1. [20%] In general, the operator $U$ of a continuous (unitary) transformation can be written in the form $U = \exp(-iGa)$, where $G$ is the generator of the transformation and $a$ is some parameter. (For simplicity, let’s assume there is only one parameter.) Show that, if the Hamiltonian $H$ is invariant under the transformation $U$, then $G$ is a conserved quantity.

2. [30%]
   (a) Consider a system with angular momentum $j=1$. Explicitly write $\langle j=1, m' | J_z | j=1, m \rangle$ as a $3 \times 3$ matrix.
   (b) Expand $\exp(-i?J_x/\hbar)$ as a polynomial of $J_x$ with finite number of terms.
   (c) Use (a) and (b) to express $\exp(-i?J_x/\hbar)$ as a $3 \times 3$ matrix.

3. [30%] For a spin-1/2 particle, the spin-up state (along $z$-axis) is $|+\rangle = (1,0)^T$.
   (a) With the help of the rotation operators $U_y$ and $U_z$, find out the state $|\hat{n}, +\rangle$ that is spin-up along the direction $\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$.
   (b) Following (a), find out the expectation values $\langle S_x \rangle$ and $\langle S_y \rangle$ of such a state.
   (c) Assume the initial state of a particle is $|\hat{n}, +\rangle$ at $t=0$. We then apply an uniform magnetic field $B$ along the $z$ direction. Calculate the expectation value $\langle S_z \rangle$ at time $t$.

4. [20%] Consider a system made up of two spin 1/2 particles. Observer A specializes in measuring the spin components of particle-1, while observer B measures the spin component of particle-2. Suppose the system is in a spin-singlet state with $S_{\text{tot}}=0$.
   (a) What is the probability for A to obtain $S_{1z} = \hbar/2$ when B makes no measurement? Repeat the same problem for $S_{1x} = \hbar/2$.
   (b) B determines the spin of particle-2 to be $S_{2z} = \hbar/2$. What can we conclude about the outcome of A’s measurement if A measures $S_{1z}$? Repeat the same problem, what if A measures $S_{1x}$?