Quantum Mechanics Final Exam (9:10-11:00, 1/21/2005)

1. [30%] Consider a 1-dim system with two particles having masses m_1 and m_2.
   (a) Assume the wave function is $y(x_1,x_2)=\exp(ik_1x_1)\exp(ik_2x_2)$, show that the wave function remains "separable" in center-of-mass and relative coordinate (X,x).
   (b) Following (a), if these two particles are fermions, how would the wave function be modified? Is it an eigenstate of the total momentum operator? If it is, find out the eigenvalue; if not, explain why.

2. [30%] The propagator is defined as $K(x,t;x',0)=\langle x,t|x',0\rangle$ and can be calculated by using $\langle x|U(t)|x'\rangle$, in which the evolution operator depends on the Hamiltonian of the system being considered.
   (a) For the simplest case, calculate the propagator for *a free particle in one dimension*. (see the bottom of this page for the formula of Gaussian integration)
   (b) The initial state of a particle is $y(x,0)=e^{ik_0x}$. Use the propagator obtained above to find the state $y(x,t)$ at time *t*.

3. [40%] A coherent state is defined as $|z\rangle=\exp(-|z|^2/2)\exp(za^+)|0\rangle$, where z is a complex number and a^+ is the raising operator.
   (a) Show that the coherent state satisfies $a|z\rangle=z|z\rangle$.
   (b) Find out the inner product between two different coherent states $\langle z_2|z_1\rangle$.
   (c) Given the Hamiltonian $H=(a^+a+1/2)\hbar\omega$, an initial coherent state $|z\rangle$ given

      above will evolve to another coherent state $|z(t)\rangle$ that satisfies $a|z(t)\rangle=z(t)|z(t)\rangle$. Find out the function z(t).
   (d) In homework, we have shown that $|z\rangle=\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle$, where $|n\rangle$ is the eigenstate of a simple harmonic oscillator. Use this relation to show that

$$\int\frac{dxdy}{p}|z\rangle\langle z|e^{-z^*z}=\sum_{n=0}^{\infty}|n\rangle\langle n|=I,$$

      where $z=x+iy$, and the integration is over the whole *x*-*y* plane.

   Hint: If $[A,B]$ commutes with both *A* and *B*, then $e^Ae^B=e^{A+B}e^{[A,B]/2}$.
   If $[A,B]=\alpha B$, then $e^ABe^{-A}=e^{a}B$.
   Also, $\int_{-\infty}^{\infty}dx\exp(-ax^2)=\sqrt{p/a}$; $\int_0^{\infty}dx x^n e^{-x}=n!$