Quantum Mechanics Final Exam (9:10-11:00, 6/15/2005)

1. [30%] In the WKB approximation, we write the wave function in the form $y(x) = r(x)\exp(i\tilde{f}(x)) = \exp(if(x))$, where $f(x) = \tilde{f}(x) - i\ln r(x)$.
   (a) Substitute this into the Schroedinger equation and find out the zeroth order solution $f_0(x)$.
   (b) When solving for $f_0(x)$ in (a), we assumed that $f(x)$ is a very smooth function. Under what condition can we say that the first order term is negligible? (express your answer using $f'(x), f''(x)\ldots$ etc.)

2. [30%] If the finite size of the proton is taken into consideration, the potential for an electron in a hydrogen atom is (R is the radius of the proton)

$$V(\vec{r}) = \begin{cases} -\dfrac{e^2}{R}\left(\dfrac{3}{2} - \dfrac{1}{2}\dfrac{r^2}{R^2}\right) & \text{for } 0 \le \text{r} \le \text{R}, \\ -\dfrac{e^2}{r} & \text{for r} \ge \text{R}. \end{cases}$$

   (a) Write down explicitly the perturbation potential V' to the Coulomb potential of a point charge source.
   (b) Find out the first order energy shift for the ground state of the hydrogen with

$$\Psi_{100}(\vec{r}) = \frac{1}{\sqrt{pa_0^3}} e^{-r/a_0}, \quad \text{where } a_0 = \frac{\hbar^2}{me^2}.$$

3. [40%] A particle of mass m is initially in the ground state of the (1-dim) infinite square-well potential. At t=0, a "brick" is dropped into the well so the potential becomes $V(x) = \begin{cases} V_0 & if \quad 0 \le x \le a/2 \\ 0 & if \quad a/2 < x \le a \\ \infty & otherwise \end{cases}$, where $V_0 \ll$ ground state energy E_1.
   After a time T, the brick is removed, and the energy of the particle is measured.
   (a) First find out the lowest two eigenenergies (E_1 and E_2) and normalized eigenstates of the *unperturbed* system (in which V_0=0).
   (b) Find out the probability that the energy of the particle (when being measured) is E_2 after the time t=T.

Hint: $d_f^1(t) = \dfrac{1}{i\hbar}\displaystyle\int_{t_0}^{t} dt' \left\langle f^0 | H'(t') | i^0 \right\rangle e^{iw_{fi}t'}$; $R_{i\to f} = \dfrac{2p}{\hbar} |V_{fi}|^2 d(e_{fi} - \hbar w)$