mathbf{k}+\mathbf{G},-s} = \varepsilon_{n\mathbf{k}-s}. \tag{1.62}$$

Typical TRIM are located at the corners of a BZ, $\mathbf{k} = \mathbf{G}/2$. They play important roles in the theory of **topological insulator**.

Note: For a crystal *without* space-inversion symmetry, we often still have $\varepsilon_{n\mathbf{k}} = \varepsilon_{n-\mathbf{k}}$ (Fig. 2(b)). This is due to the fact that, with time-reversal symmetry, $\varepsilon_{n\mathbf{k}s} = \varepsilon_{n-\mathbf{k}-s}$. In the absence of spin-orbit interaction (SOI), $\varepsilon_{n-\mathbf{k}-s} = \varepsilon_{n-\mathbf{k}s}$ and we have a symmetric energy spectrum with global two-fold degeneracy. A SOI breaks the two-fold degeneracy (except at TRIM), but the energy spectrum still looks symmetric because of the Kramer degeneracy.

### C. Rotation symmetry

A crystal can be invariant under a rotation with respect to a lattice point. Because of the rotation, a position vector $\mathbf{r}$ changes to $\mathbf{r}'$,

$$\mathbf{r} \to \mathbf{r}' = \mathsf{R}\mathbf{r}, \tag{1.63}$$

where $\mathsf{R}$ is the rotation matrix. A quantum state $\psi$ changes as,

$$\begin{aligned} \psi(\mathbf{r}) \to \psi'(\mathbf{r}) &= \psi(\mathsf{R}^{-1}\mathbf{r}) = D_R\psi(\mathbf{r}), && (1.64)\\ \text{or} \quad |\psi'\rangle &= D_R|\psi\rangle. && (1.65) \end{aligned}$$

where $D_R$ is a rotation operator. We have used a *passive rotation* $\mathsf{R}^{-1}\mathbf{r}$ to simulate an active rotation of $\psi$.

After rotation, an operator $O$ changes to $O'$, and we demand that

$$\langle\psi'|O|\psi'\rangle = \langle\psi|O'|\psi\rangle. \tag{1.66}$$

Thus,

$$O \to O' = D_R^\dagger O D_R. \tag{1.67}$$

For example, a Hamiltonian transforms as,

$$H' = D_R^\dagger H D_R. \tag{1.68}$$

The Hamiltonian is invariant under rotation, $H' = H$, if $H$ commutes with $D_R$.

The Schrödinger equation for cell-periodic state is,

$$\tilde{H}_{\mathbf{k}}(\mathbf{r},\mathbf{p})u_{\mathbf{k}}(\mathbf{r}) = \varepsilon_{\mathbf{k}}u_{\mathbf{k}}(\mathbf{r}), \tag{1.69}$$

where

$$\tilde{H}_{\mathbf{k}} = \frac{p^2}{2m} + \frac{\mathbf{p}\cdot\mathbf{k}}{m} + \frac{k^2}{2m} + V_L(\mathbf{r}). \tag{1.70}$$

After rotation (Kittel, 1963),

$$\tilde{H}_{\mathbf{k}}(\mathsf{R}^{-1}\mathbf{r}, \mathsf{R}^{-1}\mathbf{p})u_{\mathbf{k}}(\mathsf{R}^{-1}\mathbf{r}) = \varepsilon_{\mathbf{k}}u_{\mathbf{k}}(\mathsf{R}^{-1}\mathbf{r}). \tag{1.71}$$

Since $p^2$ and $k^2$ are invariant under rotation, and

$$\mathbf{k}\cdot\mathsf{R}^{-1}\mathbf{p} = \mathsf{R}\mathbf{k}\cdot\mathbf{p}, \tag{1.72}$$

we have

$$\tilde{H}_{\mathsf{R}\mathbf{k}}(\mathbf{r},\mathbf{p})D_R u_{\mathbf{k}}(\mathbf{r}) = \varepsilon_{\mathbf{k}}D_R u_{\mathbf{k}}(\mathbf{r}). \tag{1.73}$$

If a Hamiltonian is invariant under $D_R$,

$$D_R^\dagger \tilde{H}_{\mathsf{R}\mathbf{k}} D_R = \tilde{H}_{\mathbf{k}}, \tag{1.74}$$

then its Bloch energy is invariant under the rotation.

In general, symmetry operations on a crystal consist of translation, rotation, mirror reflection, space inversion, or their combinations. These operations form a **space group**. Rotation, mirror reflection, and space inversion leaves a point fixed. These operations form a **point group**.

### Exercise

1. Suppose $Q = UK$ is an anti-unitary operator, prove that $Q^2$ can only be $+1$ or $-1$.
Hint: Since performing time reversal twice would get us back to the original state, differing at most by a phase factor, we can assume $Q^2 = e^{i\delta}$. Check the consistency between mathematical operations to find out $e^{i\delta}$.
2. Show that, if an operator $O$ transforms as,

$$\Theta O \Theta^{-1} = O^\dagger, \tag{1.75}$$

and $\Theta^2 = -1$, then $\langle\psi|O|\Theta\psi\rangle = 0$.

For example, if an electron is scattered by a scalar potential $V(\mathbf{r})$, then to the first-order approximation (**Born approximation**), the scattering amplitude for $\psi$ being scattered to its time-reversal state is zero.

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