Quantum Mechanics 6/9/2014 (9:30-11:30)

1. (30%) A 2-state system has the following Hamiltonian, $H_0 = B\sigma_z$, where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(a) Find out the two energy eigenstates χ_+, χ_- and eigenvalues $\varepsilon_+, \varepsilon_-$ $(\varepsilon_+ > \varepsilon_-)$.

(b) At t=0, a particle is in state χ_-. At t>0, it is subject to a perturbation $H' = b\sin(\Omega t)\sigma_x$,

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Find out the probability $P_+(t)$, accurate to first order, of finding the

particle in state χ_+ at time t. (Hint: $c_b^{(1)}(t) = -\frac{i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt'$, where $\hbar\omega_0 = \varepsilon_+ - \varepsilon_-$)

(c) Plot $P_+(t)$ as a function of t. What's the *period* and the *maximum value* of $P_+(t)$?

2. (45%) (a) Explain briefly, but clearly (use figures if necessary) the following two terms: "stimulated emission" and "spontaneous emission".
(b) Comparing the following two formulas (one derived by Einstein, the other is the Planck formula),

$$\rho(\omega) = \frac{A}{e^{\hbar\omega/k_B T}B_{ab} - B_{ba}} \quad vs \quad \rho(\omega) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\hbar\omega/k_B T} - 1},$$

what mathematical relations between the A, B coefficients can we obtain? Briefly explain the meaning of these relations.
(c) Explain briefly, but clearly (use figures if necessary), about the Aharonov-Bohm phase of an electron circling a long solenoid with a magnetic flux Φ. *No derivation is required.*

3. (25%) A 2-state system has the following Hamiltonian, $H = \vec{B}\cdot\vec{\sigma}$, where

$\vec{B} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $\vec{\sigma}$ are the Pauli matrices. One of the eigenstate is

$$\chi_- = \begin{pmatrix} \sin\theta/2 \\ -e^{i\phi}\cos\theta/2 \end{pmatrix}, \quad \text{or} \quad \chi'_- = \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \quad \text{[both choices are valid]}$$

(a) Find out $\vec{A}_- = i\langle\chi_-|\nabla\chi_-\rangle$. Hint: $\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{\partial}{r\partial\theta}\hat{\theta} + \frac{\partial}{r\sin\theta\partial\phi}\hat{\phi}$

(b) Another choice of eigenstate gives $\vec{A}'_- = i\langle\chi'_-|\nabla\chi'_-\rangle$, which differs from $\vec{A}_-$ by a gradient term: $\vec{A}'_{-} = \vec{A}_{-} + \nabla\Lambda$. Find out the function Λ.