Quantum Mechanics Final Exam (10:10-12:00, 1/13/2003)

1. Consider a one-dimensional simple harmonic oscillator.

$$H = \frac{p^2}{2m} + \frac{mw^2}{2}x^2 = \left(a^+a + \frac{1}{2}\right)\hbar w, \text{ where } a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{mw}{\hbar}}x + \frac{i}{\sqrt{mw\hbar}}p\right).$$

The eigenstate with energy $(n+1/2)\hbar\omega$ is written as $|n\rangle$.

(a) Construct a state $|\Psi\rangle$, which is a linear combination of $|0\rangle$ and $|1\rangle$, such that $\langle x\rangle$ is as large as possible.

(b) Suppose at t=0 the oscillator is in the state constructed in (a). What is the state vector $|\Psi(t)\rangle$ at t>0?

(c) Evaluate the $\langle x\rangle$ as a function of time using the $|\Psi(t)\rangle$ in (b).

2. (a) What is the physical significance of a propagator K(x,t;x',0)? Briefly explain how do you calculate it using Feynman's formula?

(b) Consider the propagator for an electron to move around an infinitely long solenoid *once* (the path is a loop). Use path integral method to show that, after we turn on the magnetic field, the propagator gains an extra phase $(e/\hbar c)\phi$, where ϕ is the magnetic flux in the solenoid. That is, $K_{B\neq 0}(x,t;x,0)=\exp[i(e/\hbar c)\phi]K_{B=0}(x,t;x,0)$.

Hint: $L(\vec{r},\vec{v}) = \frac{mv^2}{2} - \frac{e}{c}\vec{v}\cdot\vec{A}(\vec{r})$

3. Imagine a box with only 5 (single-particle) states inside. They are labeled n=1 to 5 from low energy to high energy (assuming there is no degeneracy). The energy and normalized wave function for the n-th state are written as ε_n and ϕ_n.

(a) If there are 3 noninteracting *bosons* in the box, write down the energy E and normalized wave function $\Psi(x_1,x_2,x_3)$ for the *ground state* of this system.

(b) Repeat the question in (a) if the 3 particles are *fermions*.

(c) In general, how many different ways we can put 3 fermions in this box?

4. (a) Prove that if a Hamiltonian is invariant under space reflection, then the energy eigenstates (assuming nondegenerate) have either even parity or odd parity.

(b) For every continuous unitary transformation U, we can define a generator G by considering an infinitesimal transformation (for simplicity, we consider the transformation with only one parameter ε),

$$U(e) = 1 - i\frac{e}{\hbar}G + O(e^2).$$

Prove that if the Hamiltonian H is invariant under the unitary transformation U, then the corresponding generator G is conserved.