Integer Quantum Hall effect

- basics
- theories for the quantization
- disorder in QHS
- Berry phase in QHS
- topology in QHS
- effect of lattice
- effect of spin and electron interaction

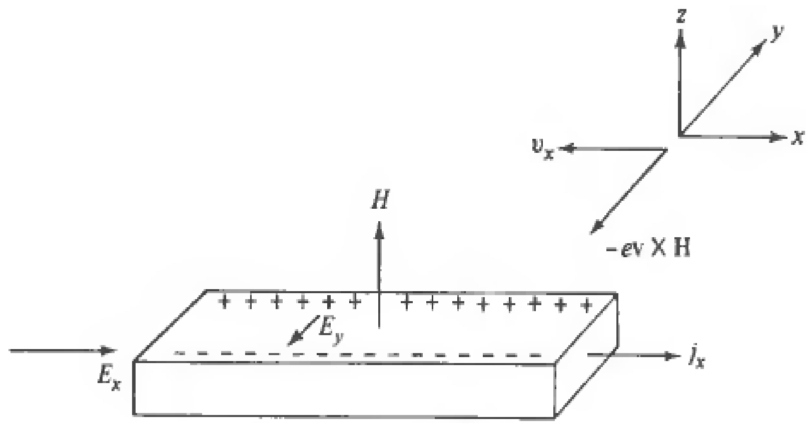
Dept of Phys

M.C. Chang

Hall effect (1879), a classical analysis

$$m^* \frac{d\vec{v}}{dt} = -e\vec{E} - e\frac{\vec{v}}{c}\times\vec{B} - m^*\frac{\vec{v}}{\tau}$$

$$\vec{B} = B\hat{z};\ d\vec{v}/dt = \vec{0} \text{ at steady state}$$



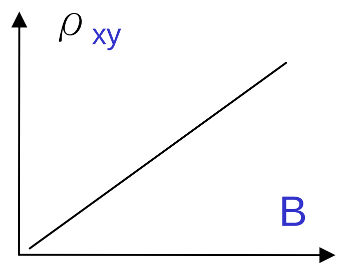
- Hall resistivity

$$\Rightarrow \begin{pmatrix} m^*/\tau & eB/c \\ -eB/c & m^*/\tau \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -e\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$\vec{j} = -en\vec{v}$$

$$\Rightarrow \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \frac{m^*}{ne^2\tau} & \frac{B}{nec} \\ -\frac{B}{nec} & \frac{m^*}{ne^2\tau} \end{pmatrix}\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \rho_0\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}\begin{pmatrix} j_x \\ j_y \end{pmatrix}$$

$$\rho_0 = \frac{m^*}{ne^2\tau},\ \omega_c = \frac{eB}{m^*c}$$



- Hall conductivity

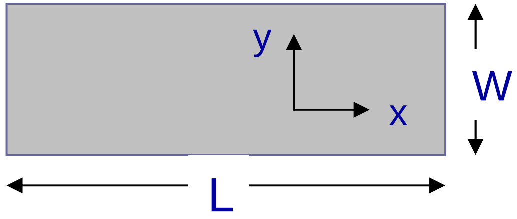
$$\boldsymbol{\sigma} = \boldsymbol{\rho}^{-1} = \frac{\sigma_0}{1+(\omega_c\tau)^2}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}$$

$$\sigma_0 = \frac{ne^2\tau}{m^*}$$

$$\xrightarrow{\omega_c\tau<<1} \sigma_0\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}$$

$$\xrightarrow{\omega_c\tau>>1} \begin{pmatrix} 0 & -nec/B \\ nec/B & 0 \end{pmatrix}$$

Resistance and conductance



$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{pmatrix} \begin{pmatrix} I_x \\ I_y \end{pmatrix}, \quad \begin{pmatrix} I_x \\ I_y \end{pmatrix} = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix}$$

Note:

$$R_{xx} = \frac{\Sigma_{yy}}{\det \Sigma}$$

So it's possible to have R_{xx} and Σ_{xx} simultaneously be zero (provided R_{xy} and Σ_{xy} are nonzero).

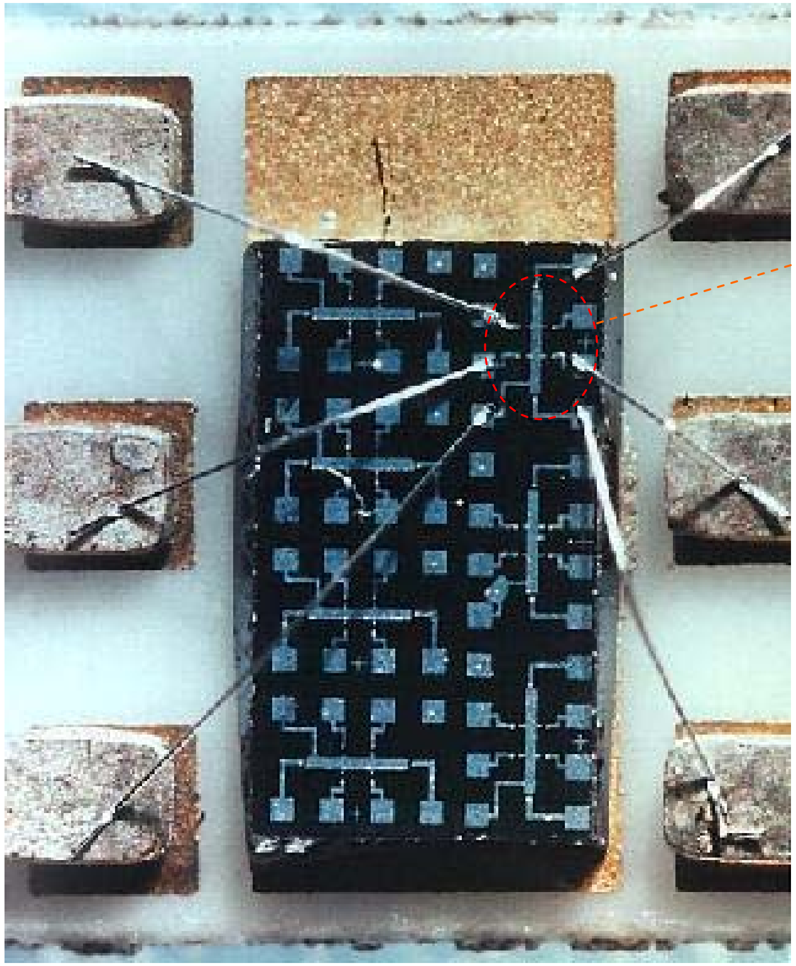
$$3D: \quad R_{xx} = \rho_{xx} \frac{L}{A} \qquad R_{yx} \equiv \left. \frac{V_y}{I_x} \right|_{I_y=0} = \frac{E_y W}{J_x A} = \rho_{yx} \frac{W}{A}$$

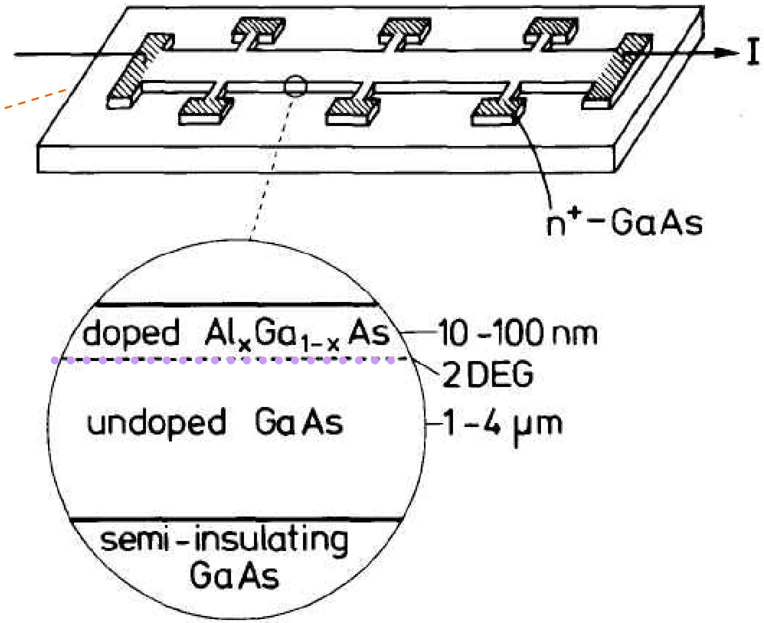
$$2D: \quad R_{xx} = \rho_{xx} \frac{L}{W} \qquad R_{yx} = \frac{E_y W}{J_x W} = \rho_{yx}$$

Quantum Hall effect $\quad \rho_{xx} = 0, \quad \rho_{yx} = const.$

$$\to \quad \Sigma_{yx} = \frac{R_{yx}}{\det R} = \frac{1}{R_{yx}} = \frac{1}{\rho_{yx}} = \frac{\rho_{yx}}{\det \rho} = \sigma_{yx}$$

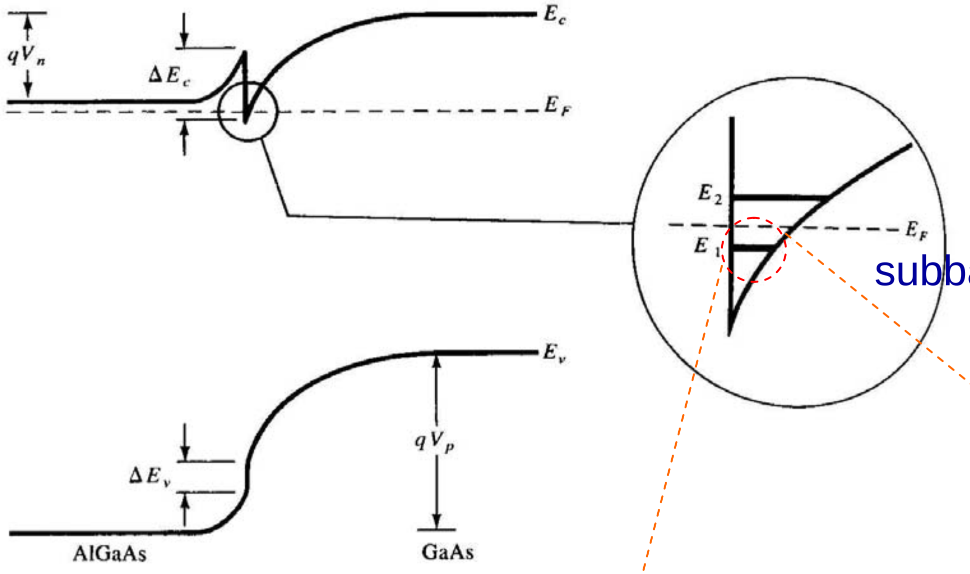
Measurement of Hall resistance





2-dim electron gas (2DEG)

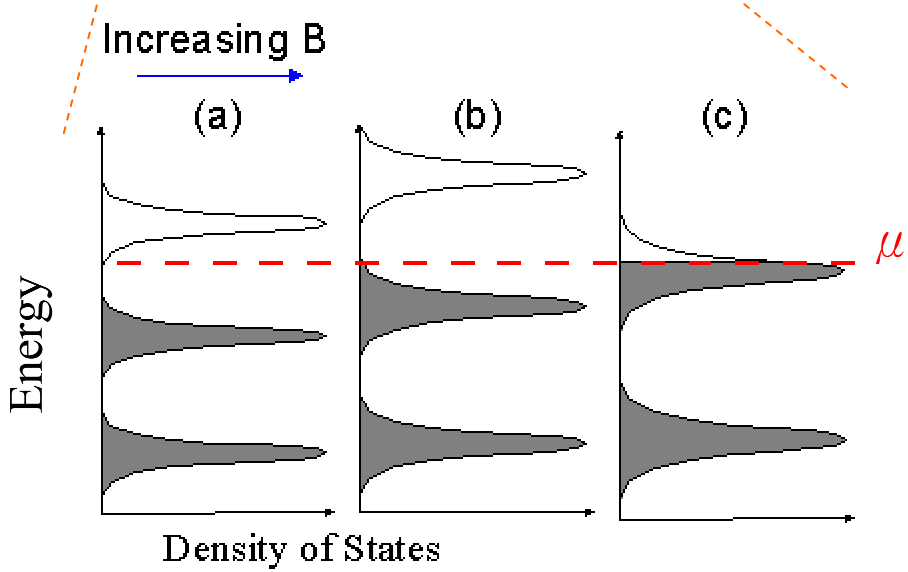
GaAs/AlGaAs heterojunction



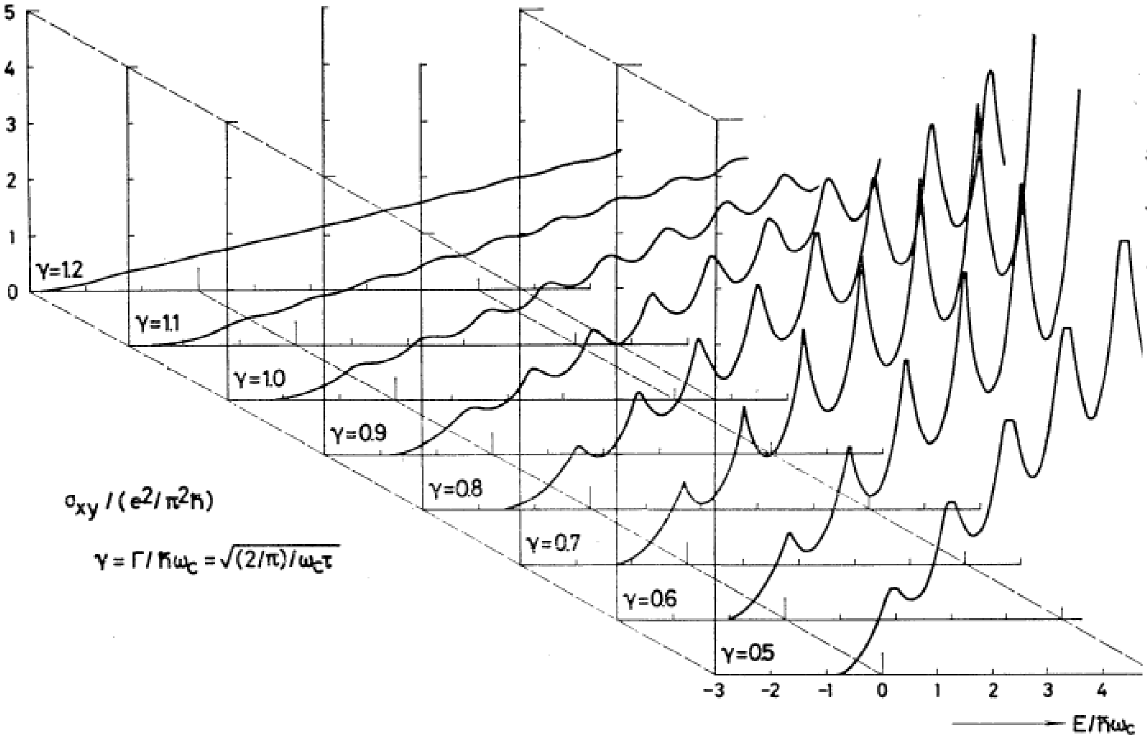
Dynamics along z-direction is frozen in the ground state

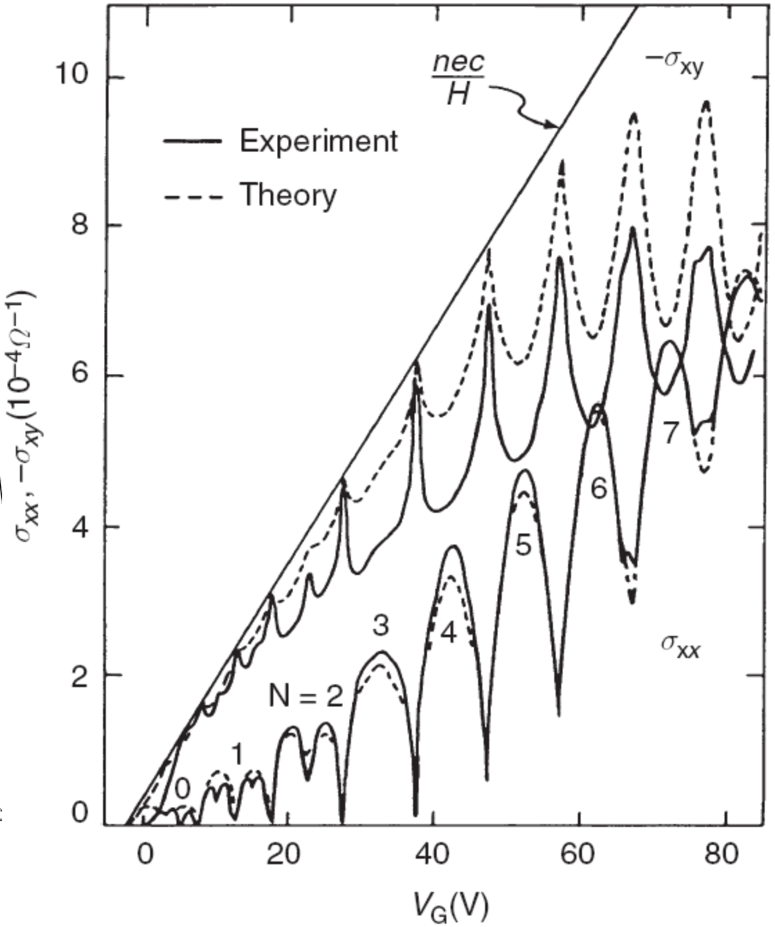
subband

(broadened) Landau levels in a magnetic field



Effect of disorder on σ_{xy} (theoretical prediction before 1980)





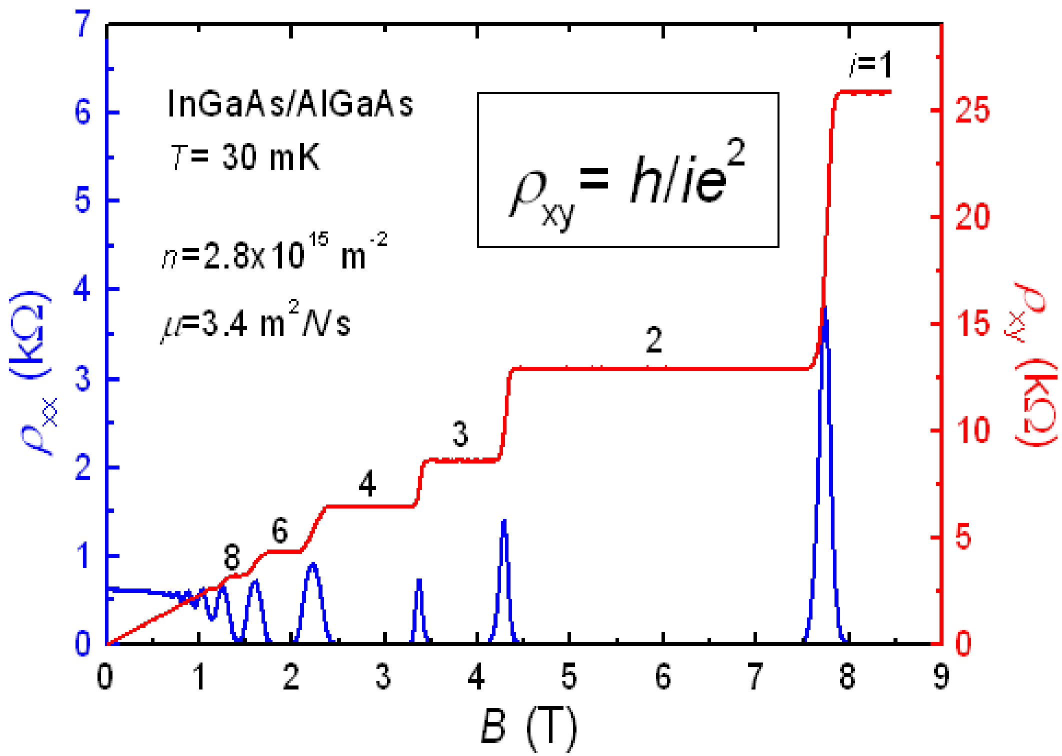
Si(100) MOS inversion layer

9.8 T, 1.6 K

Ando, Matsumoto, and Uemura JPSJ 1975

Kawaji et al, Supp PTP 1975

Quantum Hall effect (von Klitzing, 1980)



1985

ρ_{xy} deviates from $(h/e^2)/n$ by less than 3 ppm on the very first report.

- This result is independent of the shape/size of sample.
- Different materials lead to the same effect (Si MOSFET, GaAs heterojunction…)

→ a very accurate way to measure $\alpha^{-1} = h/e^2c = 137.036$ (no unit)

→ a very convenient resistance standard.

An accurate and stable resistance standard (1990)

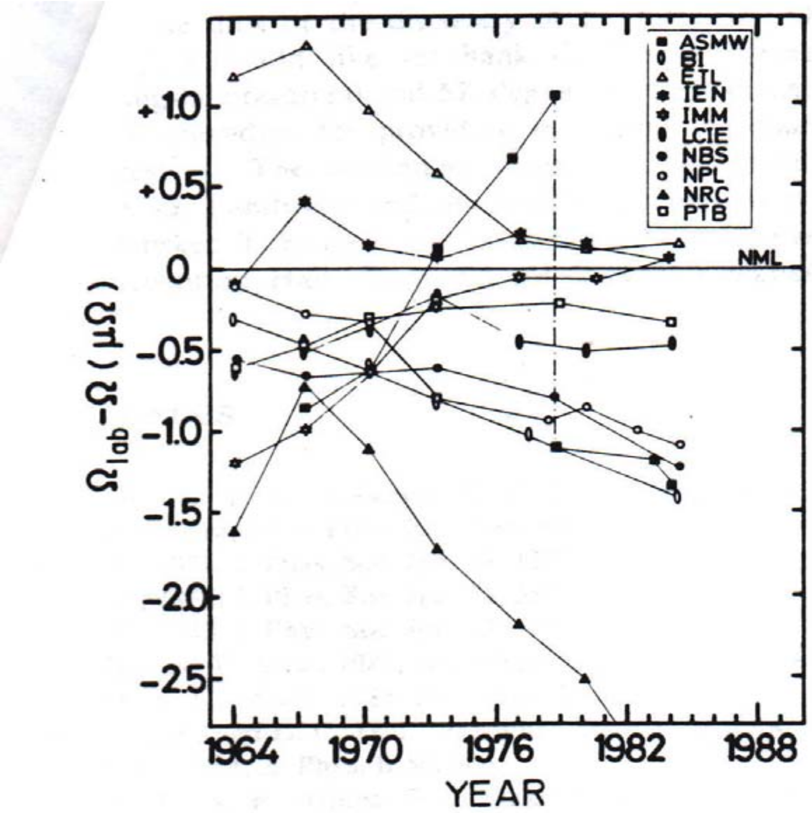


FIG. 26. Time dependence of the 1-Ω standard resistors maintained at the different national laboratories.

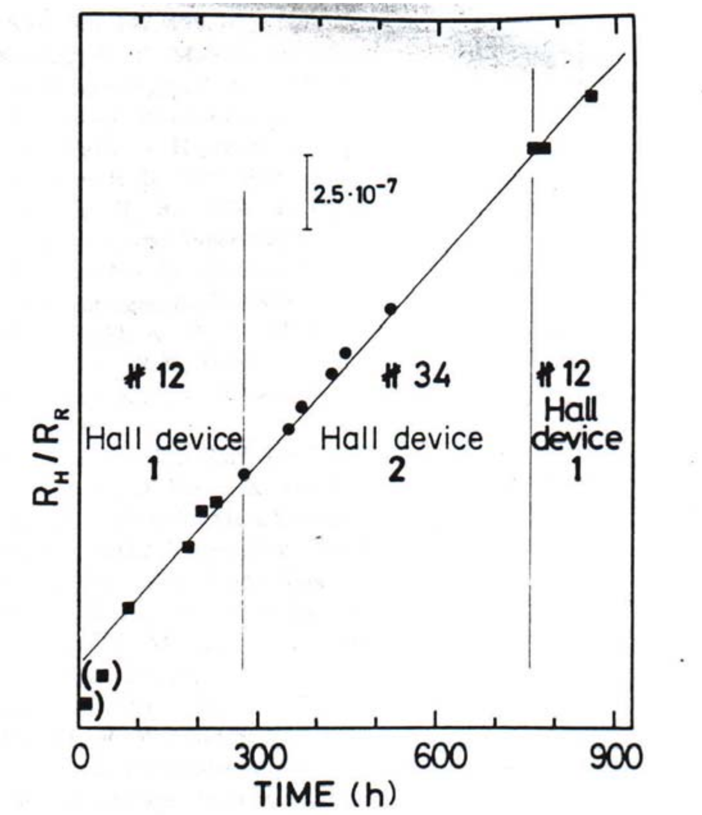


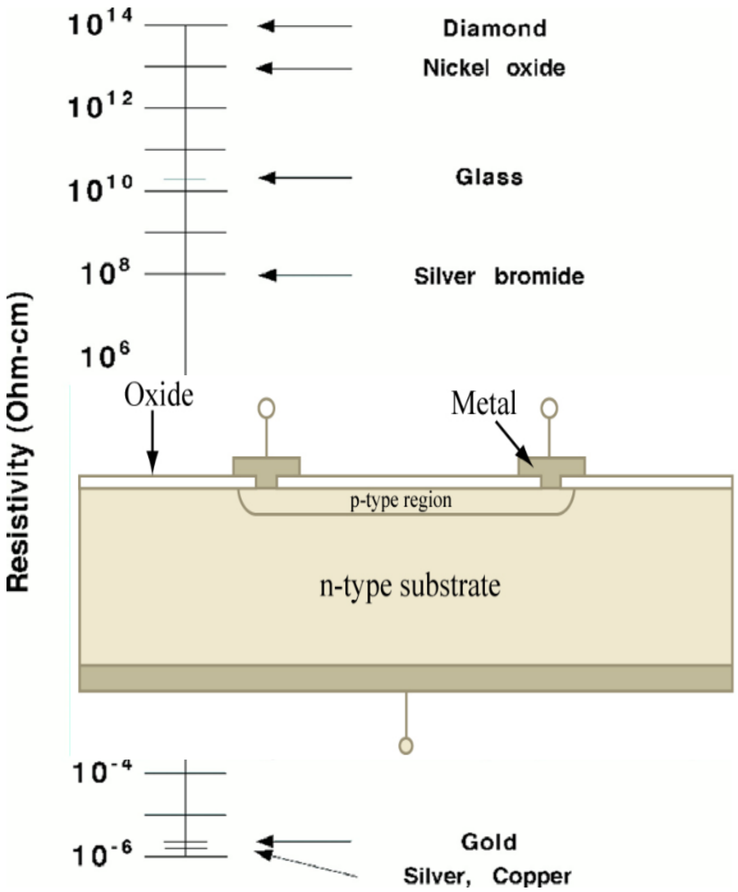
FIG. 27. Ratio R_H/R_R between the quantized Hall resistance R_H and a wire resistor R_R as a function of time. The result is time dependent but independent of the Hall device used in the experiment.

- experiment
$$\alpha^{-1}(\text{q. Hall}) = 137.035\,997\,9(32) \quad (0.024\text{ ppm}),$$
$$\alpha^{-1}(\text{acJ}) = 137.035\,977\,0(77) \quad (0.056\text{ ppm}),$$
$$\alpha^{-1}(h/m_n) = 137.036\,010\,82(524) \quad (0.039\text{ ppm}).$$
- theory
$$\alpha^{-1}(a_e) = 137.035\,999\,44(57) \quad (0.0042\text{ ppm}).$$

Kinoshita,
Phys. Rev. Lett. 1995

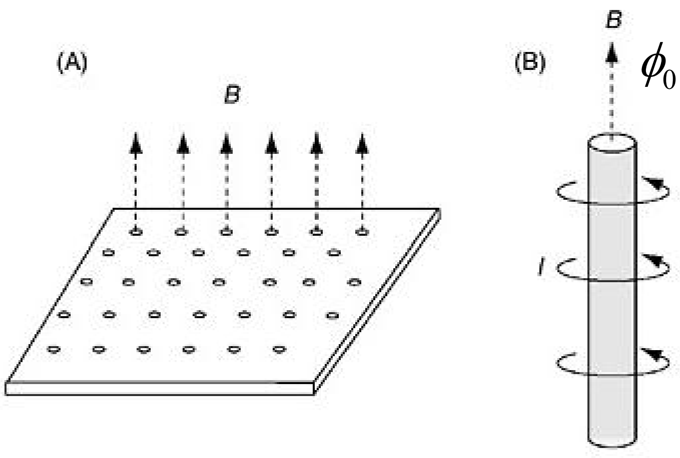
Condensed matter physics is physics of dirt - Pauli

dirty



clean

- Flux quantization

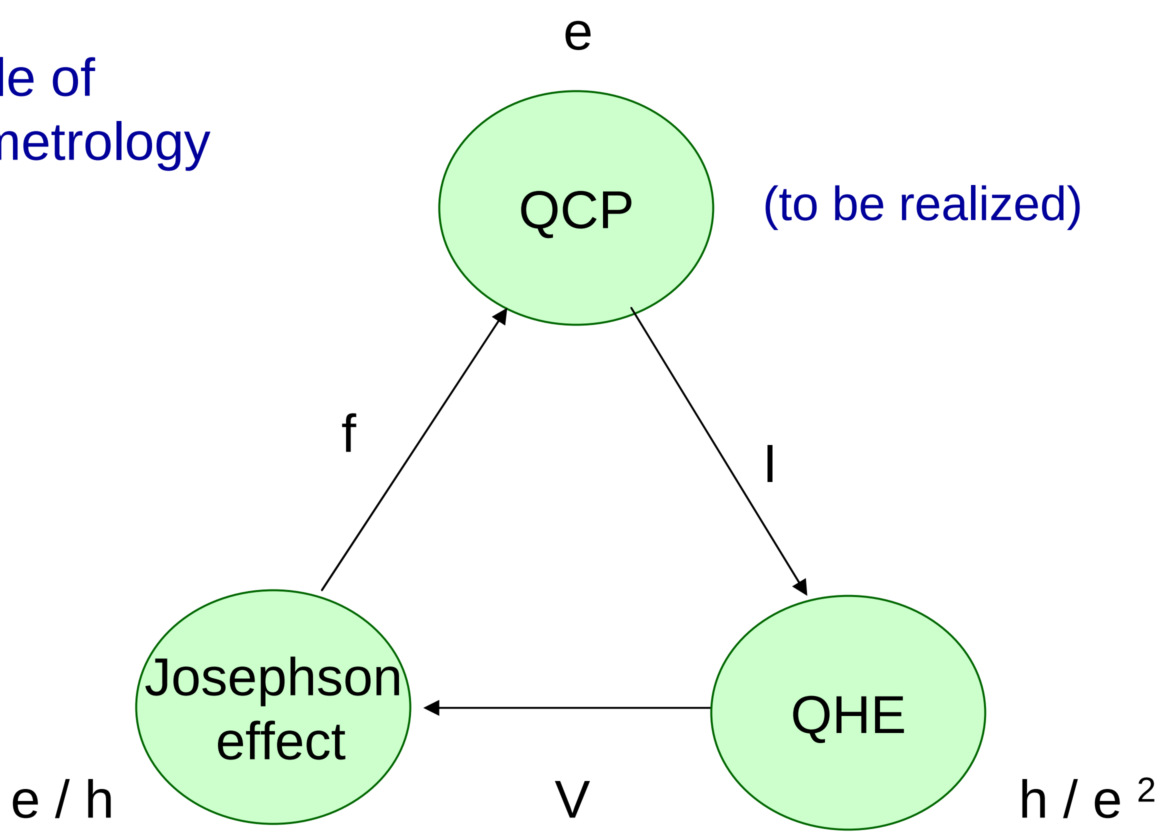


$$\phi_0 = \frac{h}{2e}$$

- Quantum Hall effect
- …

Often protected by topology,
but not vice versa.

The triangle of quantum metrology



Josephson constant $K_J = 2e/h$	
Value	483 597.879 x 10^9 Hz V^{-1}
Standard uncertainty	0.041 x 10^9 Hz V^{-1}
Relative standard uncertainty	8.5 x 10^{-8}
Concise form	483 597.879(41) x 10^9 Hz V^{-1}

von Klitzing constant $R_K = h/e^2 = \mu_0 c/2\alpha$	
Value	25 812.807 449 Ω
Standard uncertainty	0.000 086 Ω
Relative standard uncertainty	3.3 x 10^{-9}
Concise form	25 812.807 449(86) Ω

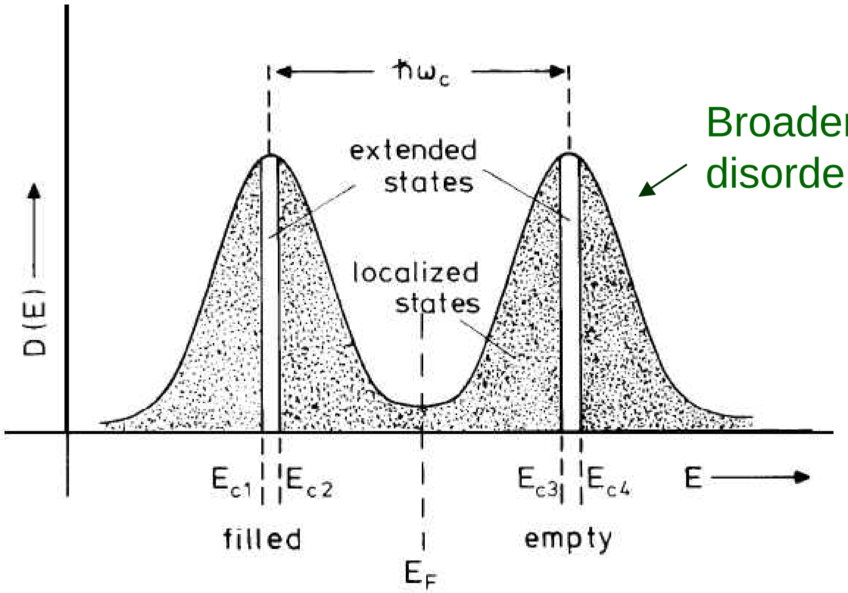
Quantum Hall effect requires

- Two-dimensional electron gas
- strong magnetic field
- low temperature $(k_B T < \hbar\omega_c)$

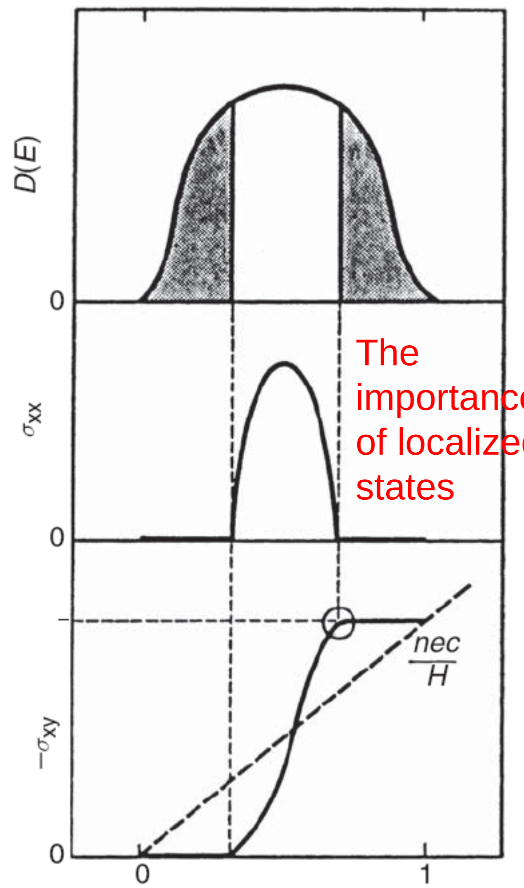
Note: Room Temp QHE in graphene (Novoselov et al, Science 2007)

Aoki, CMST 2011

Plateau and the importance of disorder



Broadened LL due to disorder



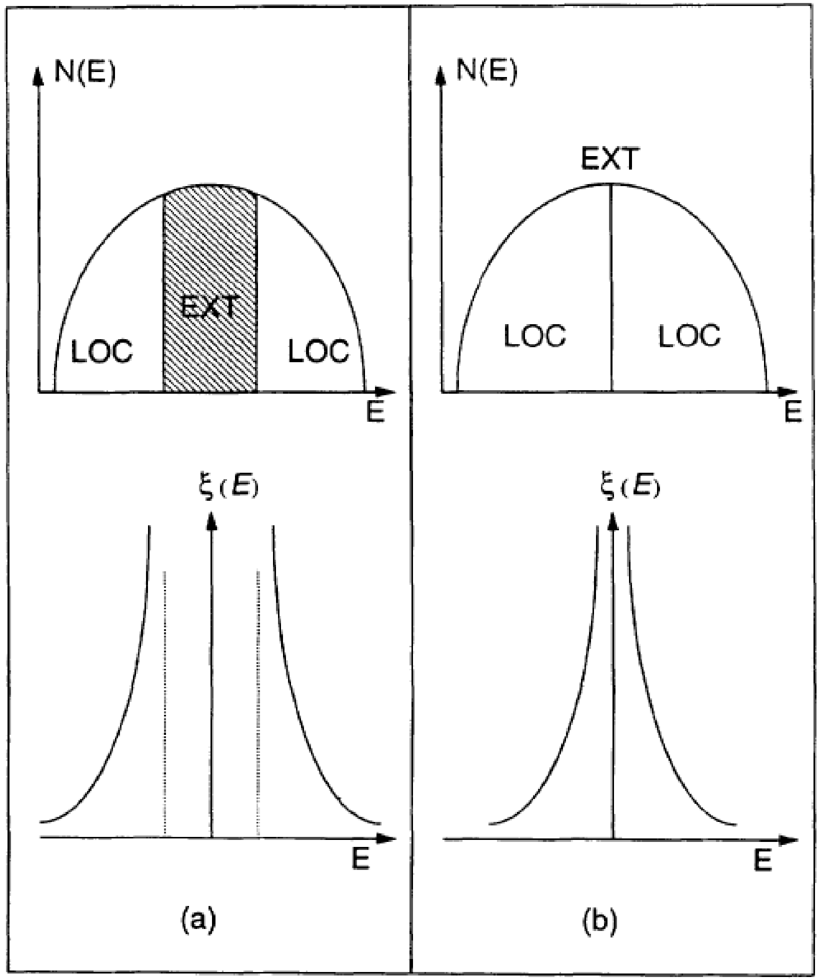
The importance of localized states

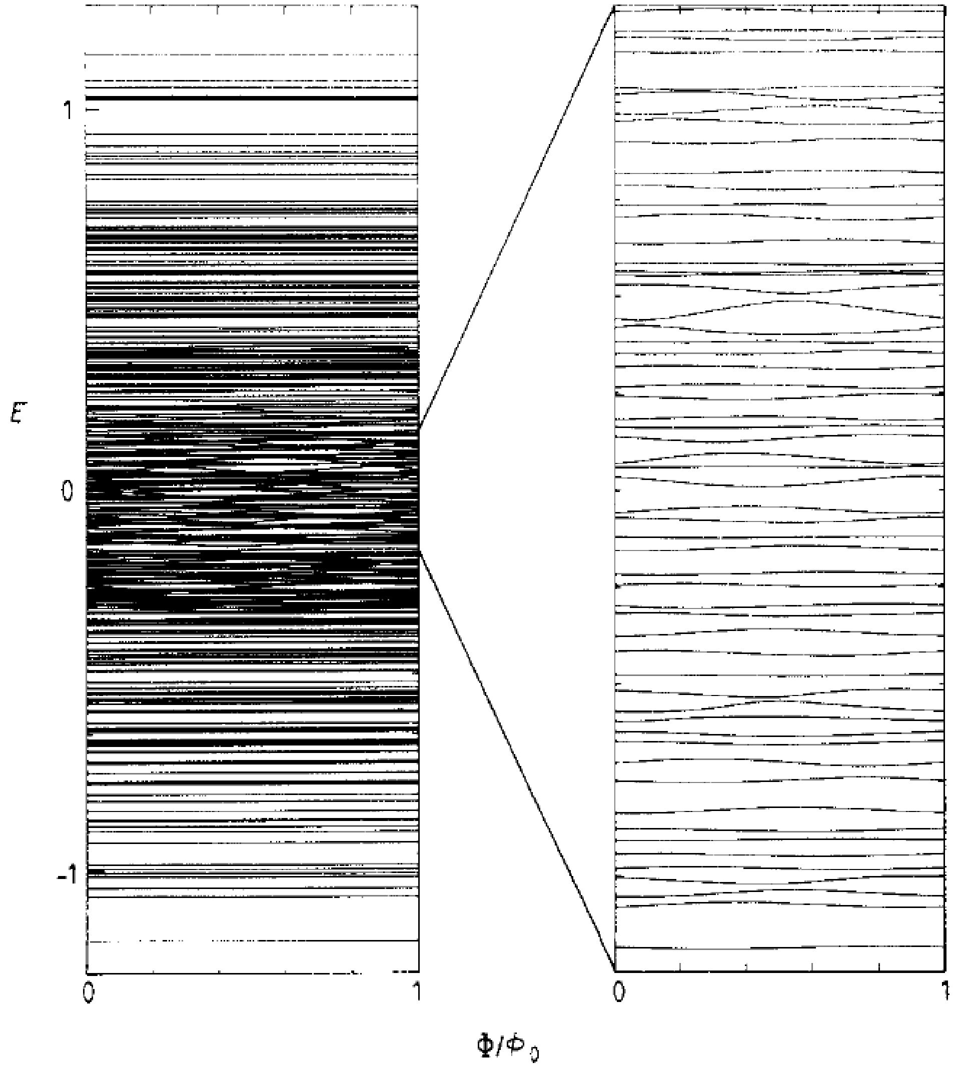
Filling factor

Why R_H has to be exactly $(h/e^2)/n$?

- see Laughlin's argument below

Width of extended states?

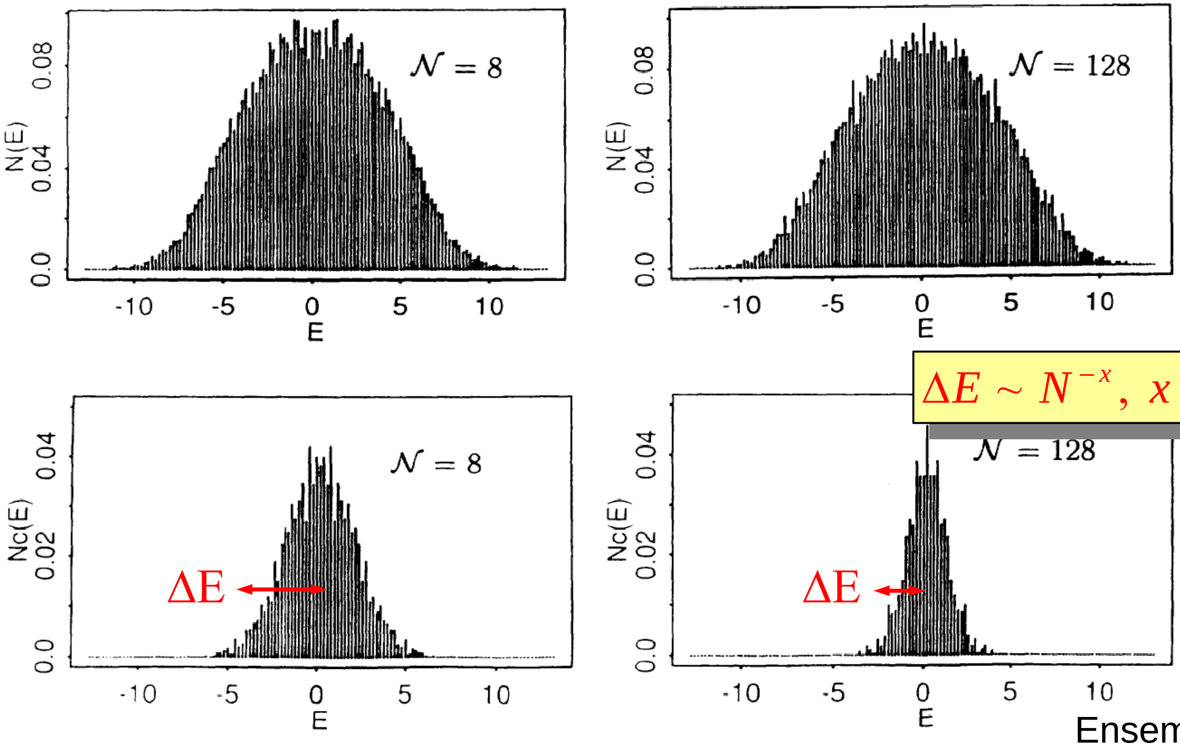




256 states in the LLL. $\varepsilon(\Phi)$ periodic in Φ_0

Aoki 1983

- Finite-Size Scaling



States that can carry Hall current (with non-zero Chern number)

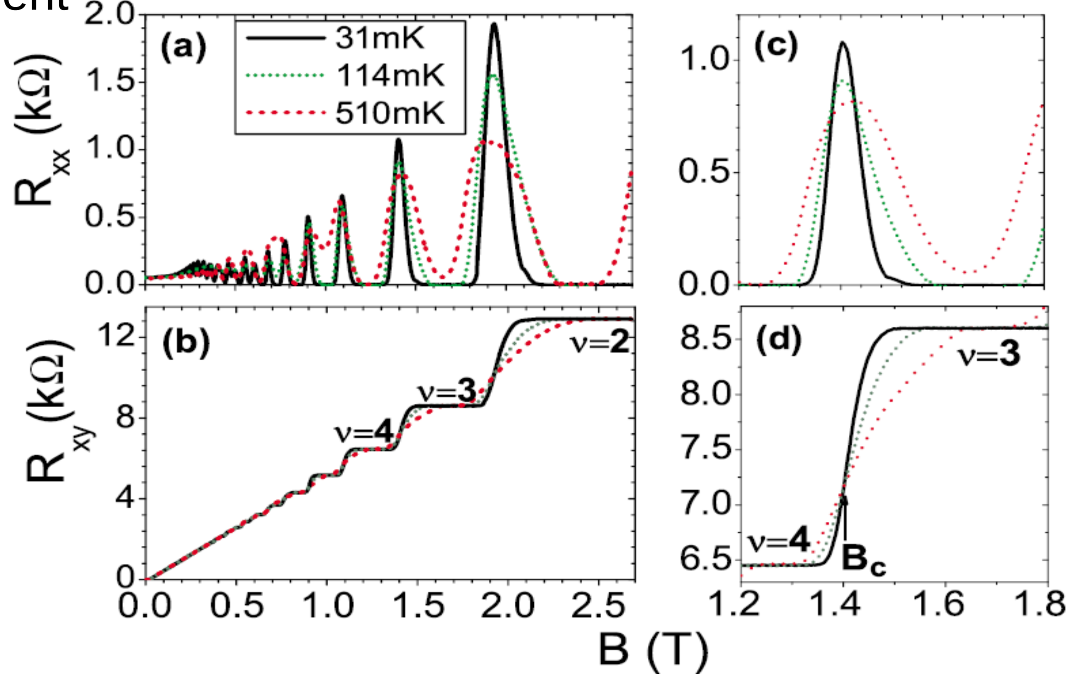
Exponent for correlation length

$$\Delta E \sim N^{-x},\ x = 1/2\nu$$

Ensemble average over 100-2000 disorder configurations

Huo and Bhatt PRL 1992

- experiment



Li et al PRL 2005

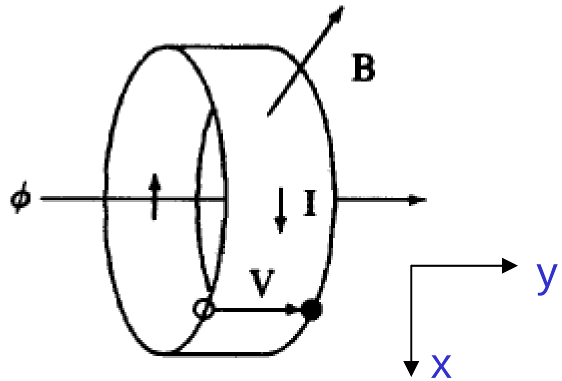
Quantization of Hall conductance, Laughlin's gauge argument (1981)

$$H=\sum_i \frac{1}{2m}\left(\vec{p}_i+\frac{e}{c}\vec{A}(\vec{r}_i)\right)^2+V(\vec{r}_i)+V_{ee}$$

- Simulate a longitudinal EMF by a fictitious time-dependent flux Φ

$$j_x=\frac{-e}{m}\frac{1}{L_xL_y}\sum_i\left[\frac{\hbar}{i}\frac{\partial}{\partial x}+\frac{e}{c}A_x(\vec{r}_i)\right]$$

$$=-\frac{c}{L_xL_y}\frac{\partial H}{\partial A_x}=-\frac{c}{L_y}\frac{\partial H_\Phi}{\partial \Phi} \qquad \Phi=A_xL_x$$



solve $\quad H_\Phi|\psi_\Phi>=E_\Phi|\psi_\Phi>$

By the Hellman-Feynman theorem, one has

$$<\psi_\Phi|\frac{\partial H_\Phi}{\partial\Phi}|\psi_\Phi>=\frac{\partial}{\partial\Phi}<\psi_\Phi|H_\Phi|\psi_\Phi>=\frac{\partial E_\Phi}{\partial\Phi}$$

$$\therefore \quad j_x=-\frac{c}{L_y}\frac{\partial E_\Phi}{\partial\Phi}$$

- Due to gauge symmetry, the system needs to be invariant under $\Phi\rightarrow\Phi+\Phi_0$,
- E_F at localized states, no charge transfer whatever Φ is.
- E_F at extended states, only integer charges may transfer along y when Φ is changed by one Φ_0.

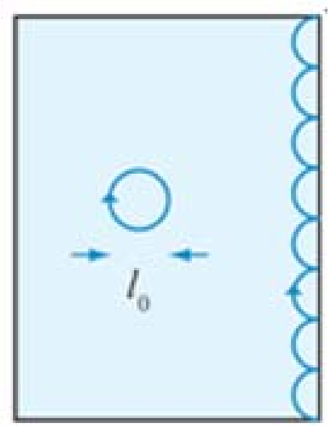
$$j_x=-c\frac{n(-e)}{\Phi_0}\frac{V_y}{L_y}=n\frac{e^2}{h}E_y$$

Edge state in quantum Hall system

• Classical picture

Chiral edge state

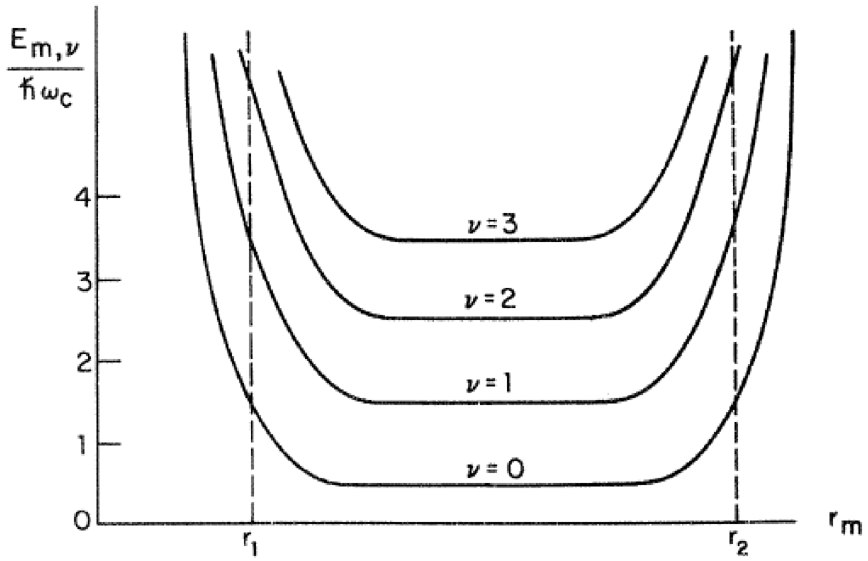
(skipping orbit)



• Robust against disorder
(no back-scattering)

• Bending of LLs

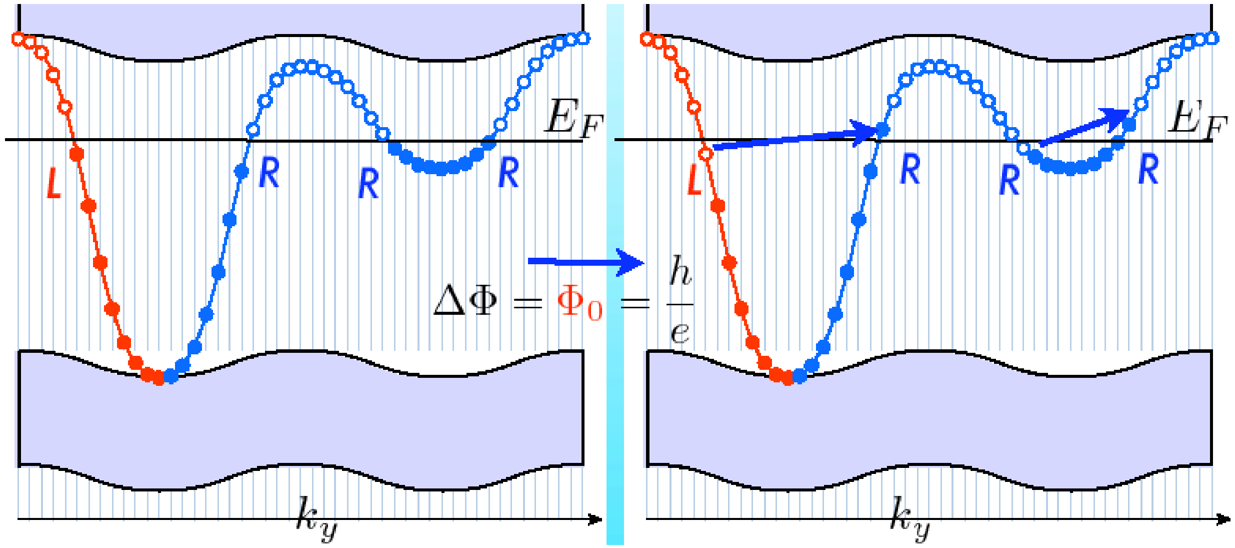
Gapless excitations at the edges

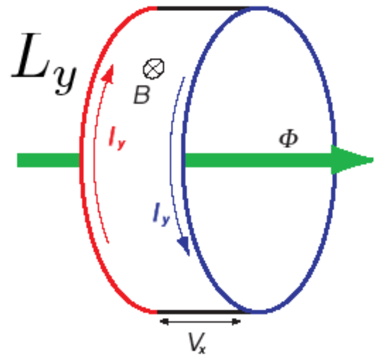


• number of edge modes = n

Inclusion of lattice (more details later)

• Bulk states: $E_n(k_x,k_y)$ (projected to k_y); Edge states: $E_n(k_y)$





Figs from Hatsugai's ppt

• when the flux is changed by 1 Φ_0, the states should come back.

$\rightarrow$ Only integer charges can be transported.

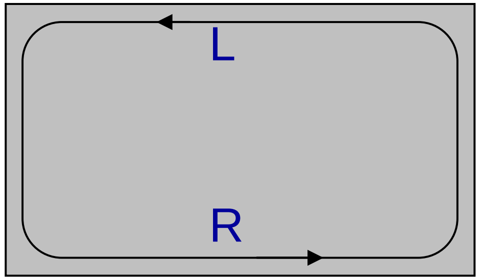
Streda formula (1982)

$$\sigma_H = ec\left(\frac{\partial n}{\partial B}\right)_\mu$$

$$\vec{j}(\vec{r}) = c\vec{\nabla} \times \vec{M}(\vec{r})$$

$$= c\nabla M \times \hat{z}$$

Nonzero along edge



$$\longrightarrow \quad I = cM(\mu, B)$$

$$\delta I_R = c\frac{\partial M(\mu, B)}{\partial \mu}(-eV_R)\,,$$

$$\delta I_L = c\frac{\partial M(\mu, B)}{\partial \mu}(eV_L)\,.$$

$$\frac{\partial M(\mu, B)}{\partial \mu} = -\frac{\partial^2 \Omega(\mu, B)}{\partial \mu \partial B} = -\left(\frac{\partial n}{\partial B}\right)_\mu$$

$$\longrightarrow \quad I_H = \delta I_R - \delta I_L = ec\left(\frac{\partial n}{\partial B}\right)_\mu V_H$$

Degeneracy of a LL: $D = BA/\Phi_0$

- If ν bands are filled, then the number of electrons per unit area is $n = \nu\, eB/hc$

$\therefore\ \sigma_H = \nu\, e^2/h$

Giuliani and Vignale, Sec 10.3.3

Current response: conductivity

- Vector potential of an uniform electric field

$$\vec{E}(t) = -\frac{1}{c}\frac{\partial \vec{A}(t)}{\partial t}$$

$$\vec{E}(t) = \vec{E}_\omega e^{-i\omega t}, \text{ then } \vec{A}(t) = \vec{A}_\omega e^{-i\omega t}; \vec{E}_\omega = \frac{i\omega}{c}\vec{A}_\omega$$

$$H = \frac{1}{2m_0}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + V_{latt} = H_0 + \frac{e}{m_0 c}\vec{A}\cdot\vec{p} + O(A^2)$$

$$H' = \frac{e}{m_0 c}\vec{p}\cdot\vec{A}_\omega e^{-i\omega t}$$

- 1st order perturbation in E $\rightarrow$ $j_\alpha = \sigma_{\alpha\beta}E_\beta$

$$\sigma_{\alpha\beta}(\omega) = \frac{e^2}{iV}\sum_{\ell m}\frac{f_\ell - f_m}{\hbar\omega_{\ell m}}\frac{v^\alpha_{\ell m}v^\beta_{m\ell}}{\omega_{\ell m} + \omega}$$

Kubo-Greenwood formula

$$\omega_{\ell m} \equiv \omega_\ell - \omega_m, \quad v^\alpha_{\ell m} \equiv \left\langle \psi_\ell \middle| v^\alpha \middle| \psi_m \right\rangle$$

Quantization of Hall conductance

Thouless et al's argument (1982)

$$\sigma^{DC}_{\alpha\neq\beta} = \frac{e^2}{im_0^2}\frac{1}{\hbar V}\sum_{\ell m} f_\ell \frac{p^\alpha_{\ell m} p^\beta_{m\ell} - p^\beta_{\ell m} p^\alpha_{m\ell}}{\omega^2_{\ell m}}$$

ℓ, m = (n, k)

$$= \frac{2e^2}{i\hbar V}\sum_{nk} f_{nk}\left(\left\langle \frac{\partial u_{nk}}{\partial k_\alpha} \middle| \frac{\partial u_{nk}}{\partial k_\beta} \right\rangle - \left\langle \frac{\partial u_{nk}}{\partial k_\beta} \middle| \frac{\partial u_{nk}}{\partial k_\alpha} \right\rangle \right)$$

$$\frac{p^\alpha_{\ell m}}{m_0} = \frac{1}{m_0}\left\langle u_\ell \middle| \frac{\hbar}{i}\partial_\alpha + \hbar k_\alpha \middle| u_m \right\rangle$$

$$= \frac{\partial \varepsilon_\ell}{\hbar \partial k_\alpha}\delta_{\ell m} + \omega_{\ell m}\left\langle \frac{\partial u_\ell}{\partial k_\alpha} \middle| u_m \right\rangle$$

cell-periodic function u_m

- Berry curvature

$$\Omega_{n\gamma}(\vec{k}) \equiv i\left(\left\langle \frac{\partial u_{nk}}{\partial k_\alpha} \middle| \frac{\partial u_{nk}}{\partial k_\beta} \right\rangle - \left\langle \frac{\partial u_{nk}}{\partial k_\beta} \middle| \frac{\partial u_{nk}}{\partial k_\alpha} \right\rangle \right)$$

$(\alpha,\beta,\gamma$ are cyclic)

- Hall conductivity for the n-th band

$$(\sigma_H)_n = \frac{e^2}{h}\left[\frac{1}{2\pi}\int_{BZ} d^2k \, (\Omega_n)_z(\vec{k}) \right]$$

an integer for a filled band

- Berry curvature (for n-th band)

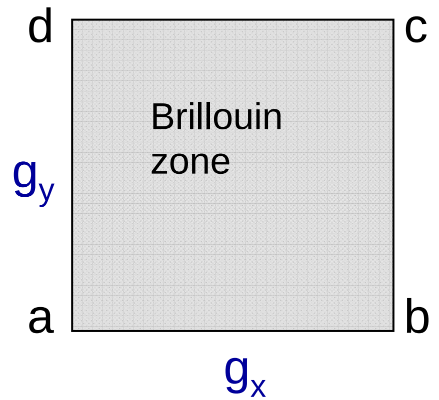
$$\vec{\Omega}_n(\vec{k}) = i\left\langle \nabla_{\vec{k}} u_n \middle| \times \middle| \nabla_{\vec{k}} u_n \right\rangle$$

$$= \nabla_{\vec{k}} \times \vec{A}_n(\vec{k})$$

- Berry connection

$$\vec{A}_n(\vec{k}) \equiv i\left\langle u_n \middle| \nabla_{\vec{k}} \middle| u_n \right\rangle$$

$$\frac{1}{2\pi}\int_{BZ} d^2k\; \Omega_z(\vec{k}) = \text{integer } n$$



Pf:

$$\int_{BZ} d^2k\; \nabla \times \vec{A}$$

$$= \int_a^b d\vec{k}\cdot\vec{A} + \int_b^c d\vec{k}\cdot\vec{A} + \int_c^d d\vec{k}\cdot\vec{A} + \int_d^a d\vec{k}\cdot\vec{A}$$

$$= \int_{\rightarrow} dk_x \left[A_x(k_x,0) - A_x(k_x,g_y)\right] + \int_{\uparrow} dk_y \left[A_y(g_x,k_y) - A_y(0,k_y)\right]$$

$$u_{\vec{k}} = e^{i\theta_1(k_y)} u_{\vec{k}+g_x\hat{x}}, \quad u_{\vec{k}} = e^{i\theta_2(k_x)} u_{\vec{k}+g_y\hat{y}}$$

$$\int_{\rightarrow} dk_x \left[A_x(k_x,0) - A_x(k_x,g_y)\right] = \theta_2(a) - \theta_2(b)$$

$\cdots$ etc

$$\int_{BZ} d^2k\; \nabla \times \vec{A}$$

$$= \theta_2(a) - \theta_2(b) + \theta_1(d) - \theta_1(a)$$

$$= 2\pi n$$

total vorticity in the BZ

$$u_a = e^{i\theta_1(a)} u_b$$

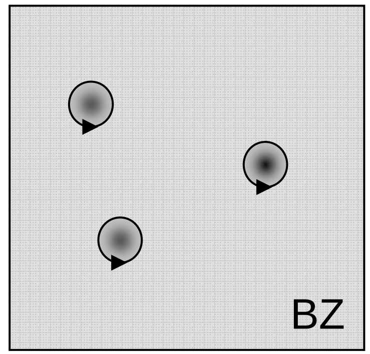
$$u_b = e^{i\theta_2(b)} u_c$$

$$u_c = e^{-i\theta_1(d)} u_d$$

$$u_d = e^{-i\theta_2(a)} u_a$$

$$\therefore u_a = e^{i[\theta_1(a)+\theta_2(b)-\theta_1(d)-\theta_2(a)]} u_a$$

Zeros and vortices



- Niu-Thouless-Wu generalization to system with disorder and electron interaction (PRB 1985).

Czerwinski and Brown, PRS (London) 1991

Connection with localization in disordered system (Anderson, 1958)

- For large g (good conductor)

  conductance

$$g(L) = \sigma_0 L^{d-2}, \quad \beta(g) = d - 2$$

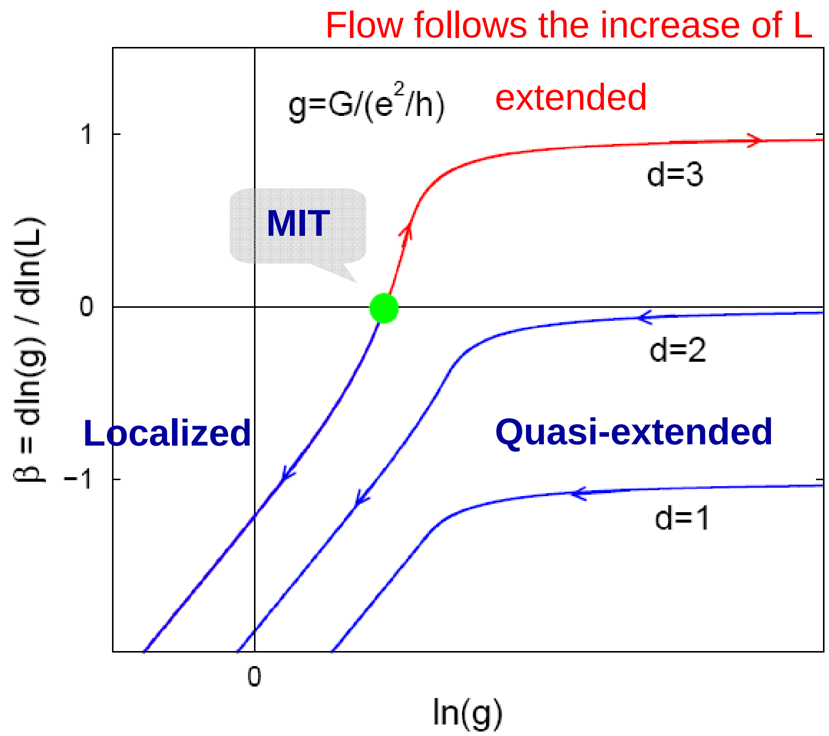
- For small g (insulator)

$$g(L) = g_c e^{-L/\xi}, \quad \beta(g) = \ln \frac{g}{g_c}$$

- one-parameter scaling hypothesis

(Abrahams et al, 1979 < Thouless, Landauer…):

assume β (g) depends only on g

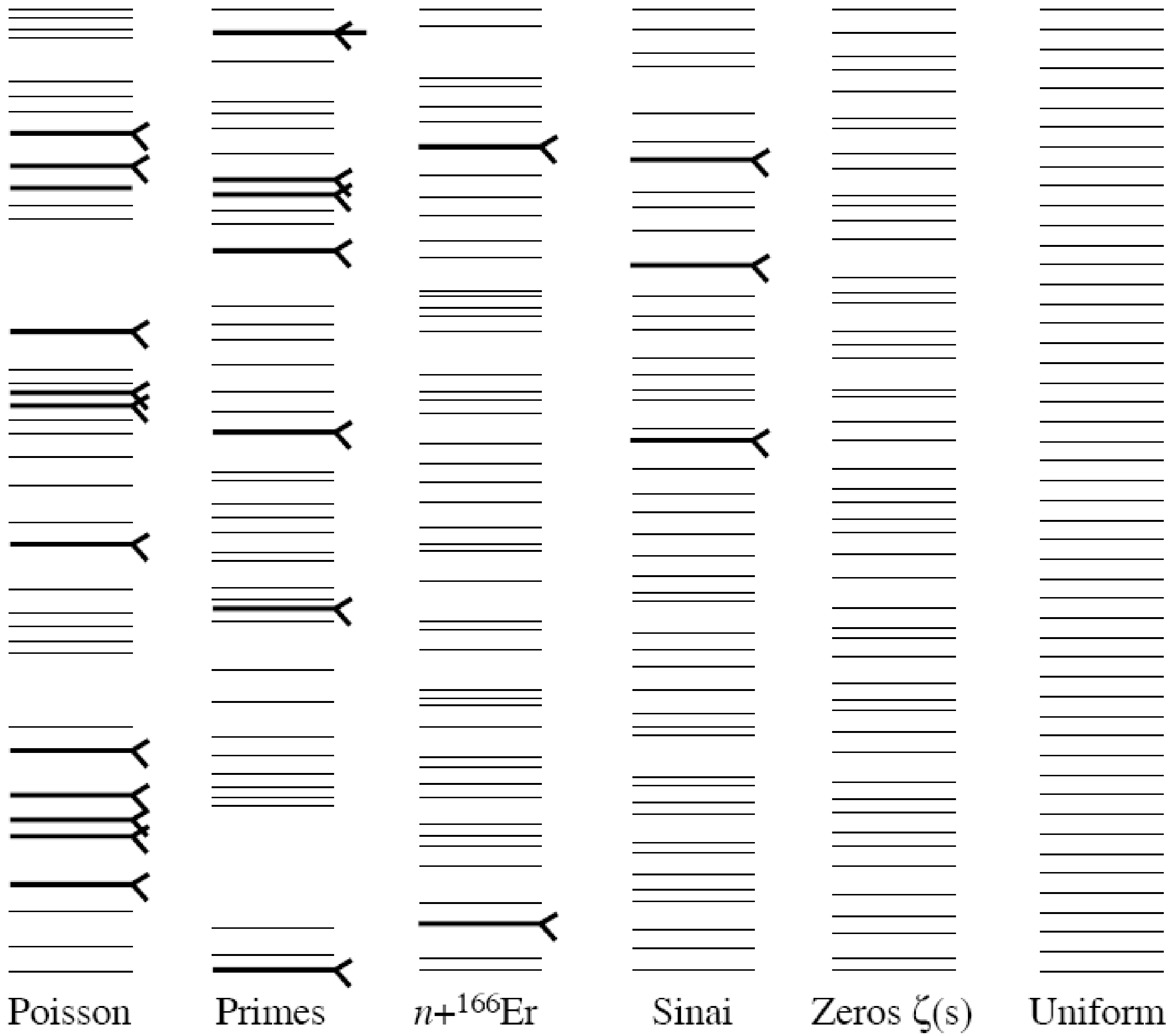


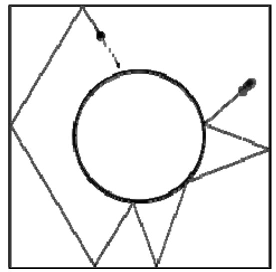
→ • All wave functions of disordered systems in 1D and 2D are localized.

This analysis does not apply to the QHS, since the extended states are crucial there.

→ • QHE belongs to a new class of disordered systems.

Lagendijk et al, Phys Today 2009



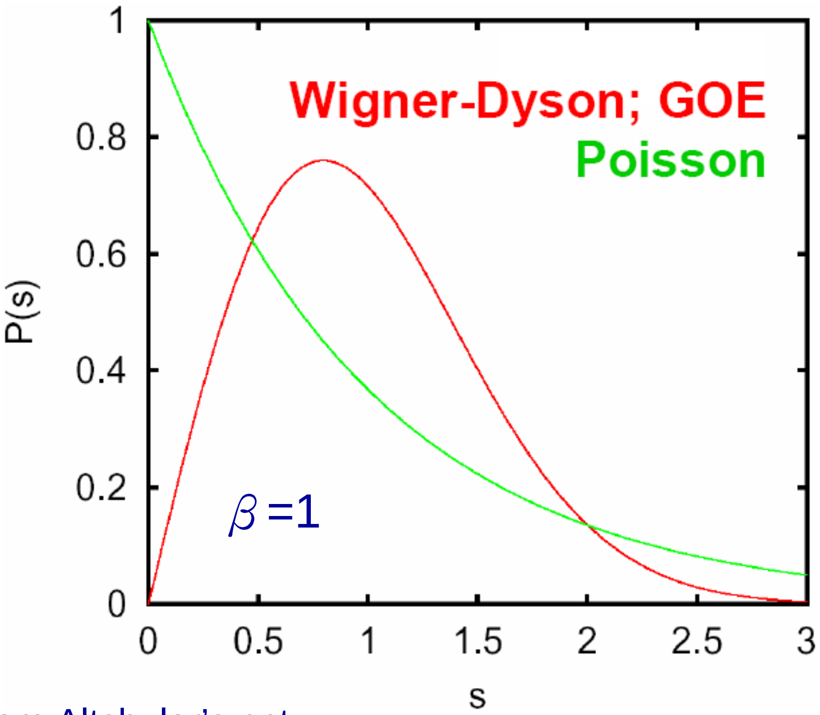


$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

"Random" Gaps. *The statistics of nearest-neighbor spacings range from random to uniform (<'s indicate spacings too close for the figure to resolve). The second column shows the primes from 7,791,097 to 7,791,877. The third column shows energy levels for an excited heavy (Erbium) nucleus. The fourth column is a "length spectrum" of periodic trajectories for Sinai billiards. The fifth column is a spectrum of zeroes of the Riemann zeta function. (Figure courtesy of Springer-Verlag New York, Inc., "Chaotic motion and random matrix theories" by O. Bohigas and M. J. Giannoni in* Mathematical and Computational Methods in Nuclear Physics, *J. M. Gomez et al., eds., Lecture Notes in Physics, volume 209 (1984), pp. 1–99.)*

Spectral distribution of random matrix (rank N>>1)

- eigenvalues E_i
- mean level spacing $d_1=\langle E_{i+1}-E_i\rangle$ (taking ensemble average)
- spacing between NN $s=(E_{i+1}-E_i)/d_1$
- $P(s)$: distribution function of s
- spectral rigidity: $P(0)=0$
- level repulsion: $P(s\ll 1) \sim s^{\beta}$



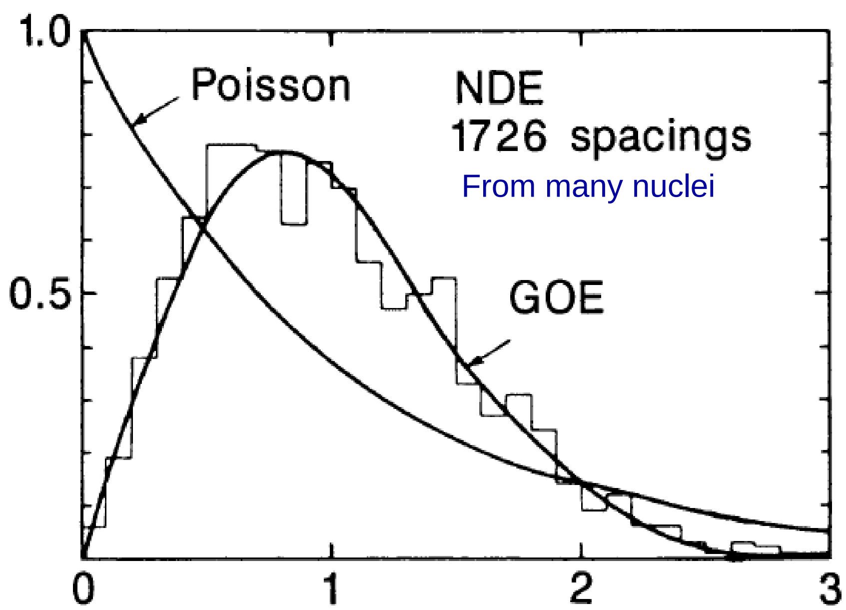


Fig from Altshuler's ppt

Wigner-Dyson classes

TABLE I. Summary of Dyson's threefold way. The Hermitian matrix $\mathcal{H}$ (and its matrix of eigenvectors U) are classified by an index $\beta \in \{1,2,4\}$, depending on the presence or absence of time-reversal (TRS) and spin-rotation (SRS) symmetry.

	β	TRS	SRS	$\mathcal{H}_{nm}$	U	**Altland-Zirnbauer classes**
GOE	1	yes	yes	real	orthogonal	AI
GUE	2	no	irrelevant	complex	unitary	A
GSE	4	yes	no	real quaternion	symplectic	AII

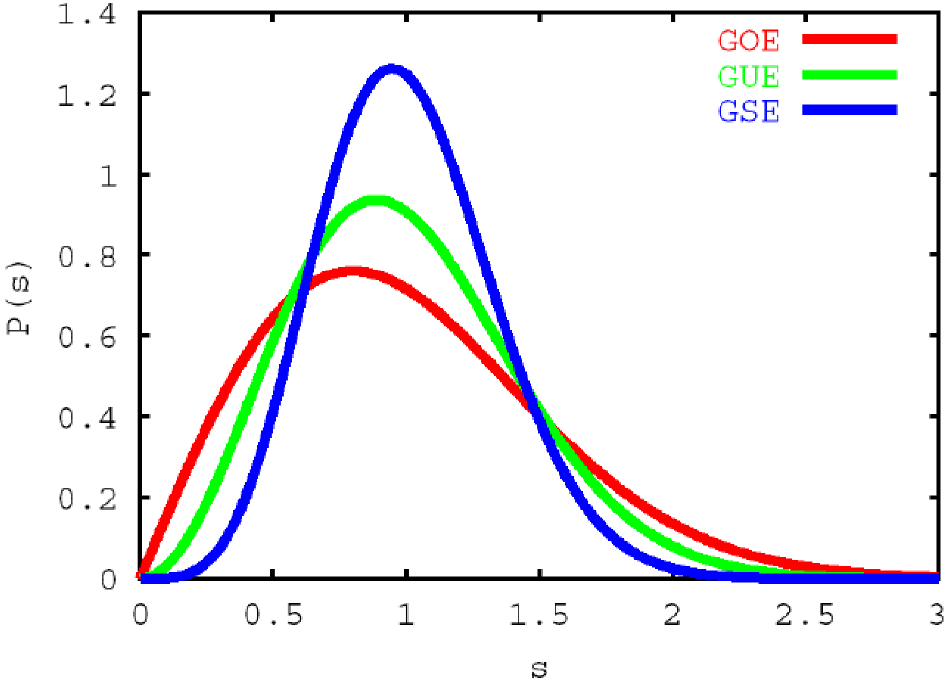
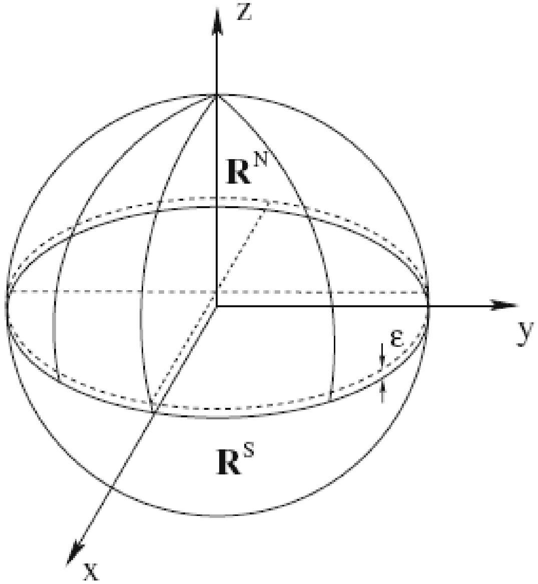


Fig from Altshuler's ppt

YM Shnir, Magnetic monopoles

Quantization of magnetic monopole

(see Sakurai Sec 2.6)



• Vector potential (use 2 "atlas" to avoid Dirac string)

$$\begin{cases} \mathbf{A}^N = \quad g\dfrac{1-\cos\theta}{r\sin\theta}\hat{\mathrm{e}}_\varphi & \Longrightarrow 0 \le \theta < \frac{\pi}{2} + \frac{\varepsilon}{2} \\ \mathbf{A}^S = -g\dfrac{1+\cos\theta}{r\sin\theta}\hat{\mathrm{e}}_\varphi & \Longrightarrow \frac{\pi}{2} - \frac{\varepsilon}{2} < \theta \le \pi \end{cases}$$

• gauge transformation between 2 atlas

$$\vec{A}^N - \vec{A}^S = -ie^{-2ig\varphi}\nabla e^{2ig\varphi}$$

$$\psi^N = \psi^S e^{2ieg/\hbar c\cdot\varphi}$$

$\rightarrow$ monopole charge is quantized

$$\frac{2eg}{\hbar c} = n$$

Note:

$$\nabla f = \hat{\boldsymbol{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\nabla\times\boldsymbol{u} = \hat{\boldsymbol{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(u_\phi\sin\theta) - \frac{\partial u_\theta}{\partial\phi}\right] + \hat{\boldsymbol{\theta}}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(ru_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right]$$

Analogy in QH system

- Gauge transformation

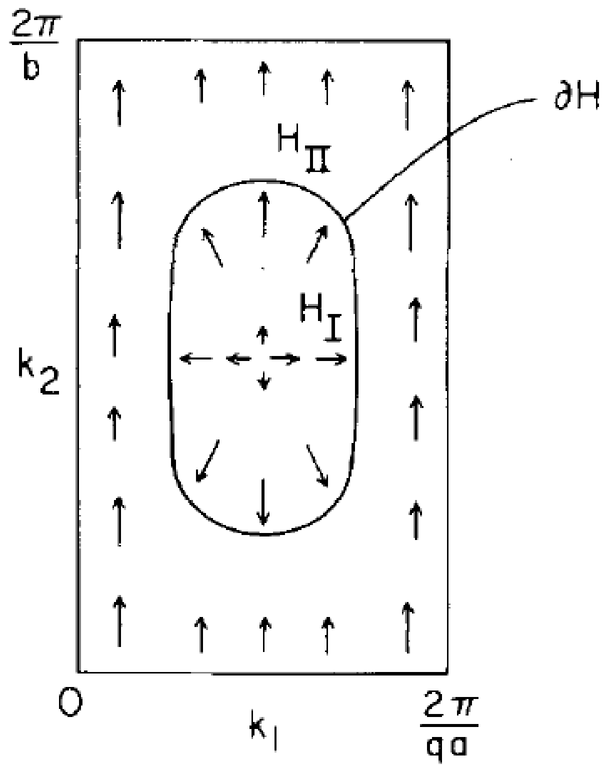
$$u'_{k_1k_2}(x, y) = u_{k_1k_2}(x, y)\exp[if(k_1, k_2)]$$

$$\hat{\mathbf{A}}'(k_1, k_2) = \hat{\mathbf{A}}(k_1, k_2) + i\nabla_k f(k_1, k_2)$$

- Two atlases

$$|u^{\mathrm{II}}_{k_1k_2}\rangle = \exp[i\chi(k_1, k_2)]|u^{\mathrm{I}}_{k_1k_2}\rangle$$

$$\hat{\mathbf{A}}_{\mathrm{II}}(k_1, k_2) = \hat{\mathbf{A}}_{\mathrm{I}}(k_1, k_2) + i\nabla_k \chi(k_1, k_2)$$



$$\sigma^{(\alpha)}_{xy} = \frac{e^2}{h}\frac{1}{2\pi i}\left\{\int_{H_I} d^2k[\nabla_k \times \hat{\mathbf{A}}_{\mathrm{I}}(k_1, k_2)]_3 + \int_{H_{\mathrm{II}}} d^2k[\nabla_k \times \hat{\mathbf{A}}_{\mathrm{II}}(k_1 k_2)]_3\right\}$$

$$= \frac{e^2}{h}\frac{1}{2\pi i}\int_{\partial H} d\mathbf{k}\cdot[\hat{\mathbf{A}}_{\mathrm{I}}(k_1, k_2) - \hat{\mathbf{A}}_{\mathrm{II}}(k_1, k_2)] = \frac{e^2}{h} n$$

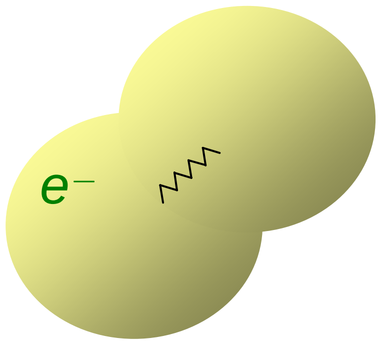
Kohmoto, Ann. Phys, 1985

Connection with Berry phase

First, a brief review of Berry phase:

- Fast variable and slow variable

  H^+_2 molecule



$$H(\vec{r},\vec{p};\{\vec{R}_i,\vec{P}_i\})$$

electron; {nuclei}

nuclei move thousands of times slower than the electron

Instead of solving time-dependent Schroedinger eq., one uses

Born-Oppenheimer approximation

- "Slow variables R_i" are treated as *parameters* $\lambda(t)$

(Kinetic energies from P_i are neglected)

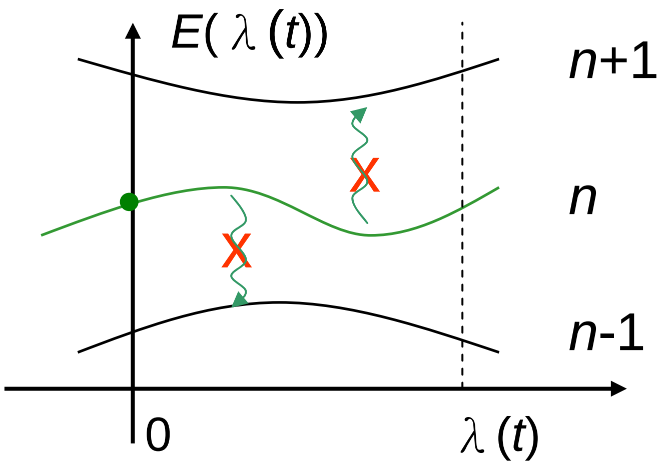
- solve time-*in*dependent Schroedinger eq.

$$H(\vec{r},\vec{p};\vec{\lambda})\psi_{n,\vec{\lambda}}(\vec{x})=E_{n,\vec{\lambda}}\psi_{n,\vec{\lambda}}(\vec{x})$$

"snapshot" solution

Adiabatic evolution of a quantum system $H(\vec{r}, \vec{p}; \vec{\lambda})$

• Energy spectrum:



• After a *cyclic* evolution

$$\vec{\lambda}(T) = \vec{\lambda}(0)$$

$$\psi_{n,\vec{\lambda}(T)} = e^{-\frac{i}{\hbar}\int_0^T dt' E_n(t')} \psi_{n,\vec{\lambda}(0)}$$

Dynamical phase

• Phases of the snapshot states at different λ's are independent and can be arbitrarily assigned

$$\psi_{n,\vec{\lambda}(t)} \rightarrow e^{i\gamma_n(\vec{\lambda})} \psi_{n,\vec{\lambda}(t)}$$

• Do we need to worry about this phase?

No!
- Fock, Z. Phys 1928
- Schiff, Quantum Mechanics (3rd ed.) p.290

Pf : Consider the *n*-th level,

$$\Psi_{\vec{\lambda}}(t) = e^{i\gamma_n(\vec{\lambda})} e^{-i\int_0^t dt' E_n(t')} \psi_{n,\vec{\lambda}}$$

Stationary, snapshot state

$$H\psi_{n,\vec{\lambda}} = E_n \psi_{n,\vec{\lambda}}$$

$$H\Psi_{\vec{\lambda}}(t) = i\hbar \frac{\partial}{\partial t} \Psi_{\vec{\lambda}}(t)$$

$$\Rightarrow \quad \dot{\gamma}_n = i\left\langle \psi_{n,\vec{\lambda}} \left| \frac{\partial}{\partial \vec{\lambda}} \right| \psi_{n,\vec{\lambda}} \right\rangle \cdot \dot{\vec{\lambda}} \neq 0$$

$\equiv \mathbf{A}_n(\lambda)$

Redefine the phase,

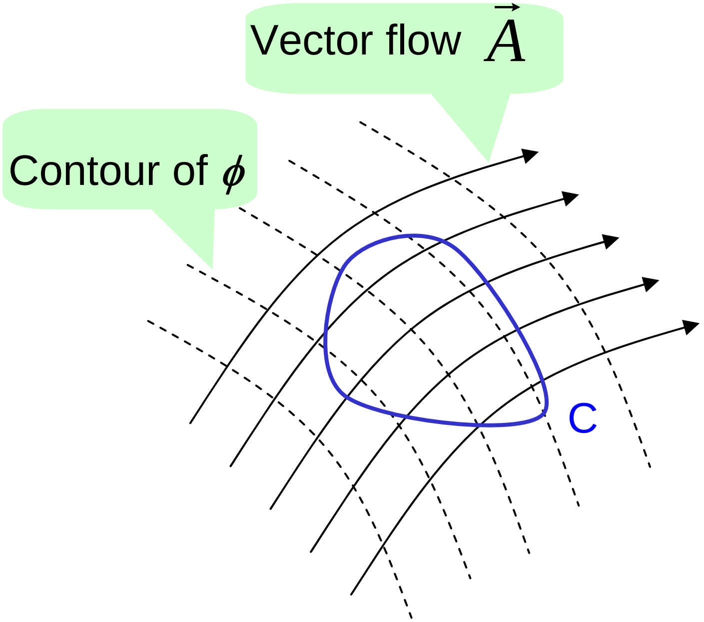
$$\psi'_{n,\vec{\lambda}} = e^{i\phi_n(\vec{\lambda})} \psi_{n,\vec{\lambda}}$$

$$\Rightarrow \quad \mathbf{A}_n'(\lambda) = \mathbf{A}_n(\lambda) - \frac{\partial \phi_n}{\partial \vec{\lambda}}$$

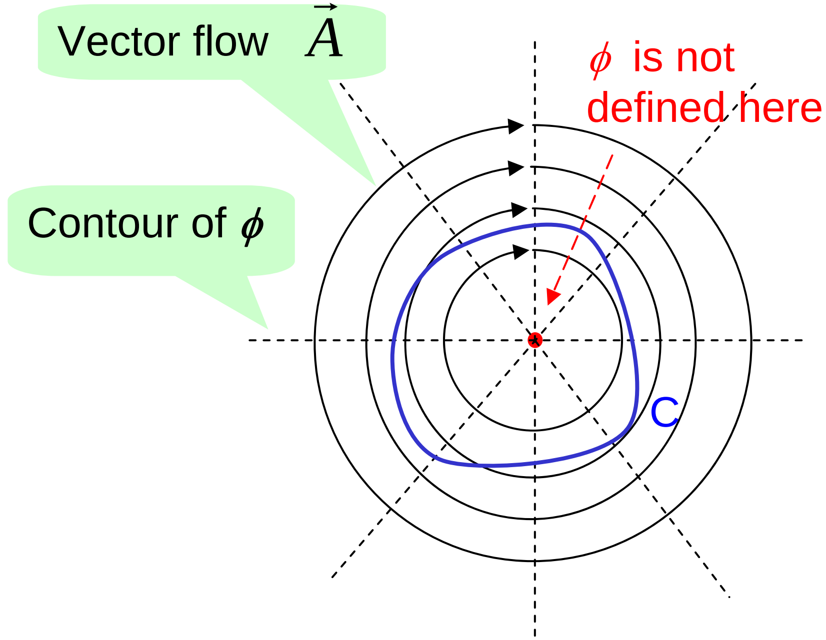
Choose a $\phi(\lambda)$ such that,

$\mathbf{A}_n'(\lambda) = 0$ Thus removing the extra phase

• One problem: $\nabla_{\vec{\lambda}}\phi = \vec{A}(\vec{\lambda})$ does not always have a well-defined (global) solution.



$$\oint_C \vec{A}\cdot d\vec{\lambda} = 0$$



$$\oint_C \vec{A}\cdot d\vec{\lambda} \neq 0$$

M. Berry, 1984 :

• Parameter-dependent phase $\psi_{\vec{\lambda}(T)} = e^{i\gamma_C} e^{-\frac{i}{\hbar}\int_0^T dt' E(t')} \psi_{\vec{\lambda}(0)}$ Index *n* neglected

NOT always removable!

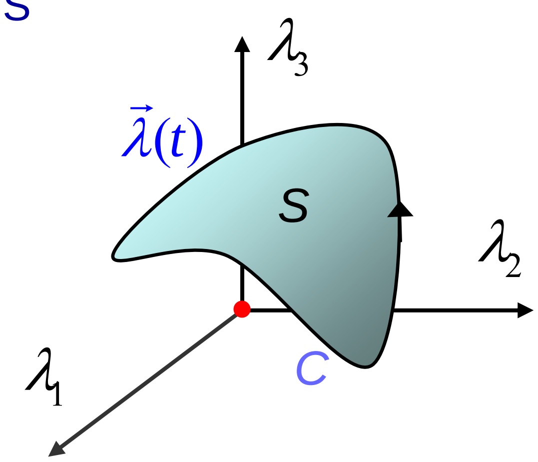
• Berry phase (path dependent) $\gamma_C = \oint_C \left\langle \psi_{\vec{\lambda}} \middle| i\frac{\partial}{\partial\vec{\lambda}} \middle| \psi_{\vec{\lambda}} \right\rangle \cdot d\vec{\lambda} \neq 0$

Some terminology

- **Berry connection** (or Berry potential)

$$\vec{A}(\vec{\lambda}) \equiv i\left\langle \psi_{\vec{\lambda}} \middle| \nabla_{\lambda} \middle| \psi_{\vec{\lambda}} \right\rangle$$

$\lambda \rightarrow$ k in QHS



- Stokes theorem (3-dim here, can be higher)

$$\gamma_C = \oint_C \vec{A} \cdot d\vec{\lambda} = \int_S \nabla_{\vec{\lambda}} \times \vec{A} \cdot d\vec{a}$$

- **Berry curvature** (or Berry field)

$$\vec{F}(\vec{\lambda}) \equiv \nabla_{\lambda} \times \vec{A}(\vec{\lambda}) = i\left\langle \nabla_{\lambda}\psi_{\vec{\lambda}} \middle| \times \middle| \nabla_{\lambda}\psi_{\vec{\lambda}} \right\rangle$$

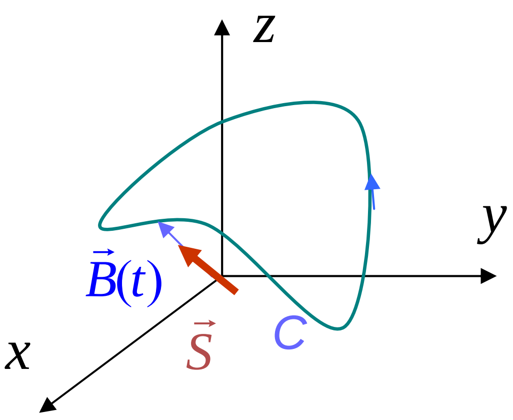
- Gauge transformation
  - $\left|\psi_{\vec{\lambda}}\right\rangle \rightarrow e^{i\phi(\vec{\lambda})}\left|\psi_{\vec{\lambda}}\right\rangle$
  - $\vec{A}(\vec{\lambda}) \rightarrow \vec{A}(\vec{\lambda}) - \nabla_{\lambda}\phi$
  - $\vec{F}(\vec{\lambda}) \rightarrow \vec{F}(\vec{\lambda})$
  - $\gamma_C \rightarrow \gamma_C$

Redefine the phases of the snapshot states

Berry curvature and Berry phase are gauge invariant

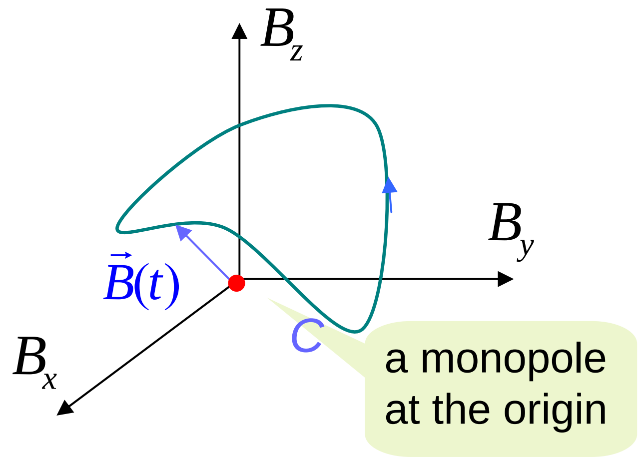
Example: spin-1/2 particle in slowly changing *B* field

- Real space

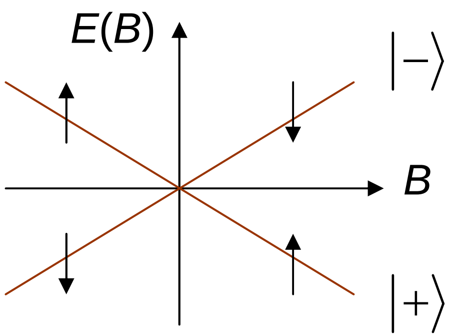


$$H_{\vec{\lambda}=\vec{B}} = \mu_B \vec{B}\cdot\vec{\sigma}$$

- Parameter space



Level crossing at *B*=0



Berry curvature

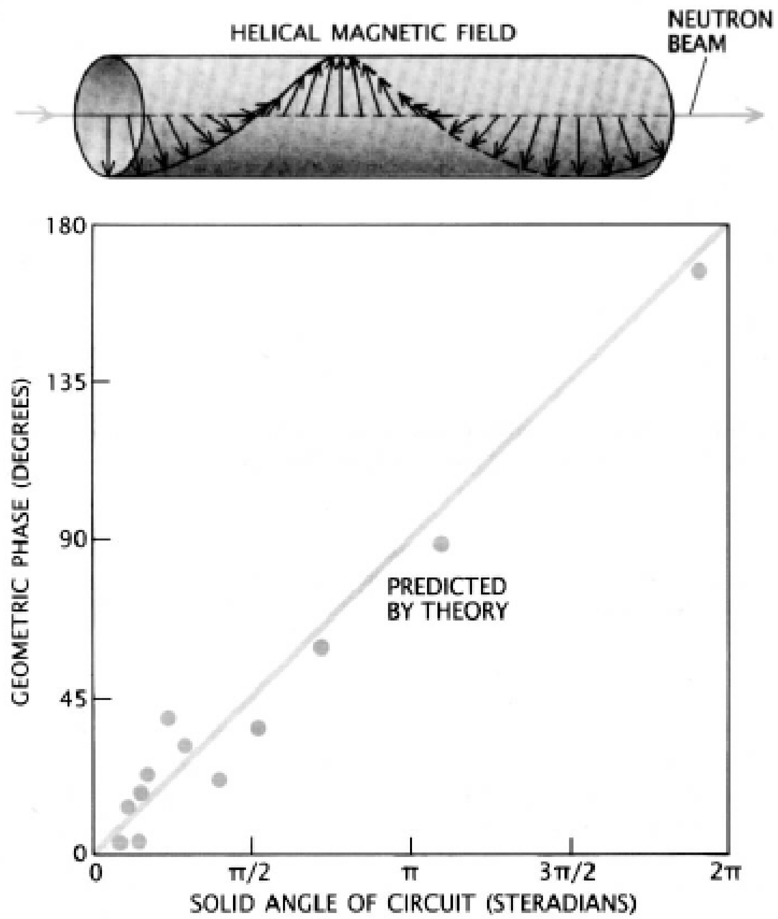
$$\vec{F}_\pm(\vec{B}) = i\left\langle \nabla_B \psi_{\pm,\vec{B}} \middle| \times \middle| \nabla_B \psi_{\pm,\vec{B}} \right\rangle = \mp \frac{1}{2}\frac{\hat{B}}{B^2}$$

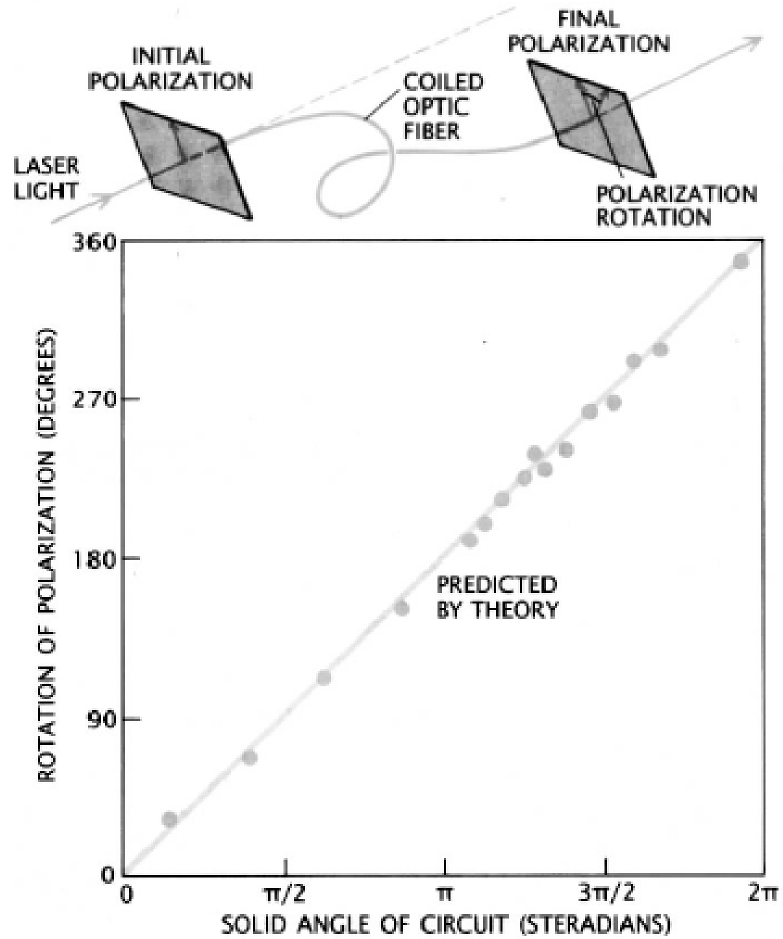
Berry phase

$$\gamma_\pm = \int_S \vec{F}_\pm \cdot d\vec{a} = \mp \frac{1}{2}\Omega(C)$$

spin × solid angle

Examples of the Berry phase:





Magnetic monopole / Berry phase / fiber bundle

in real space	in parameter space	U(1) fiber bundle
Vector potential	Berry connection	connection
$\vec{A}(\vec{r})$	$\vec{A}(\vec{k}) \equiv i\langle\psi_\lambda\|\nabla_\lambda\|\psi_\lambda\rangle$	A
Magnetic field	Berry curvature (in 3D)	curvature
$\vec{B}(\vec{r}) \equiv \nabla\times\vec{A}(\vec{r})$	$\vec{F}(\vec{\lambda}) \equiv \nabla_\lambda\times\vec{A}(\vec{\lambda})$	F
Magnetic flux	Berry phase	horizontal lift (along a U(1) fiber)
$\Phi = \int_C \vec{A}(\vec{r})\cdot d\vec{r} = \int_S \vec{B}\cdot d\vec{a}$	$\gamma_C = \int_C \vec{A}(\vec{\lambda})\cdot d\vec{\lambda} = \int_S \vec{F}\cdot d\vec{a}$	γ
Monopole charge	Total curvature	1st Chern number
$\frac{1}{4\pi}\int\vec{B}(\vec{r})\cdot d\vec{a} = \text{integer}$	$\frac{1}{2\pi}\int\vec{F}(\vec{\lambda})\cdot d\vec{a} = \text{integer}$	C_1
	(QHE: $\lambda\rightarrow$ **k** in BZ)	

Connection with geometry

First, a brief review of topology:

外在 • extrinsic curvature K vs

內在 • intrinsic (Gaussian) curvature G

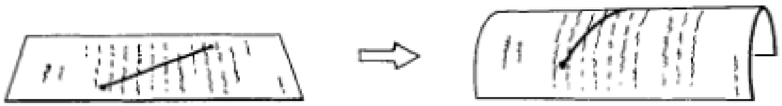
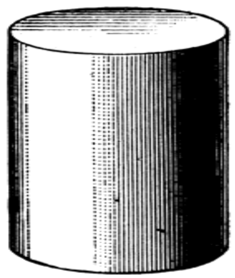
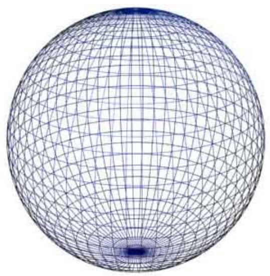


Figure 3.6 Bending a sheet of paper changes its extrinsic— but not its intrinsic—geometry.



$K \neq 0$

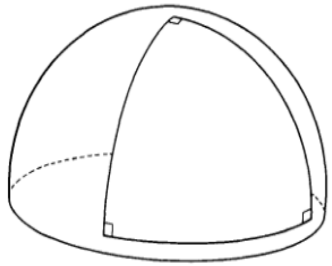
$G=0$



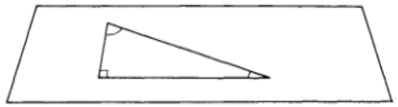
$K \neq 0$

$G \neq 0$

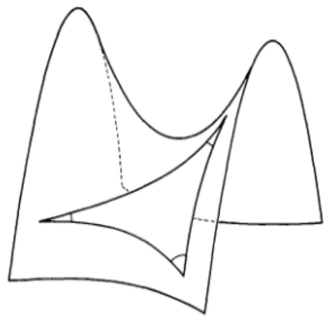
• Positive and negative Gaussian curvature



$G>0$



$G=0$



$G<0$

• **anholonomy angle** α = 內角和-180

• Gaussian curvature $G \equiv \lim_{A\to 0}\frac{\alpha}{A} = \frac{1}{R^2}$

• Berry phase ≒ *anholonomy angle* in differential geometry

• Berry curvature ≒*Gaussian curvature*

The most beautiful theorem in differential topology

- Gauss-Bonnet theorem (for a 2-dim closed surface)

$$\int_M da\, G = 2\pi\chi(M), \quad \chi = 2(1-g)$$

Euler characteristic
歐拉特徵數

$g = 0$

$g = 1$

$g = 2$

- Gauss-Bonnet theorem (for a surface with boundary)

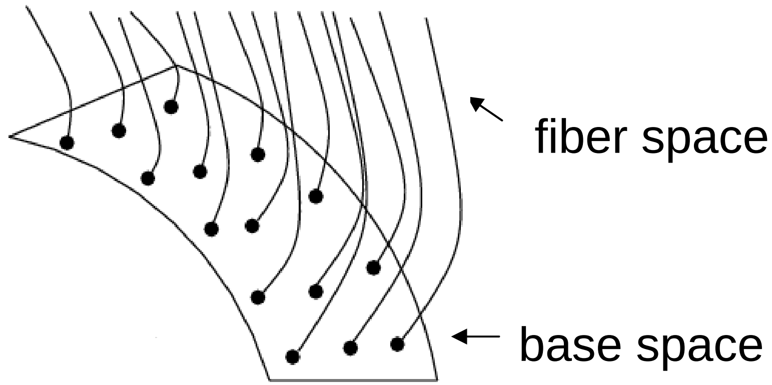
$$\int_M da\, G + \int_{\partial M} ds\, k_g = 2\pi\, \chi(M, \partial M)$$

- Can be generalized to higher dimension.

Marder, Phys Today, Feb 2007

Fiber bundle: a generalization of product space

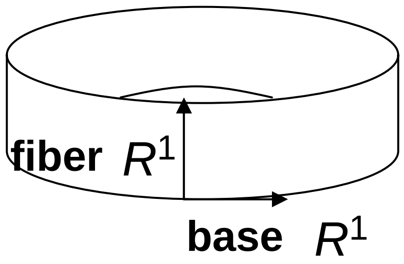
• Fiber bundle
~ base space × fiber space



Simplest examples:

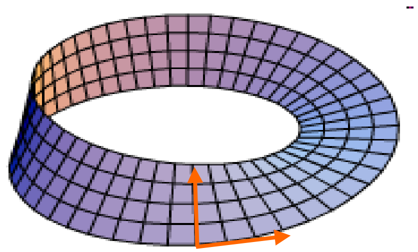
• **Trivial** fiber bundle

(a product space R^1 x R^1)



• **Nontrivial** fiber bundle

Möbius band



• In physics, a fiber bundle ~ Physical space × Inner space
(spin, gauge field…)

• In QHS, we have T^2 x U(1)

• The topology of a fiber bundle is classified by Chern numbers

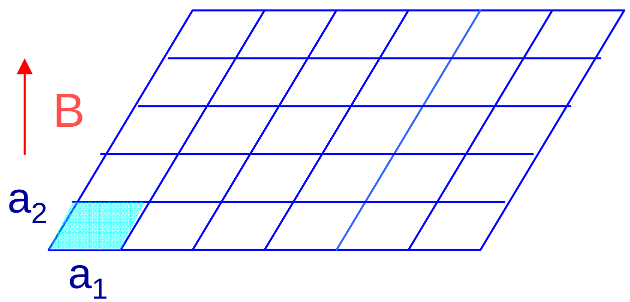
∼the topology of a closed surface is classified by Euler characteristics

Lattice electron in a magnetic field: magnetic translation symmetry

consider a uniform B field

$$\left\{\frac{1}{2m}[\boldsymbol{p}+e\mathbf{A}(\boldsymbol{r})]^2+V_L(\boldsymbol{r})\right\}\psi(\boldsymbol{r})=E\psi(\boldsymbol{r})$$

$$\rightarrow \left\{\frac{1}{2m}[\boldsymbol{p}+e\mathbf{A}(\boldsymbol{r}+\boldsymbol{a})]^2+V_L(\boldsymbol{r})\right\}\psi(\boldsymbol{r}+\boldsymbol{a})=E\psi(\boldsymbol{r}+\boldsymbol{a})$$



where $V_L(\boldsymbol{r}+\boldsymbol{a})=V_L(\boldsymbol{r})$ has been used. One can write

$$\mathbf{A}(\boldsymbol{r}+\boldsymbol{a})=\mathbf{A}(\boldsymbol{r})+\nabla f(\boldsymbol{r}),$$

where $\nabla f(\boldsymbol{r})=\mathbf{A}(\boldsymbol{r}+\boldsymbol{a})-\mathbf{A}(\boldsymbol{r})\equiv\Delta\mathbf{A}(\boldsymbol{a})$.

$f=\Delta\mathbf{A}\cdot\boldsymbol{r}$

Indep of r

The extra vector potential ∇f can be removed by a gauge transformation,

$$\left\{\frac{1}{2m}[\boldsymbol{p}+e\mathbf{A}(\boldsymbol{r})]^2+V_L(\boldsymbol{r})\right\}e^{i(e/\hbar)f}\psi(\boldsymbol{r}+\boldsymbol{a})=Ee^{i(e/\hbar)f}\psi(\boldsymbol{r}+\boldsymbol{a}).$$

- Magnetic translation operator

$$T_{\boldsymbol{a}}\psi(\boldsymbol{r})=e^{i(e/\hbar)\Delta\mathbf{A}\cdot\boldsymbol{r}}\psi(\boldsymbol{r}+\boldsymbol{a})$$

$$[H,T_{\boldsymbol{a}}]=0$$

$$T_{\boldsymbol{a}_2}T_{\boldsymbol{a}_1}=T_{\boldsymbol{a}_1}T_{\boldsymbol{a}_2}\exp\left(i\frac{e}{\hbar}\oint\mathbf{A}\cdot d\boldsymbol{r}\right)$$

Commute if this is 1

Xiao et al, RMP 2010

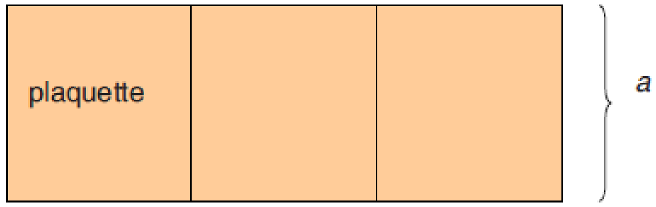
Simultaneous eigenstates:

magnetic Bloch states

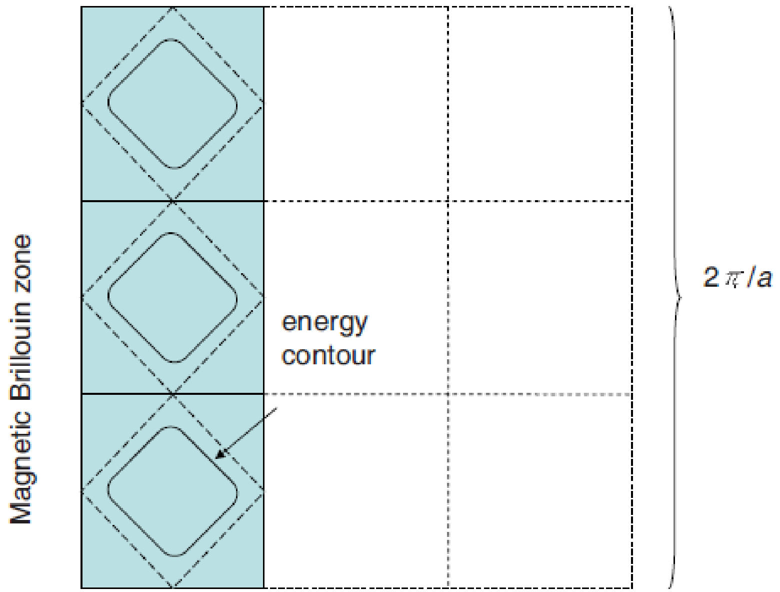
$$\begin{cases} H\psi_{nk} = E_{nk}\psi_{nk}, \\ T_{qa_1}\psi_{nk} = e^{ik\cdot qa_1}\psi_{nk}, \\ T_{a_2}\psi_{nk} = e^{ik\cdot a_2}\psi_{nk}. \end{cases}$$

• If Φ=(p/q)Φ_0 per plaquette, then Magnetic Brillouin zone = BZ/q.

e.g., p/q=1/3



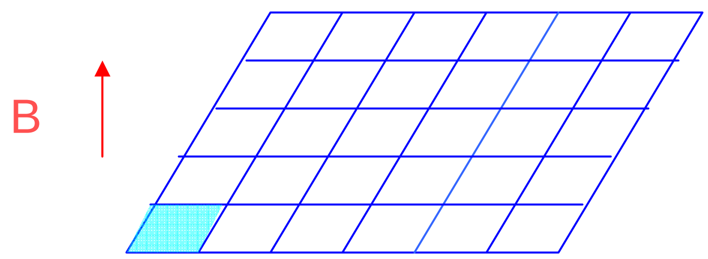
Magnetic unit cell



Hofstadter spectrum

Band structure of a 2DEG subjects to both a periodic potential V(x,y) and a magnetic field B.

$$\left\{\frac{1}{2m}[\boldsymbol{p}+e\mathbf{A}(\boldsymbol{r})]^2+V_L(\boldsymbol{r})\right\}\psi(\boldsymbol{r})=E\psi(\boldsymbol{r})$$



☐ Can be studied using the tight-binding model (TBM).

☐ Surprisingly complex spectrum!

Split of energy band depends on flux/plaquette.

If Φ_{plaq}/Φ_0= p/q, where p, q are co-prime integers, then a Bloch band splits to q subbands (for TBM).

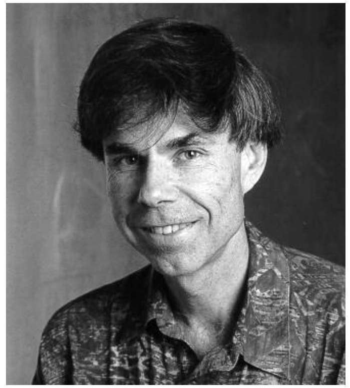
☐ The tricky part:

$$\frac{1}{3-\frac{1}{10}}=\frac{10}{\boxed{29}}=\frac{1}{\boxed{3}}+\frac{1}{87}$$

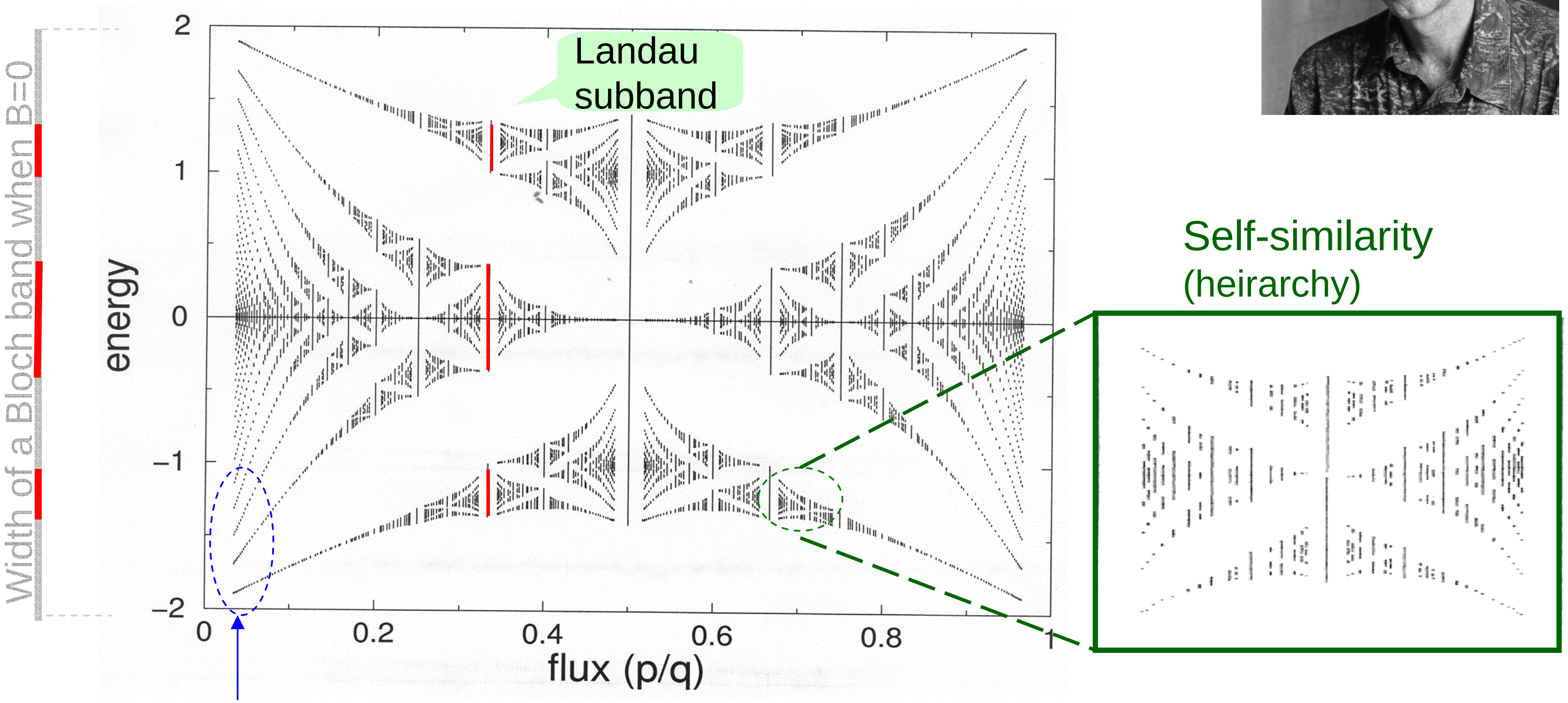
q=3 → q=29 upon a small change of B!

Also, when B → 0, q can be very large.

Hofstadter's butterfly (Hofstadter, PRB 1976)



• A fractal spectrum with self-similarity structure

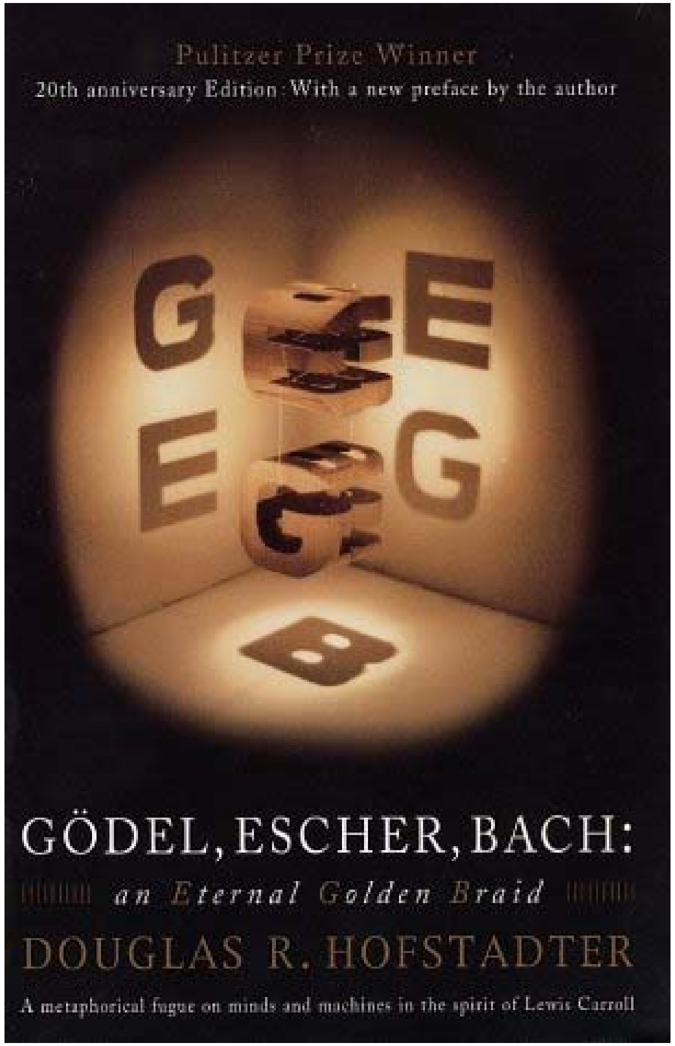


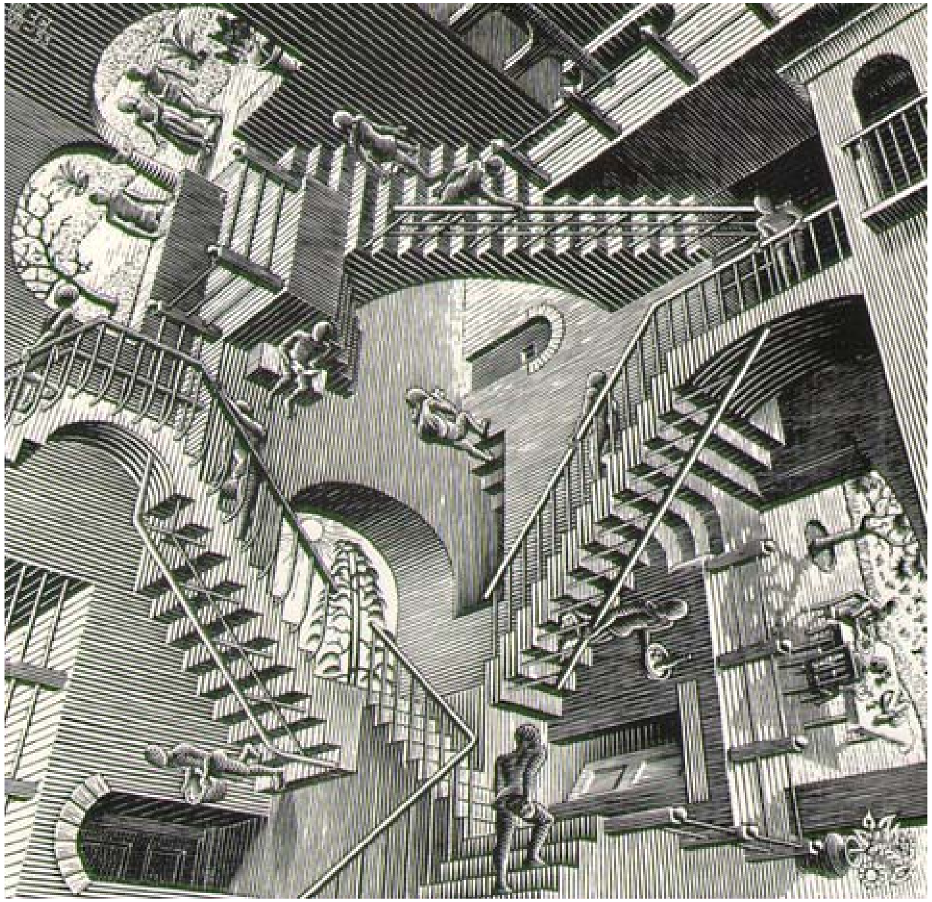
B $\rightarrow$ 0 near band button, evenly-spaced LLs

• The total band width for an irrational q is of measure zero (as in a Cantor set).

Pulitzer 1980

MIT: http://ocw.mit.edu/high-school/courses/godel-escher-bach/

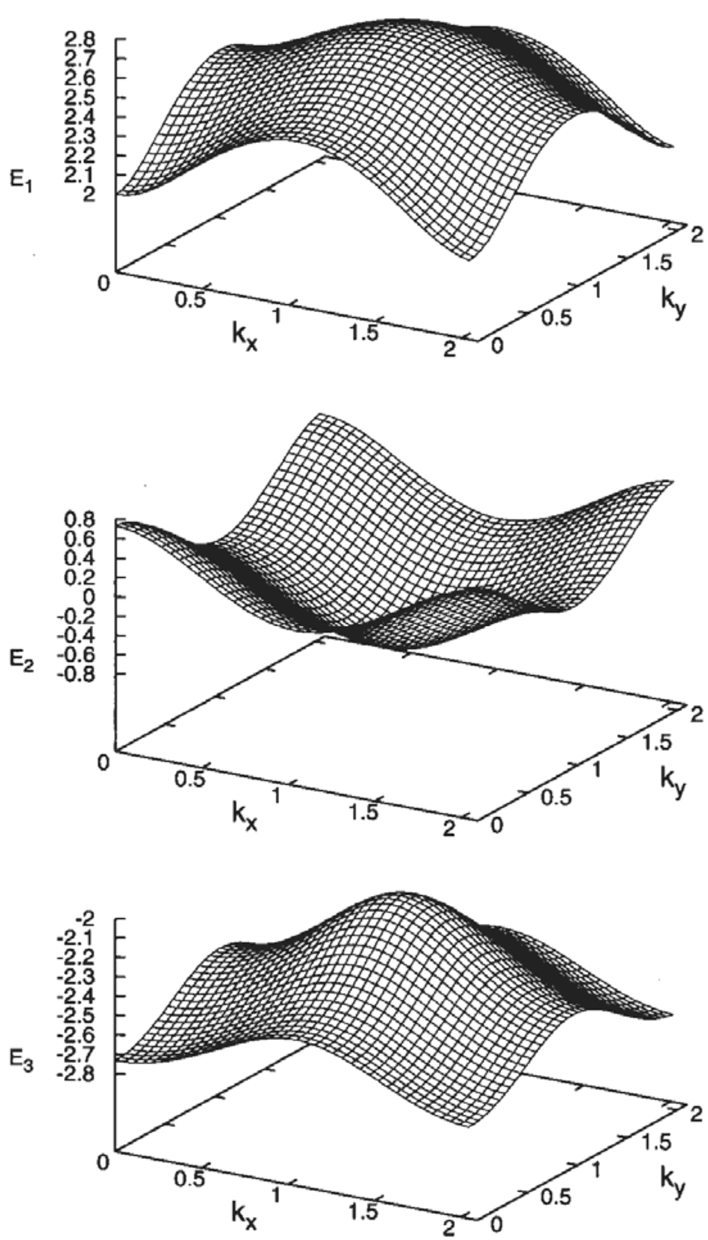


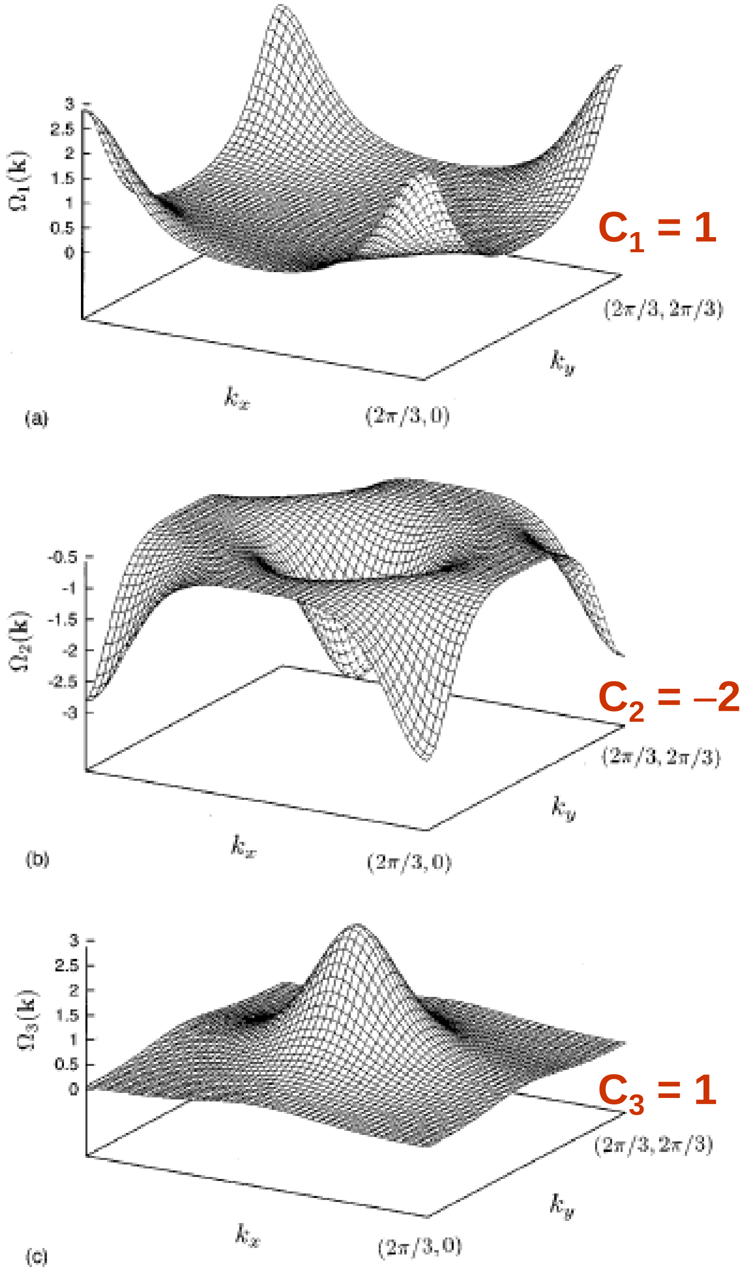


集異璧

著作：Douglas R. Hofstadter
翻譯：郭維德

p/q=1/3 Bloch energy $E(k)$ Berry curvature $\Omega(k)$





Distribution of Hall conductance among subbands

(Thouless et al PRL 1982)

- Diophantine equation

$$r = pt_r + qs_r$$

$$\text{e.g.,}\quad p/q = 2/5$$

$$r = 2t_r + 5s_r$$

$$5 = 2(0) + 5(1)$$

$$4 = 2(2) + 5(0)$$

$$3 = 2(-1) + 5(1)$$

$$2 = 2(1) + 5(0)$$

$$1 = 2(-2) + 5(1)$$

$$0 = 2(0) + 5(0)$$

- Streda formula

$$\sigma_H = ec\frac{\partial n}{\partial B}$$

$$n = \frac{N_{tot}}{A_{tot}}\frac{r}{q} = \frac{1}{A_{plaq}}\frac{r}{q}$$

$$\frac{p}{q} = \frac{BA_{plaq}}{hc/e}$$

$$\therefore \sigma_H = t_r$$

p/q	$(\sigma_1, \sigma_2, \ldots, \sigma_q)$ in units of e^2/h
1/3	$(1, -2, 1)$
2/3	$(-1, 2, -1)$
1/4	$(1, 1, -3, 1)$
1/5	$(1, 1, -4, 1, 1)$
2/5	$(-2, 3, -2, 3, -2)$

- for rectangular lattice:
s_r should be as small as possible

- for triangular lattice:
s_r and t_r cannot both be odd
(Thouless, Surf Sci 1984)

- for weak magnetic field:
$(\sigma_H)_r = t_r - t_{r-1}$

- for strong magnetic field:
$(\sigma_H)_r = s_r - s_{r-1}$

See Xiao et al RMP 2010 for another derivation

Jump of Hall conductance induced by band-crossing

p/q	$(\sigma_1, \sigma_2, \ldots, \sigma_q)$, in units of e^2/h		
1/3	$(1,-2,1)$	0.267949 $\rightarrow$	$(1,1,-2)$
		1.0 $\rightarrow$	$(1,1,-2)$
2/3	$(-1,2,-1)$	0.267949 $\rightarrow$	$(2,-1,-1)$
		1.0 $\rightarrow$	$(2,-1,-1)$
1/4	$(1,1,-3,1)$	0.382683 $\rightarrow$	$(1,1,1,-3)$
		0.707107 $\rightarrow$	$(1,1,1,-3)$
		0.92388 $\rightarrow$	$(1,-3,5,-3)$
1/5	$(1,1,-4,1,1)$	0.21296 $\rightarrow$	$(1,1,1,-4,1)$
		0.432325 $\rightarrow$	$(1,1,1,1,-4)$
		0.618034 $\rightarrow$	$(1,1,1,1,-4)$
		0.685096 $\rightarrow$	$(1,1,-4,6,-4)$
2/5	$(-2,3,-2,3,-2)$	0.413418 $\rightarrow$	$(-2,3,3,-2,-2)$
		0.618034 $\rightarrow$	$(-2,3,3,-2,-2)$
		0.743729 $\rightarrow$	$(-2,3,3,-7,3)$

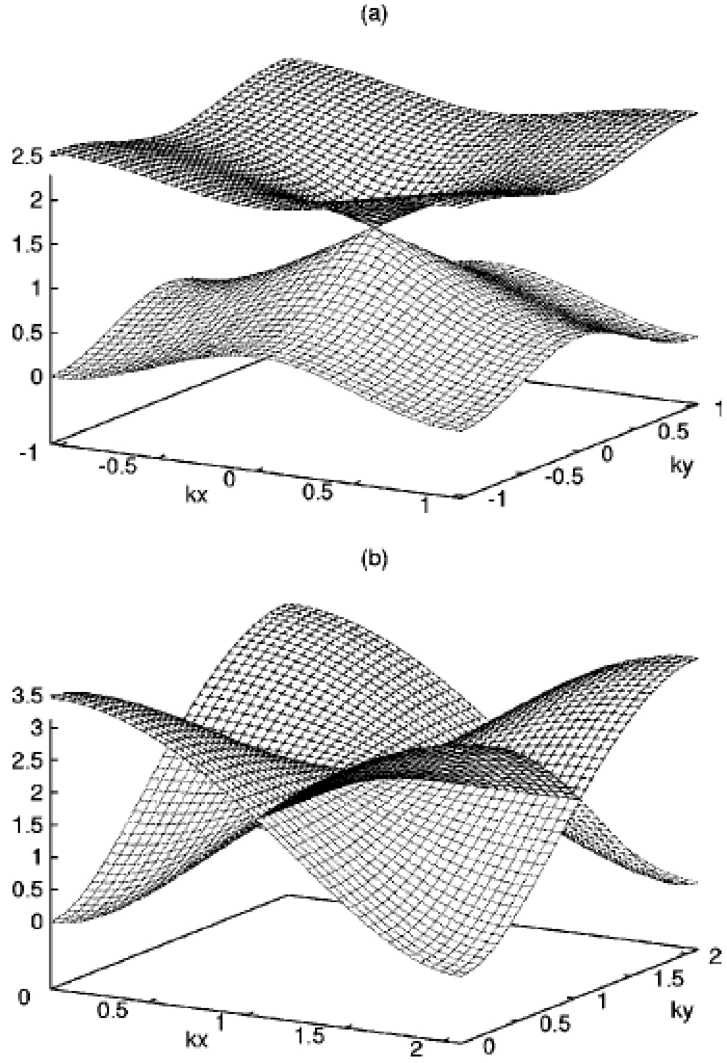
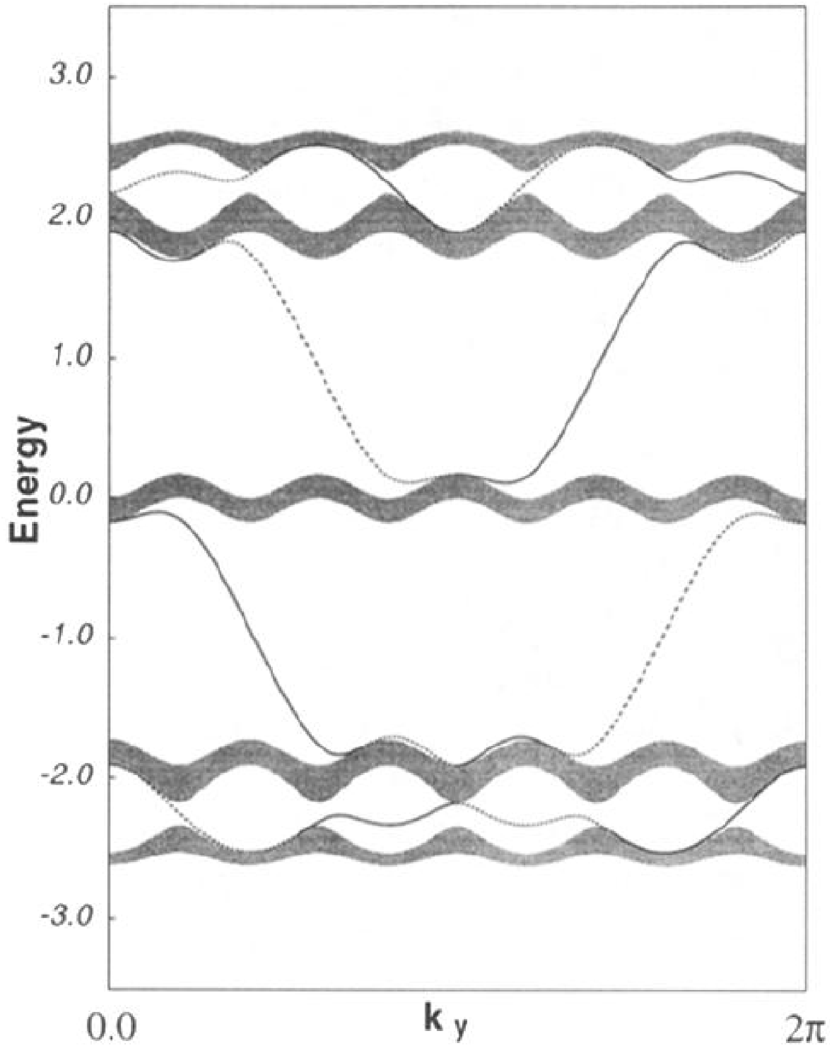


FIG. 3. The local energy dispersion $\mathcal{E}_n(\mathbf{k})$ near the degenerate point at (a) $t^*_{xy} \approx 0.267\,949$ and (b) $t^*_{xy} = 1.0$ (for a magnetic flux $\phi = 1/3$). Because of the threefold degeneracy in a MBZ, only one-third of the MBZ is shown.

Lattice with edges

- Energy dispersion of edge states



Φ=2/5

Hatusgai, J Phys 1997