Lecture notes on topological insulators

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Contents

I. 1D SPIN PUMP

We now consider Bloch electrons with spin degree of freedom. Just as the 1D charge pump can be related to the 2D quantum anomalous Hall system, a 1D spin pump can be linked with 2D topological insulators (TI). Before embarking on the TI, we discuss the spin pump in this chapter.

A. Bloch state with spin

At the beginning of this course, we have discussed the Kramer degeneracy of spin-1/2 systems. Now we study their roles in energy bands. The spin of an electron in a solid is often coupled with the electron's orbital motion via the **spin-orbit interaction** (SOI),

$$H_{so} = \lambda_{so}\boldsymbol{\sigma} \times \mathbf{p} \cdot \nabla V_L,\ \lambda_{so} = \frac{e\hbar}{4m^2c^2}, \tag{1.1}$$

where V_L is the lattice potential. Such an interaction is invariant under time-reversal symmetry (TRS), and invariant under space-inversion symmetry (SIS) *if* $V_L(-\mathbf{r}) = V_L(\mathbf{r})$. Because of the SOI, Bloch states $\psi_{n\mathbf{k}\pm}$, which are energy eigenstates (see **??**), are in general not spin eigenstates $\psi_{n\mathbf{k}\uparrow/\downarrow}$.

Recall that the TR operator for fermion is,

$$\Theta = i\sigma_y K,\ \Theta^2 = -1. \tag{1.2}$$

In the presence of TRS, *If* the Bloch states are topologically trivial, then one can choose

$$\begin{cases} \Theta\psi_{n\mathbf{k}+} = -\psi_{n-\mathbf{k}-}, \\ \Theta\psi_{n\mathbf{k}-} = +\psi_{n-\mathbf{k}+}. \end{cases} \tag{1.3}$$

The $-$ sign in front of $\psi_{-\mathbf{k},-}$ is necessary because $\Theta^2 = -1$.

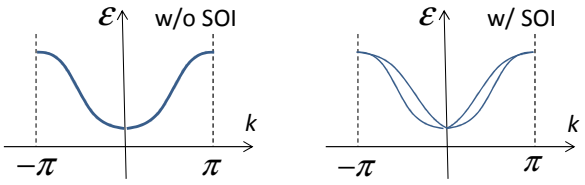

FIG. 1 Without SIS, the energy levels would cross each other at TRIM (0,π here), as long as there is TRS.

The Bloch states, which are spinors now, are of the form,

$$\psi_{n\mathbf{k}+}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}\left[+a_{n\mathbf{k}}(\mathbf{r})\chi_\uparrow + b_{n\mathbf{k}}(\mathbf{r})\chi_\downarrow\right], \tag{1.4}$$

$$\psi_{n\mathbf{k}-}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}\left[-b^*_{n\mathbf{k}}(\mathbf{r})\chi_\uparrow + a^*_{n\mathbf{k}}(\mathbf{r})\chi_\downarrow\right], \tag{1.5}$$

where $a_{n\mathbf{k}}, b_{n\mathbf{k}}$ are cell-periodic functions, and $\chi_\uparrow = (1,0)^T, \chi_\downarrow = (0,1)^T$. If the SOI is weak, then $|b_{n\mathbf{k}}(\mathbf{r})| \ll 1$, so that $(+,-) \simeq (\uparrow,\downarrow)$. It is not uncommon to refer to $\pm$ simply as spin up/down.

As we have learned in Chap. 1, in the presence of TRS and SIS, one has

$$\varepsilon_{n\mathbf{k}+} = \varepsilon_{n\mathbf{k}-}. \tag{1.6}$$

That is, the energy band n is two-fold degenerate at each Bloch momentum $\mathbf{k}$ (global two-fold degeneracy).

If the lattice does not have SIS, then only $\varepsilon_{n\mathbf{k}\pm} = \varepsilon_{n-\mathbf{k}\mp}$ is guaranteed. Now spin-orbit interaction would break the global two-fold degeneracy. Still, the energy levels need to cross at TRIM $\mathbf{\Lambda}$, where $\varepsilon_{n-\mathbf{\Lambda}\pm} = \varepsilon_{n\mathbf{\Lambda}\pm}$ (see Fig. 1).

B. Berry curvature of multiple energy levels

Based on the discussion of the non-Abelian connection in **??**, replace the parameters $\boldsymbol{\lambda}$ with the Bloch momentum $\mathbf{k}$, then the Berry connection and the Berry curvature for band n are,

$$\mathbf{A}^n_{\alpha\beta}(\mathbf{k}) = i\langle u_{n\mathbf{k}\alpha}|\frac{\partial}{\partial\mathbf{k}}|u_{n\mathbf{k}\beta}\rangle, \tag{1.7}$$

$$\mathsf{F}^n_{k\ell} = \partial_k \mathsf{A}^n_\ell - \partial_\ell \mathsf{A}^n_k - i[\mathsf{A}^n_k, \mathsf{A}^n_\ell], \tag{1.8}$$

where α,β are "spin" indices $\pm$, and k,ℓ are space indices 1,2,3. Since $\mathsf{F}^n_{k\ell}$ is antisymmetric in k,ℓ, it can also be represented as a (matrix-valued) vector $\vec{\mathsf{F}}^n$,

$$\mathsf{F}^n_j \equiv \frac{1}{2}\varepsilon_{jk\ell}\mathsf{F}^n_{k\ell}, \tag{1.9}$$

$$\vec{\mathsf{F}}^n = \nabla \times \vec{\mathsf{A}}^n - i\vec{\mathsf{A}}^n \times \vec{\mathsf{A}}^n. \tag{1.10}$$

In the presence of TRS, the states with momentum $-\mathbf{k}$ are related to their time-reversed states as follows,

$$\begin{cases} \Theta\psi_{n\mathbf{k}+} = -e^{i\chi_{n-k}}\psi_{n-\mathbf{k}-}, \\ \Theta\psi_{n\mathbf{k}-} = +e^{i\chi_{nk}}\psi_{n-\mathbf{k}+}. \end{cases} \tag{1.11}$$

Compared to Eq. (1.3), we have allowed a $\mathbf{k}$-dependent phase $\chi_{n\mathbf{k}}$ (single-valued mod 2π) in the transformation. Such a phase turns out to be crucial in the study of topological insulator.

Note: It's possible *not* to have such a phase (in the so-called **TR-smooth gauge**). However, this would result in points of gauge singularity within the BZ. See Sec. 10.5 of Bernevig and Hughes, 2013 for more details.

Since $\psi_{n\mathbf{k}\alpha} = e^{i\mathbf{k}\cdot\mathbf{r}}u_{n\mathbf{k}\alpha}$, the TR transformation can also be written as,

$$\begin{cases} \Theta u_{n\mathbf{k}+} = -e^{i\chi_{n-k}}u_{n-\mathbf{k}-}, \\ \Theta u_{n\mathbf{k}-} = +e^{i\chi_{nk}}u_{n-\mathbf{k}+}, \end{cases} \tag{1.12}$$

or,

$$\begin{cases} u_{n-\mathbf{k}+} = +e^{-i\chi_{nk}}\Theta u_{n\mathbf{k}-}, \\ u_{n-\mathbf{k}-} = -e^{-i\chi_{n-k}}\Theta u_{n\mathbf{k}+}. \end{cases} \tag{1.13}$$

Introduce the **sewing matrix**.

$$w_{n\alpha\beta}(\mathbf{k}) \equiv \langle u_{n-\mathbf{k}\alpha}|\Theta|u_{n\mathbf{k}\beta}\rangle. \tag{1.14}$$

Note that the inner product between different bands are zero,

$$\langle u_{m-\mathbf{k}\alpha}|\Theta|u_{n\mathbf{k}\beta}\rangle = 0, \quad \text{if } m \neq n. \tag{1.15}$$

It follows that,

$$\Theta|u_{n\mathbf{k}\alpha}\rangle = \sum_\beta |u_{n-\mathbf{k}\beta}\rangle w_{n\beta\alpha}(\mathbf{k}), \tag{1.16}$$

or,

$$|u_{n-\mathbf{k}\alpha}\rangle = \sum_\beta w^*_{n\alpha\beta}(\mathbf{k})\Theta|u_{n\mathbf{k}\beta}\rangle, \tag{1.17}$$

It is not difficult to see that $w_{n\alpha\beta}$ is an *unitary* matrix,

$$\mathsf{w}_n^\dagger \mathsf{w}_n = \mathsf{w}_n \mathsf{w}_n^\dagger = 1. \tag{1.18}$$

For one Kramer pair of band n,

$$\mathsf{w}_n = \begin{pmatrix} 0 & e^{i\chi_{nk}} \\ -e^{i\chi_{n-k}} & 0 \end{pmatrix}. \tag{1.19}$$

Also,

$$\begin{aligned} w_{n\alpha\beta}(-\mathbf{k}) &= \langle u_{n\mathbf{k}\alpha}|\Theta|u_{n-\mathbf{k}\beta}\rangle && (1.20)\\ &= -\langle u_{n-\mathbf{k}\beta}|\Theta|u_{n\mathbf{k}\alpha}\rangle && (1.21)\\ &= -w_{n\beta\alpha}(\mathbf{k}), && (1.22) \end{aligned}$$

in which we have used $\langle\psi_1|\Theta\psi_2\rangle = \langle\Theta^2\psi_2|\Theta\psi_1\rangle = -\langle\psi_2|\Theta\psi_1\rangle$ to get Eq. (1.21). Therefore, at a TRIM, where the states at $\mathbf{\Lambda}$ and $-\mathbf{\Lambda}$ are considered equivalent, w is *antisymmetric*,

$$\mathsf{w}_n(\mathbf{\Lambda}) = w_n(\mathbf{\Lambda})\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, w_n(\mathbf{\Lambda}) \equiv e^{i\chi_{n\Lambda}}. \tag{1.23}$$

Under TR transformation, there is a relation between $\vec{\mathsf{A}}^n(\mathbf{k})$ and $\vec{\mathsf{A}}^n(-\mathbf{k})$ (see Nomura, 2016),

$$\vec{\mathsf{A}}^n(-\mathbf{k}) = \mathsf{w}_n(\mathbf{k})\vec{\mathsf{A}}^{n*}(\mathbf{k})\mathsf{w}_n^\dagger(\mathbf{k}) - i\mathsf{w}_n(\mathbf{k})\frac{\partial}{\partial\mathbf{k}}\mathsf{w}_n^\dagger(\mathbf{k}). \tag{1.24}$$

Pf: The proof works for every Kramer pair, hence n is dropped for brevity.

$$\begin{aligned} &\mathbf{A}_{\alpha\beta}(-\mathbf{k}) \\ =\ & i\langle u_{-\mathbf{k}\alpha}|\frac{\partial}{\partial(-\mathbf{k})}|u_{-\mathbf{k}\beta}\rangle \\ =\ & -i\left(\sum_{\alpha'} w_{\alpha\alpha'}(\mathbf{k})\langle\Theta u_{\mathbf{k}\alpha'}|\right)\frac{\partial}{\partial\mathbf{k}}\left(\sum_{\beta'} w^*_{\beta\beta'}(\mathbf{k})|\Theta u_{\mathbf{k}\beta'}\rangle\right) \\ =\ & \sum_{\alpha'\beta'} w_{\alpha\alpha'}\mathbf{A}^*_{\alpha'\beta'}(\mathbf{k})w^*_{\beta\beta'} - i\sum_{\alpha'} w_{\alpha\alpha'}\frac{\partial}{\partial\mathbf{k}}w^*_{\beta\alpha'}. \end{aligned} \tag{1.25}$$

End of proof.

For the diagonal elements in one Kramer pair, one has

$$\begin{aligned} \mathbf{A}^n_+(-\mathbf{k}) &= \mathbf{A}^n_-(\mathbf{k}) - \frac{\partial\chi_k}{n\partial\mathbf{k}}, && (1.26)\\ \mathbf{A}^n_-(-\mathbf{k}) &= \mathbf{A}^n_+(\mathbf{k}) - \frac{\partial\chi_{n-k}}{\partial\mathbf{k}}. && (1.27) \end{aligned}$$

There is a relation similar to Eq. (1.24) if the system has SI symmetry. See Prob. 2 for details.

For the Berry curvature, it is left as an exercise for you to show that,

$$\vec{\mathsf{F}}^n(-\mathbf{k}) = \mathsf{w}_n(\mathbf{k})\left(-\vec{\mathsf{F}}^{n*}(\mathbf{k})\right)\mathsf{w}_n^\dagger(\mathbf{k}). \tag{1.28}$$

In summery, because of TRS, the Bloch states at $\mathbf{k}$ and $-\mathbf{k}$ are related. So are the Berry connections and the Berry curvatures. The total Chern number of an energy band with one Kramer-pair vanishes because of the pairwise cancellation of tr $\vec{\mathsf{F}}^n$ at $\mathbf{k}$ and $-\mathbf{k}$. However, we will see below and next lecture that an alternative topological number can be defined over *half* of the Brillouin zone.

C. Time-reversal polarization in 1D

We now focus on 1D system. Recall that in the modern theory of electric polarization,

$$P(\lambda) = \sum_{n\ \text{filled}}\int_{-\pi}^{\pi}\frac{dk}{2\pi}A_n(k,\lambda). \tag{1.29}$$

With spin degree of freedom,

$$P_\pm(\lambda) = \sum_{n\ \text{filled}}\int_{-\pi}^{\pi}\frac{dk}{2\pi}A_{n\pm}(k,\lambda). \tag{1.30}$$

The *charge* polarization is given by

$$P_c = P_+ + P_-. \tag{1.31}$$

In the absence of SOI, S_z is conserved, $P_\pm = P_{\uparrow\downarrow}$, and the *spin* polarization is

$$P_s = P_\uparrow - P_\downarrow. \tag{1.32}$$

In the presence of SOI, spin is no longer conserved, and we call (following Fu and Kane, 2006)

$$P_\theta = P_+ - P_- \tag{1.33}$$

as the **time-reversal polarization**.

With the help of Eq. (1.27), one has

$$\begin{aligned} & P_\theta & (1.34)\\ = & \sum_n \int_0^\pi \frac{dk}{2\pi} \left[A_{n+}(k) + A_{n+}(-k) - A_{n-}(k) - A_{n-}(-k)\right] & \\ = & -\sum_n \int_0^\pi \frac{dk}{2\pi} \left(\frac{\partial \chi_{nk}}{\partial k} - \frac{\partial \chi_{n-k}}{\partial k}\right) & (1.35)\\ = & -\frac{1}{2\pi}\sum_n (\chi_{n\pi} - \chi_{n-\pi}) \in Z. & (1.36) \end{aligned}$$

It is an integer since χ_{nk} (mod 2π) is single-valued. A method will be presented later to evaluate P_θ.

Like P_c, this P_θ is gauge dependent. For example, if one redefines the phase of the Bloch state (of band-n) as (see Fu *et al.*, 2007),

$$\begin{aligned} u_{k+} \to u'_{k+} &= e^{i\varphi_k} u_{k+}, & (1.37)\\ u_{k-} \to u'_{k-} &= u_{k-}, & (1.38) \end{aligned}$$

in which φ_k (mod 2π) is single-valued, then

$$\begin{aligned} \Theta u'_{k+} &= -e^{i\chi'_{-k}} u'_{-k-}, & (1.39)\\ \Theta u'_{k-} &= +e^{i\chi'_k} u'_{-k+}, & (1.40) \end{aligned}$$

with $\chi'_k = \chi_k - \varphi_{-k}$.

It follows that

$$A'_+ = A_+ - \frac{\partial \varphi_k}{\partial k}, \; A'_- = A_-, \tag{1.41}$$

and,

$$\begin{aligned} P'_\theta &= P_\theta - \frac{1}{2\pi}(\varphi_\pi - \varphi_{-\pi}) & (1.42)\\ &= P_\theta + m, \; m \in Z. & (1.43) \end{aligned}$$

Like electric polarization, such an ambiguity can be removed if one calculates the difference, ΔP_θ, between two states. [Note: In Bernevig and Hughes, 2013, it is stated that a gauge transformation would change P_θ by an even integer. So the proof here may need be revised.]

We now calculate P_θ for 1D systems following Fu and Kane's approach. The key is to restrict the integration of the Berry connection to *half* of the BZ using (the 1D version of) Eq. (1.26). For band-n,

$$\begin{aligned} P_{n+} &= \int_{-\pi}^{\pi} \frac{dk}{2\pi} A_{n+}(k) & (1.44)\\ &= \int_0^\pi \frac{dk}{2\pi} [A_{n+}(k) + A_{n+}(-k)] & (1.45)\\ &= \int_0^\pi \frac{dk}{2\pi} A_n(k) - \frac{\chi_{n\pi} - \chi_{n0}}{2\pi}, & (1.46) \end{aligned}$$

where $A_n(k) \equiv A_{n+}(k) + A_{n-}(k)$. Similarly,

$$\begin{aligned} P_{n-} &= \int_{-\pi}^{0} \frac{dk}{2\pi} [A_{n-}(k) + A_{n-}(-k)] & (1.47)\\ &= \int_{-\pi}^{0} \frac{dk}{2\pi} A_n(k) - \frac{\chi_{n0} - \chi_{n\pi}}{2\pi}. & (1.48) \end{aligned}$$

Different from Eq. (1.46), here the P_{n-} integral is restricted to $[-\pi, 0]$. It follows that,

$$P_{n+} - P_{n-} = \int_0^\pi \frac{dk}{2\pi} [A_n(k) - A_n(-k)] - \frac{\chi_{n\pi} - \chi_{n0}}{\pi}. \tag{1.49}$$

Taking the trace of the 1D version of Eq. (1.24), one has

$$A_n(-k) = A_n(k) - i\mathrm{tr}\left[\mathsf{w}_n(k)\frac{\partial}{\partial k}\mathsf{w}_n^\dagger(k)\right]. \tag{1.50}$$

Therefore, the first term on the RHS of Eq. (1.49) becomes

$$\begin{aligned} & \int_0^\pi \frac{dk}{2\pi i} \mathrm{tr}\left[\mathsf{w}_n^\dagger(k)\frac{\partial}{\partial k}\mathsf{w}_n(k)\right] & \\ = & \frac{1}{2\pi i}\int_0^\pi dk \; \mathrm{tr}\frac{\partial}{\partial k}\log \mathsf{w}_n(k) & (1.51)\\ = & \frac{1}{2\pi i}\int_0^\pi dk \frac{\partial}{\partial k}\log\det \mathsf{w}_n(k) & (1.52)\\ = & \frac{1}{2\pi i}\log\frac{w_n^2(\pi)}{w_n^2(0)}, & (1.53) \end{aligned}$$

in which $w_n(k) = e^{i\chi_{nk}}$ is the off-diagonal element in the w_n matrix.

Finally,

$$\begin{aligned} P_\theta &= \sum_n (P_{n+} - P_{n-}) & (1.54)\\ &= \sum_n \left[\frac{1}{2\pi i}\log\frac{w_n^2(\pi)}{w_n^2(0)} + \frac{i}{\pi}\log\frac{w_n(\pi)}{w_n(0)}\right] & (1.55)\\ &= \frac{1}{\pi i}\log\prod_n \frac{\sqrt{w_n^2(\pi)}}{w_n(\pi)}\frac{w_n(0)}{\sqrt{w_n^2(0)}} + 2m, m \in Z. & \end{aligned}$$

Due to the ambiguity of $\log z$ ($z \in C$), P_θ is only defined modulo 2.

Furthermore, given $z = re^{i\theta}$, if $\theta \in [0, \pi]$, then

$$\frac{z}{\sqrt{z^2}} = \frac{re^{i\theta}}{r\left(e^{2i\theta}\right)^{1/2}} = 1. \tag{1.56}$$

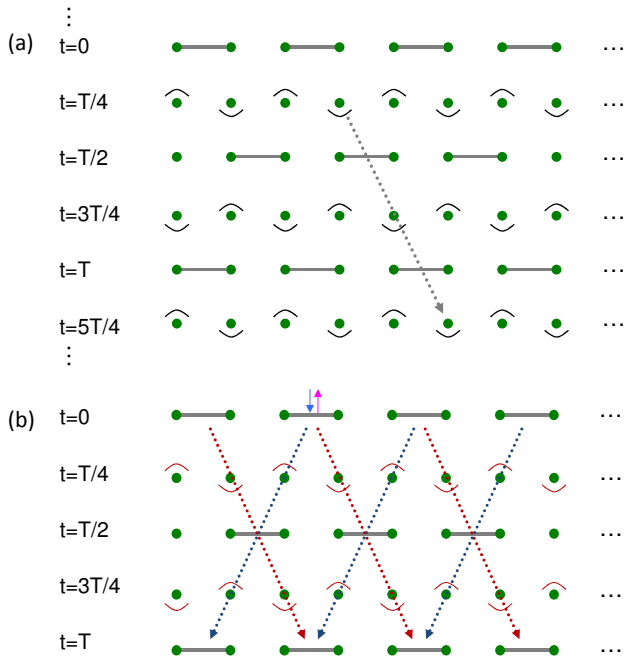

FIG. 2 (a) Visualization of a charge pump based on the Rice-Mele model. After half of a period, the stronger bonds (in grey) become weaker bonds (not drawn), and vice versa. Also the positive on-site potentials (upper caps) become negative on-site potentials (lower caps), and vice versa. (b) Visualization of a spin pump based on two copies of the Rice-Mele model. The on-site potentials for spin-up and spin-down electrons are moving out of phase. As a result, opposite spins move to opposite directions.

However, if $\theta \in [\pi, 2\pi]$, then

$$\frac{z}{\sqrt{z^2}} = \frac{re^{i(\pi+\tilde{\theta})}}{r\left(e^{2i\tilde{\theta}}\right)^{1/2}},\ \tilde{\theta} \in [0,\pi] \tag{1.57}$$

$$= -1. \tag{1.58}$$

Thus, $z/\sqrt{z^2}$ can be $+1$ or -1.

Since each of the ratio $w_n/\sqrt{w_n^2} = \pm 1$, it is alright to flip one of the fraction, and write

$$P_\theta = \frac{1}{\pi i}\log \prod_{n \text{ filled}} \frac{w_n(\pi)}{\sqrt{w_n^2(\pi)}}\frac{w_n(0)}{\sqrt{w_n^2(0)}} \mod 2 \tag{1.59}$$

$$= 0, 1 \mod 2. \tag{1.60}$$

It has the gauge ambiguity mentioned earlier. But the difference ΔP_θ between two electronic states is gauge invariant.

D. Fu-Kane spin pump

Analogous to the connection between charge polarization and charge pump (see **??**), there is also a close connection between time-reversal polarization and spin pump. For example, Fu and Kane proposed the following 1D model with spin-dependent interactions (Fu and Kane, 2006),

$$H = \frac{1}{2}\sum_{i,\alpha}\left(t+\delta(-1)^i\right)\left(c^\dagger_{i\alpha}c_{i+1\alpha} + h.c\right) + \Delta\sum_{i,\alpha\beta}(-1)^i c^\dagger_{i\alpha}\sigma^z_{\alpha\beta}c_{i\beta} + V_{so}, \tag{1.61}$$

where

$$V_{so} = \sum_{i,\alpha\beta} i\mathbf{e}_{so}\cdot\boldsymbol{\sigma}_{\alpha\beta}\left(c^\dagger_{i\alpha}c_{i+1\beta} - c^\dagger_{i+1\alpha}c_{i\beta}\right). \tag{1.62}$$

It is an extension of the Rice-Mele model (see Fig. 2). The on-site potentials now become spin-dependent, and a spin-orbit interaction V_{so} characterized by a vector $\mathbf{e}_{so}$ has been added.

Let the parameters (δ, Δ) vary with a period T,

$$(\delta, \Delta) = \left(\delta_0 \cos\frac{2\pi t}{T}, \Delta_0 \sin\frac{2\pi t}{T}\right). \tag{1.63}$$

The Hamiltonian has time-reversal symmetry at $t = 0, T/2$ when the on-site potential is zero. Therefore, there are corresponding P_θ's at these two particular instants. Their difference,

$$\Delta P_\theta = P_\theta(T/2) - P_\theta(0) \tag{1.64}$$

$$= 0, 1 \mod 2. \tag{1.65}$$

This is gauge invariant, provided that the gauge at $t = 0$ evolves to the one at $t = T/2$ continuously (Grusdt *et al.*, 2014). Alternatively, we can write

$$(-1)^{\Delta P_\theta} = \prod_n \frac{w_n(0,0)}{\sqrt{w_n^2(0,0)}}\frac{w_n(\pi,0)}{\sqrt{w_n^2(\pi,0)}}\frac{w_n(0,T/2)}{\sqrt{w_n^2(0,T/2)}}\frac{w_n(\pi,T/2)}{\sqrt{w_n^2(\pi,T/2)}}$$

$$= \pm 1. \tag{1.66}$$

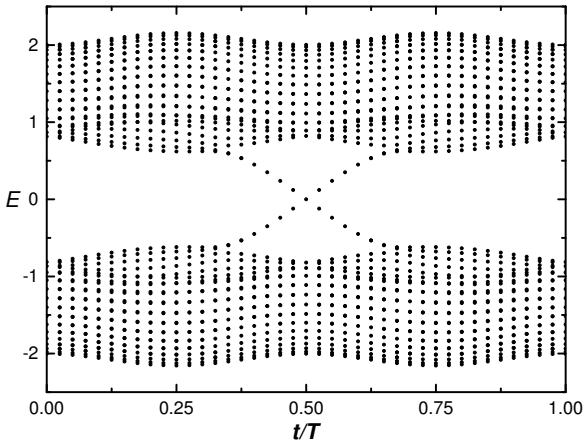

FIG. 3 The energy levels for a finite spin pump vary as time evolves. The energy levels of edge states are inside the energy gap and cross each other at $t = T/2$.

Instead of calculating the ΔP_θ of a Fu-Kane spin pump with periodic boundary condition, one can judge its value based on the existence of edge states in a system with open ends. Fig. 3 shows the evolution of the energy spectrum for a finite Fu-Kane spin pump with open boundary condition. From Fig. 2(b), we expect that at $t = 0$ and $t = T/2$, the chain is dimerized between different pairs of sites. If the initial state has no edge state, then the chain at $t = T/2$ would have an unpaired spin on each end. This implies that $\Delta P_\theta = 1$. Furthermore, the energy levels of the edge states have to cross each other at $t = T/2$ because of the Kramer degeneracy.

At $t > T/2$, the filled state from the valence band rises above $E = 0$. At $t = T$, the system is in an excited state (instead of the initial state) with *two* unpaired spins on each end. There is an important difference between the Kramer degeneracies at $t = 0, T$ and those at $t = T/2, 3T/2$. That is, the former could be lifted by electron-electron interaction (and the manybody ground state becomes non-degenerate), while the latter would not (Fu and Kane, 2006).

If there is no spin-orbit interaction V_{so}, then the electron spin would be conserved. After half a period $T/2$, a spin-up electron is pumped to (say) the right; a spin-down electron is pumped to the left. As a result, the net charge transport is zero, but a quantized spin $\hbar$ has been pumped to the right. If there is spin-orbit interaction, then the electron spin is no longer conserved, and the pumped spin is not quantized (even though ΔP_θ remains an integer). For more details of such a spin pump, one can consult Fu and Kane, 2006.

Exercise

1. Verify the relation between $\vec{\mathsf{F}}(-\mathbf{k})$ and $\vec{\mathsf{F}}(\mathbf{k})$ in Eq. (1.28) due to TRS,

$$\vec{\mathsf{F}}(-\mathbf{k}) = \mathsf{w}(\mathbf{k}) \left(-\vec{\mathsf{F}}^*(\mathbf{k})\right) \mathsf{w}^\dagger(\mathbf{k}). \tag{1.67}$$

2. (a) One can define a sewing matrix for space inversion,

$$s_{\alpha\beta}(\mathbf{k}) = \langle u_{-\mathbf{k}\alpha}|\Pi|u_{\mathbf{k}\beta}\rangle, \tag{1.68}$$

$$\text{or} \quad |u_{-\mathbf{k}\alpha}\rangle = \sum_\beta s^*_{\alpha\beta}(\mathbf{k})\Pi|u_{\mathbf{k}\beta}\rangle. \tag{1.69}$$

Show that it relates $\vec{\mathsf{A}}(-\mathbf{k})$ and $\vec{\mathsf{A}}(\mathbf{k})$ as follows,

$$\vec{\mathsf{A}}(-\mathbf{k}) = -\mathsf{s}(\mathbf{k})\vec{\mathsf{A}}(\mathbf{k})\mathsf{s}^\dagger(\mathbf{k}) - i\mathsf{s}(\mathbf{k})\frac{\partial}{\partial \mathbf{k}}\mathsf{s}^\dagger(\mathbf{k}). \tag{1.70}$$

(b) What's the relation between $\vec{\mathsf{F}}(-\mathbf{k})$ and $\vec{\mathsf{F}}(\mathbf{k})$ when there is SIS.

3. Prove the equation used in Eq. (1.52). That is, given a non-singular matrix M (i.e. $\det \mathsf{M} \neq 0$), one has

$$\det e^{\mathsf{M}} = e^{\mathrm{tr}\ \mathsf{M}}, \tag{1.71}$$

or, if $\mathsf{M} = \log \mathsf{N}$, then

$$\log\det \mathsf{N} = \mathrm{tr} \log \mathsf{N}. \tag{1.72}$$

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