Quantum Mechanics 11/20/2003 (14:10-16:00)

1. [30 %] Assume that operators A and B do not commute, but A, B commute with [A,B]. We would like to show that $\exp(A)\exp(B) = \exp(A + B + \frac{1}{2}[A, B])$ (*)
   (a) First consider an auxiliary function $f(l) \equiv \exp(lA)\exp(lB)\exp\{-l(A + B)\}$, where λ is a number, prove that $\frac{df}{dl} = l[A, B]f$.
   (b) Integrate the equation in (a) over λ from 0 to 1 to prove the equation (*).

2. [40 %] Consider a physical system whose Hilbert space is spanned by the 3 kets $|u_1>$, $|u_2>$, and $|u_3>$. In this basis, the Hamiltonian operator and an observable A are written as (ω and a are positive real constants)

$$H = \hbar w \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A = a \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   The physical system at time t=0 is in the state

$$|y(0)> = \frac{1}{\sqrt{2}}|u_1> + \frac{1}{2}|u_2> + \frac{1}{2}|u_3>$$

   (a) At t=0, the energy of the system is measured. What values can be found? Calculate the mean value of <H> at t=0.
   (b) Instead of measuring H, one measures A at t=0; what results can be found, and with what probabilities?
   (c) Calculate the state $|y(t)>$ for the system at time t.
   (d) Calculate the mean value <A> at time t.

3. [30 %] Assume the wave function of a particle in an infinitely deep square well potential with width L is $y(x) = \frac{4}{\sqrt{L}} \sin\frac{px}{L} \cos^2\frac{px}{L}$
   (a) What are the possible values of the particle's energy when being measured?
   (b) What is the probability of finding the particle in the ground state?
   (c) Assume the width L of the potential suddenly expands (symmetrically) to twice its size. Find out the probability of finding the particle in the ground state of the expanded system *immediately* after (so that the wave function remains the same) the expansion.