Topics in mathematical physics

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CONTENTS

I. COMPLEX ANALYSIS

In classical physics, complex variables are sometimes used to represent real variables. For example, $\cos\omega t = Re\ e^{i\omega t}$. It could simplify a calculation since the multiplication of two sinusoidal functions is not as easy as the multiplication of two exponentials. In quantum physics, complex variable is indispensable since the imaginary number i is explicit in the Schrödinger equation. Due to its importance, we will learn about the basics of complex analysis in this note.

A. Complex function

A complex variable z consists of a real part and an imaginary part,

$$z = x + iy,\ i \equiv \sqrt{-1}. \tag{1.1}$$

It can also be written in the **polar form**,

$$z = r\cos\theta + i\sin\theta = re^{i\theta}, \tag{1.2}$$

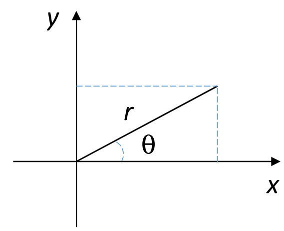

FIG. 1 Coordinates of the complex plane.

in which r and θ are the polar coordinates (Fig. 1), and

$$r = \sqrt{x^2+y^2},\ \tan\theta = \frac{y}{x}. \tag{1.3}$$

The angle θ is called the **argument** of z.

The **complex conjugate** of z is defined as

$$z^* \equiv x - iy. \tag{1.4}$$

The **norm** (or **modulus**) of z is written as

$$\begin{aligned} |z| &= \sqrt{z^* z} && (1.5)\\ &= \sqrt{x^2+y^2} = r. && (1.6)\end{aligned}$$

A complex-valued function of z consists of a real part and an imaginary part,

$$f(z) = u(x,y) + iv(x,y). \tag{1.7}$$

For example,

$$\begin{aligned} f(z) &= z^2 && (1.8)\\ &= x^2 - y^2 + i2xy. && (1.9)\end{aligned}$$

Suppose $z = re^{i(\bar\theta+2\pi n)}$ $(n \in Z)$, in which $\bar\theta \in [0, 2\pi)$, then

$$\begin{aligned} f(z) &= z^{1/2} && (1.10)\\ &= r^{1/2} e^{i\bar\theta/2} e^{i\pi n}. && (1.11)\end{aligned}$$

There are two possible outcomes, depending on whether n is an even integer or an odd integer. We can take the so-called **principle value** of z by restricting its argument to the interval $[0, 2\pi)$ (or $(-\pi, \pi]$). In what follows, the square root of the principle value of z will be written as $\sqrt{z}$, instead of $z^{1/2}$.

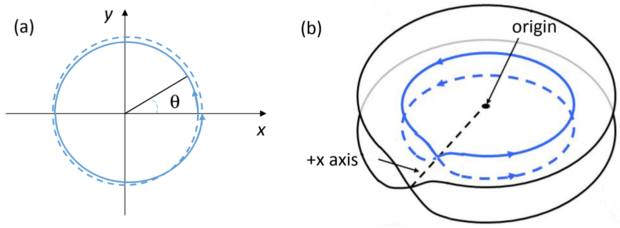

FIG. 2 (a) Circle around the origin two times. (b) The Riemann surface of the square-root function.

Given $z = re^{i\bar{\theta}}$, $\bar{\theta} \in [0, 2\pi)$, one has

$$\begin{aligned} \sqrt{z^2} &= \sqrt{r^2 e^{2i\bar{\theta}+i2\pi n}}, \quad n = 0, -1 &(1.12)\\ &= +re^{i\bar{\theta}} \text{ if } \bar{\theta} \in [0, \pi) &(1.13)\\ & \quad -re^{i\bar{\theta}} \text{ if } \bar{\theta} \in [\pi, 2\pi). &(1.14)\end{aligned}$$

That is, depending on the argument of z,

$$\sqrt{z^2} = \pm z. \tag{1.15}$$

In general, if $f(z) = \sqrt[n]{z}$, where n is a positive integer, then $f(z)$ is n-valued.

The second example is the logarithmic function,

$$\begin{aligned} f(z) = \ln z &= \ln[re^{i(\bar{\theta}+2\pi n)}], \; \bar{\theta} \in [0, 2\pi) &(1.16)\\ &= \ln r + i\bar{\theta} + 2\pi n i. &(1.17)\end{aligned}$$

This is a function with infinite possible values depending on the integer n. If z is restricted to its principle value with $n = 0$, then we write the logarithmic function as $\mathrm{Ln}\ z$, instead of $\ln z$.

Given a complex number $z = e^{i\bar{\theta}}$,

$$\mathrm{Ln}\ z^2 = \mathrm{Ln}\ e^{2i\bar{\theta}} = 2i\bar{\theta}, \quad \text{if } \bar{\theta} \in [0, \pi). \tag{1.18}$$

This is not equal to $2\mathrm{Ln}\ z = 2i\bar{\theta}$ if $\bar{\theta}$ is within $[\pi, 2\pi)$. In general, given a real number α,

$$\mathrm{Ln}\ z^\alpha = \alpha \mathrm{Ln}\ z + 2\pi n i, \; n \in Z. \tag{1.19}$$

It is possible to have the square root function single-valued by extending its domain. Suppose one starts from a point x on the positive x-axis ($\theta = 0$), moves along a circular path and comes back to x, then $z^{1/2}$ acquires a negative sign when $\theta = 2\pi$. To prevent the double-valuedness, let $z^{1/2}$ be defined on the second sheet of complex plane when $\theta \in [2\pi, 4\pi)$ (Fig. 2). The signs of $z^{1/2}$ for $\theta = \bar{\theta} + 2\pi n$ ($\bar{\theta} \in [0, 2\pi)$) depend on the integer n being odd or even. Thus, only two sheets of complex plane are needed.

On the other hand, to retain the single-valuedness of the logarithmic function, an infinite number of sheets, each corresponding to an integer n, are required. These sheets of complex planes connected along some line (here the positive x-axis) are called **Riemann surfaces**.

B. Differentiable complex function

Recall that a real function $f(x)$ is **differentiable** at x if its *right* derivative equals its *left* derivative,

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}. \tag{1.20}$$

Then one has a well defined derivative $\frac{df}{dx}$.

Now, a complex function $f(z)$ is differentiable if

$$\frac{df}{dz} \equiv \lim_{\delta z\to 0} \frac{f(z+\delta z) - f(z)}{\delta z} \quad \text{is independent of } \delta z. \tag{1.21}$$

Given the increment $\delta z = \delta x + i\delta y$, the change of the function

$$\delta f = \delta u + i\delta v, \tag{1.22}$$

hence

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}. \tag{1.23}$$

Consider two special directions of taking the limit: The first is $\delta z = \delta x$, then

$$\lim_{\delta z\to 0} \frac{\delta f}{\delta z} = \lim_{\delta x\to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{1.24}$$

The second is $\delta z = i\delta y$, then

$$\lim_{\delta z\to 0} \frac{\delta f}{\delta z} = \lim_{\delta y\to 0} \left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{1.25}$$

If f is differentiable at $z = x + iy$, then these two limits should be same. It follows that,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, &(1.26)\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. &(1.27)\end{aligned}$$

This is called the **Cauchy-Riemann condition**.

Conversely, if u, v satisfies the Cauchy-Riemann (CR) condition, then $f = u + iv$ is differentiable. That is, the CR condition is a necessary and sufficient condition of differentiability. This sufficient condition is proved as follows: Upon the shift $z \to z + dz$,

$$\begin{aligned} f(z) &\to f(z+dz), &(1.28)\\ \text{or } \; f(x,y) &\to f(x+dx, y+dy). &(1.29)\end{aligned}$$

Thus,

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy &(1.30)\\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right) dx + \underbrace{\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \right)}_{=-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}} dy &(1.31)\\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right)(dx + idy), &(1.32)\end{aligned}$$

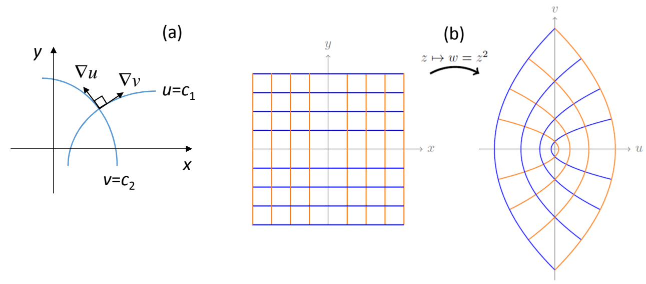

FIG. 3 (a) A u-cruve and a v-curve are tangent to each other. (b) The map of $z \to f(z) = z^2$.

in which the CR condition has been used. It follows that,

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}, \tag{1.33}$$

which is independent of the direction of dz. Thus the CR condition implies differentiability.

If $f(z)$ is *differentiable* and *single-valued* in a region M, then we say that $f(z)$ is **analytic** in M. Note that z^* is not analytic since

$$z^* = x - iy \quad (u = x, v = -y), \tag{1.34}$$

and the CR condition is not satisfied,

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1. \tag{1.35}$$

Apparently, $Re\ z$ or $|z|^2$ is not differentiable either.

One can write dx and dy in terms of dz and dz^*,

$$dx = \frac{1}{2}(dz + dz^*), \tag{1.36}$$
$$dy = \frac{1}{2i}(dz - dz^*), \tag{1.37}$$

so that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{1.38}$$
$$= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)dz^*. \tag{1.39}$$

We can define

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \tag{1.40}$$
$$\frac{\partial f}{\partial z^*} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right), \tag{1.41}$$

so that

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial z^*}dz^*. \tag{1.42}$$

Note that

$$\frac{\partial f}{\partial z^*} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{1.43}$$
$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{1.44}$$

which is zero if u, v satisfy the CR condition. That is, the CR condition can simply be written as $\partial f/\partial z^* = 0$. This is not satisfied by the examples $(z^*, Rez, |z|^2)$ given above, thus they are not differentiable.

It follows that if f is differentiable, then

$$df = \frac{\partial f}{\partial z}dz, \tag{1.45}$$

and there is no need to distinguish between df/dz and $\partial f/\partial z$. Also, from Eq. (1.40), one has

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{1.46}$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}, \tag{1.47}$$

which is consistent with Eq. (1.33).

1. Properties of differentiable function

First, by combining the two equations in the CR condition, it's not difficult to see that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{1.48}$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{1.49}$$

That is, both the real part and the imaginary part of an analytic function satisfy the 2D **Laplace equation**. They are called **harmonic functions**.

Second, the curves depicted by

$$u(x,y) = c_1, \tag{1.50}$$
$$\text{and } v(x,y) = c_2 \quad (c_1, c_2 \text{ are constants}) \tag{1.51}$$

are orthogonal to each other. This is proved as follows: First, you need to know that the gradient ∇u is perpendicular to the tangent of the curve $u = c_1$. This is so because the change of $u(\mathbf{r})$ along $d\mathbf{r}$,

$$du = \nabla u \cdot d\mathbf{r}. \tag{1.52}$$

When $d\mathbf{r}$ is along the curve of constant c_1, the change $du = 0$. Hence $\nabla u \perp d\mathbf{r}$.

Similarly, ∇v is perpendicular to $v = c_2$. Now, if u, v satisfy the CR condition, then their inner product

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \tag{1.53}$$
$$= \frac{\partial u}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 0. \tag{1.54}$$

This shows that $\nabla u \perp \nabla v$. Therefore, at an intersection point of two u, v curves, their tangent vectors must be perpendicular to each other, as shown in Fig. 3.

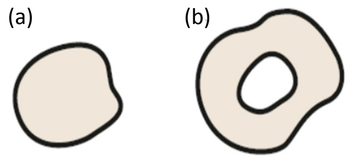

FIG. 4 (a) A simply-connected region. (b) A multiply-connected region.

Such a property of the u, v curves fit perfectly with the field lines in electrostatics. In a charge-free region M, the electric potential satisfies the Laplace equation $\nabla^2\phi = 0$. In two dimension, the equation $\phi(x, y) = c$ draws out an equipotential line. We know that the electric lines are perpendicular to the equipotential lines. Therefore, the mesh of constant u, v lines can be identified with the mesh of equipotential lines and field lines.

Third, a function being analytic is a very strong property. Later we will show that if df/dz exists, then $d^n f/dz^n$ exists for all $n \in Z^+$. This is obviously not true for *real* functions. Furthermore, if $f(z)$ is analytic around a point z_0 in a region M, then it can always be expanded with respect to z_0 as a Taylor series in that region. In fact, the original meaning of the term *analytic function* refers to a function that can be expanded by a Taylor series. Again this is not always true for *real* functions. But for a *complex* function, being differentiable (and single-valued) is enough to guarantee that it has a Taylor series expansion. That is, it is analytic (see Sec. I.E.1).

2. Derivative rules

The rules for the derivative of complex functions are similar to those for real functions. For example,

$$\begin{aligned} \frac{d}{dz} z^n &= n z^{n-1}, && (1.55) \\ \frac{d}{dz} \sin z &= \cos z, && (1.56) \\ \frac{d}{dz} \cos z &= -\sin z, && (1.57) \\ \frac{d}{dz} e^z &= e^z, && (1.58) \\ \frac{d}{dz} \ln z &= \frac{1}{z}, && (1.59) \\ \frac{d}{dz} f(g(z)) &= \frac{df}{dg}\frac{dg}{dz}, && (1.60) \\ \frac{d}{dz} f(z)g(z) &= \frac{df}{dz} g + f \frac{dg}{dz}. && (1.61) \end{aligned}$$

These can be checked with the use of real variables. For example, since $\ln z$ is an analytic function (in one sheet of the Riemann surface), one has

$$\begin{aligned} \frac{d}{dz} \ln z &= \frac{\partial}{\partial x} \ln z && (1.62) \\ &= \frac{\partial \ln r}{\partial x} + i \frac{\partial \theta}{\partial x} && (1.63) \\ &= \frac{x}{r^2} - i \frac{y}{r^2} = \frac{1}{z}. && (1.64) \end{aligned}$$

C. Cauchy's integral theorem

The integral of a complex function can be carried out with the help of real variables. For example, integrate $f(z)$ from point 1 to point 2 over a path on the complex plane,

$$\begin{aligned} \int_1^2 f(z) dz &= \int_1^2 (u + iv)(dx + idy) && (1.65) \\ &= \int_1^2 (udx - vdy) + i \int_1^2 (vdx + udy). \end{aligned}$$

The real part and the imaginary are just the usual integrals of real functions.

A region M on the complex plane can be **simply connected** or **multiply connected**. It is simply connected if any closed loop C in M can be continuously shrunk to a point. If not, then M is multiply connected (see Fig. 4).

Theo: *Cauchy's integral theorem* (in a simply connected region)

If $f(z)$ is *analytic* in a *simply connected* region M, then for any closed curve C in M,

$$\oint_C f(z) dz = 0. \tag{1.66}$$

Note that by convention, a path of integration is always *counter-clockwise*.

Pf: Recall the Stokes theorem,

$$\int_S \nabla \times \mathbf{V} \cdot d\mathbf{a} = \oint_C \mathbf{V} \cdot d\mathbf{r}, \tag{1.67}$$

in which C is the boundary of an area S. In two dimension, $\mathbf{V} = V_x \hat{\mathbf{x}} + V_y \hat{\mathbf{y}}$, and the theorem can be written as,

$$\int_S \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) dxdy = \oint_C (V_x dx + V_y dy). \tag{1.68}$$

Applying this to the loop integral above, one then has

$$\begin{aligned} \oint_C f(z) dz &= \oint_C (udx - vdy) + i \oint_C (vdx + udy) && (1.69) \\ &= -\int_S \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dxdy + i \int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy \\ &= 0. && (1.70) \end{aligned}$$

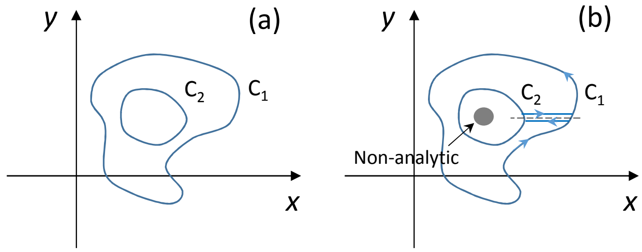

FIG. 5 Closed curves in (a) simply connected region and (b) multiply-connected region.

These two integrands are both zero because of the CR condition. QED.

The reverse statement of this theorem is also true: In a simply connected region M, if

$$\oint_C f(z)dz = 0 \text{ for any } C \in M, \tag{1.71}$$

then $f(z)$ is analytic in M. For a proof, see p. 489 of Arfken.

Ex: Consider $f(z) = z^n, (n \in Z)$, which is analytic for $n \geq 0$. If $n < 0$, then it is not analytic at the origin. Choose C to be a circle with radius r centered at the origin, then along the circle $z = re^{i\theta}$, and $dz = id\theta re^{i\theta}$. Hence

$$\oint_C z^n dz = ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}d\theta \tag{1.72}$$

$$= \begin{cases} 0 \text{ if } n \neq -1, \\ 2\pi i \text{ if } n = -1. \end{cases} \tag{1.73}$$

This result is consistent with Cauchy's integral theorem when z^n is analytic ($n \geq 0$). When it is not analytic, the integrals are still mostly zero (for $n \leq -2$). The only exception is when $n = -1$,

$$\oint_C \frac{1}{z}dz = 2\pi i. \tag{1.74}$$

The function z^n ($n \leq -1$) is not analytic at $z = 0$. So there is a "hole" in the region where the function is analytic. As a result, a closed loop circling the origin cannot be continuously shrunk to a point. That is, the function z^n ($n \leq -1$) is analytic in a *multiply-connected* region. For this type of functions, we have the following integral theorem.

Theo: *Cauchy's integral theorem* (in a multiply connected region)

Suppose a loop C_1 surrounding a non-analytic region can be continuously deformed to another loop C_2 without crossing that region, then

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz. \tag{1.75}$$

Pf: One can connect the two loops to build a closed path $C = C_1 + \leftarrow + (-C_2) + \rightarrow$ (see Fig. 5(b)). Since C can be continuously shrunk to a point without crossing the non-analytic region, one has

$$\oint_C f(z)dz = 0, \tag{1.76}$$

However, since

$$\oint_C = \int_{C_1} + \int_{\leftarrow} + \int_{-C_2} + \int_{\rightarrow} \tag{1.77}$$

$$= \oint_{C_1} - \oint_{C_2}, \tag{1.78}$$

in which $\int_{\leftarrow}$ and $\int_{\rightarrow}$ cancel with each other. Therefore,

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz. \quad \text{QED.} \tag{1.79}$$

Based on Cauchy's integral theorem, Eq. (1.73) can be generalized as follows: For any closed curve C surrounding point z_0,

$$\oint_C (z-z_0)^n dz = \begin{cases} 0 \text{ if } n \neq -1, \\ 2\pi i \text{ if } n = -1. \end{cases} \tag{1.80}$$

This is valid since a closed curve surrounding z_0 can be continuously deformed to a *circle* surrounding z_0.

D. Cauchy integral formula

Theo: Suppose $f(z)$ is analytic along a closed loop C and inside, then

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0) \text{ if } z_0 \text{ is inside } C, \\ 0 \text{ if } z_0 \text{ is outside } C. \end{cases} \tag{1.81}$$

Pf: The point z_0 is the only singularity of the integrand $f(z)/(z-z_0)$. If z_0 is outside C, then the integrand is analytic everywhere inside C. Thus, according to Cauchy's integral theorem, the integral is zero.

If z_0 is inside C, then we can deform C to a small circle C_r surrounding z_0 without changing the value of the integral. Along the circle C_r,

$$z = z_0 + re^{i\theta} \; (r \ll 1), \quad dz = ird\theta e^{i\theta}. \tag{1.82}$$

Thus,

$$\oint_{C_r} \frac{f(z)}{z-z_0}dz = \int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} ire^{i\theta}d\theta \tag{1.83}$$

$$(r \to 0) \quad = if(z_0)\int_0^{2\pi} d\theta \tag{1.84}$$

$$= 2\pi i f(z_0). \quad \text{QED.} \tag{1.85}$$

Ex 1: Evaluate

$$I = \oint_C \frac{dz}{z(z+2)}, \tag{1.86}$$

where C is a unit circle centered at the origin.
Sol'n

$$I = \oint_C \frac{1/(z+2)}{z-0}dz \quad (1.87)$$
$$= 2\pi i \times \frac{1}{2} = \pi i. \quad (1.88)$$

Ex 2: Evaluate

$$I = \oint_C \frac{dz}{4z^2-1}, \quad (1.89)$$

where C is a unit circle centered the origin.
Sol'n

$$I = \frac{1}{4}\left(\oint_C \frac{dz}{z-1/2} - \oint_C \frac{dz}{z+1/2}\right) \quad (1.90)$$
$$= \frac{1}{4}(2\pi i - 2\pi i) = 0. \quad (1.91)$$

1. Derivatives of $f(z)$

Suppose $f(z)$ is analytic along a closed loop C and inside. According to Cauchy's integral formula, if the value of $f(z)$ along C is known, then its value at any z_0 inside C is known,

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz. \quad (1.92)$$

It follows that

$$f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz, \quad (1.93)$$
$$f''(z_0) = \frac{2}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz, \quad (1.94)$$
$$\vdots$$
$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz. \quad (1.95)$$

Since there is no singularity along the path of integration, all of these derivatives exist. That is, if $f(z_0)$ is differentiable, then it is *differentiable to all orders of the derivatives.*

Ex 3: Evaluate

$$I = \oint_C \frac{\sin^2 z}{(z-a)^4}dz, \quad (1.96)$$

where C encircles the point a.
Sol'n First,

$$I_0 = \oint_C \frac{\sin^2 z}{z-a}dz = 2\pi i \sin^2 a. \quad (1.97)$$

Thus,

$$I = \frac{1}{3!}\frac{d^3}{da^3}I_0 = -\frac{8\pi i}{3}\sin a \cos a. \quad (1.98)$$

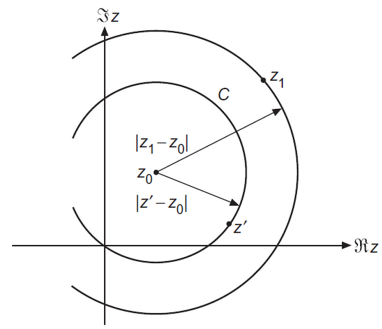

FIG. 6 A circle C in a simply-connected region.

2. Application

Lemma: Suppose $f(z) = \sum_{n=0} a_n z^n$, $|f(z)| \le M$ along a circle C with radius r around the origin, then

$$|a_n| \le \frac{M}{r^n}, \text{ for } n = 0, 1, 2, \cdots \quad (1.99)$$

Pf: Differentiate both sides of the equation to get

$$a_n = \frac{f^{(n)}(0)}{n!}. \quad (1.100)$$

Because of Eq. (1.95), one has

$$|a_n| = \frac{1}{2\pi}\left|\oint_C \frac{f(z)}{z^{n+1}}dz\right| \quad (1.101)$$
$$\le \frac{1}{2\pi}\oint_C \left|\frac{f(z)}{z^{n+1}}\right||dz| \quad (1.102)$$
$$\le \frac{1}{2\pi}\frac{M}{r^{n+1}}2\pi r \quad (1.103)$$
$$= \frac{M}{r^n}. \quad \text{QED.} \quad (1.104)$$

This property leads to **Liouville theorem**: If $f(z)$ is *analytic* and *bounded* over the entire complex plane, then $f(z)$ must be a constant.
Pf: In next section, we will know that if $f(z)$ is analytic, then it can be expanded as a Taylor series. Furthermore, if $f(z)$ is bounded over the complex plane, then $|f(z)| \le M, \forall z$. With the lemma above, let $r \gg 1$, then

$$|a_n| \le \frac{M}{r^n} \to 0, \ \forall\ n > 0. \quad (1.105)$$

Therefore, $f(z) = a_0$. QED.

Conversely, any slight deviation of a *bounded analytic* $f(z)$ from constant implies that it has at least one singularity somewhere.

From Liouville theorem, one can prove the **fundamental theorem of algebra**:A polynomial with degree n must have n roots. See p. 490 of Arfken if you're interested in the proof.

E. Series expansion

In this section, we will learn about Taylor series and Laurent series.

1. Taylor series

We'd like to expand an analytic function $f(z)$ with respect to $z = z_0$. Suppose $z = z_1$ is the nearest singular point of $f(z)$, then inside a circle C_r centered at z_0 with radius $r = |z_1 - z_0|$, $f(z)$ is analytic. It can be expanded as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad (1.106)$$

$$\text{where} \quad f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz. \quad (1.107)$$

This is a **Taylor series expansion** of $f(z)$ with respect to z_0.
Pf: Consider a circle C within the analytic region bounded by C_r (Fig. 6). That is, the radius of $C < |z_1 - z_0|$. Now,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz' \quad (1.108)$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)\left(1 - \frac{z - z_0}{z' - z_0}\right)} dz' \quad (1.109)$$

For a point z inside C, $|z - z_0|/|z' - z_0| < 1$ since z is closer to z_0 compared to a point z' located on C.

It is known from the **binomial expansion** that if $|w| < 1$, then

$$\frac{1}{1 - w} = 1 + w + w^2 + \cdots . \quad (1.110)$$

Therefore,

$$f(z) = \frac{1}{2\pi i} \oint_C dz \frac{f(z')}{z' - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0}\right)^n \quad (1.111)$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n. \quad \text{QED.} \quad (1.112)$$

Finally, because of Cauchy's integral theorem, the circle C of integration can be continuously deformed to any other closed loop, as long as it does not cross some singular point during the process.

2. Laurent series

We'd like to expand an analytic function $f(z)$ with respect to $z = z_0$. Suppose there are two singular points z_1, z_2 nearby, then inside an annular region M bounded by two circles, C_r and C_R, $f(z)$ is analytic (see Fig. 7). Given a point $z \in M$, $f(z)$ can be expanded as

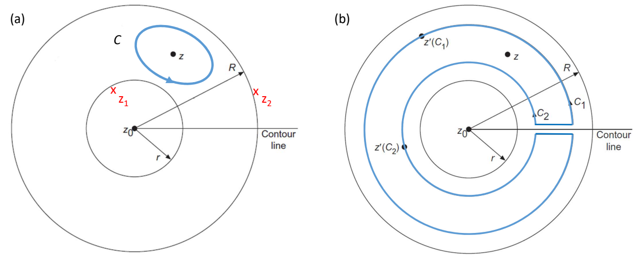

FIG. 7 (a) A loop C in a multiply-connected region. (b) The loop C is continuously deformed to another loop C' in the annular region without crossing itself and any singular point.

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad (1.113)$$

$$\text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad C \in M.$$

This is a **Laurent series expansion** of $f(z)$ with respect to z_0. Note that it is similar to the Taylor expansion, except that now the power n can be negative.
Pf: In Fig. 7, the inner and outer radii of the annular region M are $r = |z_1 - z_0|^+$ and $R = |z_2 - z_0|^-$. For a point z in M,

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz', \quad C \in M. \quad (1.114)$$

Inflate the loop C in Fig. 7(a) to occupy the annular region M, so that it becomes $C' = C_1 + \leftarrow + (-C_2) + \rightarrow$ in Fig. 7(b). Now,

$$\oint_C = \oint_{C'} = \int_{C_1} + \int_{\leftarrow} + \int_{-C_2} + \int_{\rightarrow} \quad (1.115)$$

$$= \oint_{C_1} - \oint_{C_2}. \quad (1.116)$$

Therefore,

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')dz'}{z' - z} - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z')dz'}{z' - z}. \quad (1.117)$$

For the first term, rewrite

$$\frac{1}{z' - z} = \frac{1}{z' - z_0 - (z - z_0)} = \frac{1}{(z' - z_0)\left(1 - \frac{z - z_0}{z' - z_0}\right)}.$$

Compared to $z' \in C_1$, z is closer to z_0, thus the ratio $|w| = |(z - z_0)/(z' - z_0)| < 1$ and the fraction $1/(1 - w)$ can be expanded with the binomial expansion. Similarly, for the second term, rewrite

$$\frac{1}{z' - z} = \frac{1}{z' - z_0 - (z - z_0)} = \frac{-1}{(z - z_0)\left(1 - \frac{z' - z_0}{z - z_0}\right)}.$$

Compared to $z' \in C_2$, z is farther away from z_0, thus the ratio $|w| = |(z'-z_0)/(z-z_0)| < 1$ and the fraction $1/(1-w)$ can be expanded with the binomial expansion.

It follows that

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_{C_1}\frac{f(z')dz'}{(z'-z_0)^{n+1}} \quad (1.118)$$
$$+ \underbrace{\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{n+1}}\oint_{C_2} f(z')(z'-z_0)^n dz'}_{\text{replace } n \text{ by } n-1 \text{ and } \sum_0^\infty \to \sum_1^\infty}$$
$$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad (1.119)$$

where $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$. QED.

In the last step, we have deformed C' back to the contour C that is inside M.

Ex: Expand $f(z) = 1/z(z-1)$ with respect to $z_0 = 0$, where $0 < |z| < 1$.

Sol'n: (1) Direct expansion gives

$$f(z) = -\frac{1}{1-z} - \frac{1}{z} \quad (1.120)$$
$$= -\frac{1}{z} - 1 - z - z^2 - \cdots \quad (1.121)$$
$$= -\sum_{n=-1}^{\infty} z^n. \quad (1.122)$$

(2) Alternatively, we can use the formula in Eq. (1.113),

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}dz \quad (z_0 = 0) \quad (1.123)$$
$$= -\frac{1}{2\pi i}\oint_C \frac{dz}{z^{n+2}(1-z)}, \quad \frac{1}{1-z} = \sum_{m=0}^{\infty} z^m$$
$$= -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\underbrace{\oint_C dz z^{m-n-2}}_{=2\pi i\delta_{m,n+1}} \quad (1.124)$$
$$= -\sum_{m=0}^{\infty}\delta_{m,n+1} \quad (1.125)$$
$$= \begin{cases} 0 & n < -1, \\ -1 & n \geq -1. \end{cases} \quad (1.126)$$

This then gives the series in Eq. (1.122).

F. Singularities

A singularity is a point at which a complex function is not differentiable. There are two types of singularities for a complex function: pole and branch point. This is explained below.

1. Pole

Suppose $f(z)$ is not analytic at $z = z_0$, but is analytic at neighboring points, then z_0 is called an **isolated singularity**. We can then expand $f(z)$ with respect to z_0 with the Laurent expansion,

$$f(z) = \sum_n a_n (z-z_0)^n. \quad (1.127)$$

Some terminologies:
(1) If the series terminate at $n = -1$, without more negative-power terms, then z_0 is a **simple pole**.
(2) If the negative-power terms terminate at $n < -1$, then z_0 is a **pole of order** n.
(3) If there are infinite negative-power terms, then z_0 is an **essential singularity**.

For a pole of order n at $z = z_0$,

$$\lim_{z\to z_0}(z-z_0)^n f(z) \text{ is finite.} \quad (1.128)$$

This can be easily seen if $f(z)$ is expanded with a Laurent series.

In complex analysis, the behavior of $f(z)$ at $z \to \infty$ can be identified with that of $f(1/w)$ at $w \to 0$. So the behavior of $f(z) = z$ at infinity can be identified with the behavior of $f(w) = 1/w$ at $w = 0$. Therefore, $f(z) = z$ has a singularity at infinity.

One example of essential singularity is

$$e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{z^n}. \quad (1.129)$$

Clearly $z = 0$ is an essential singularity. Another example of essential singularity appears in $\sin z$ when $z \to \infty$. As we have just mentioned, the behavior of $f(z)$ when $z \to \infty$ can be identified the behavior of $f(1/w)$ when $w \to 0$. Therefore

$$\sin\frac{1}{w} = \frac{1}{w} - \frac{1}{3!}\frac{1}{w^3} + \cdots, \quad (1.130)$$

and $w = 0$ (or $z \to \infty$) is an essential singularity.

Some more terminologies:

A function is **holomorphic** in a region M if its Laurent series has no negative powers. That is, $f(z)$ has no pole in region M, and $f(z)$ can be expanded as a Taylor series.

A function is **meromorphic** in a region M if its Laurent series has negative-power terms at isolated points. That is, it is a holomorphic function with isolated singularities.

As we mentioned before, a function is *analytic* in a region M if it is differentiable and single-valued in M. However, we know that if $f(z)$ is differentiable, then it can be differentiated infinite times. Furthermore, $f(z)$ can be expanded as a Taylor series in M. Therefore, for

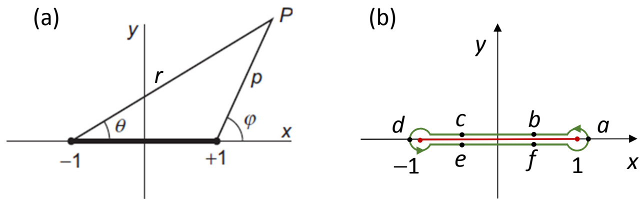

FIG. 8 (a) The variables relating a point P to the branch points. (b) Path around the branch cut in the Example.

complex functions, being holomorphic is *exactly the same* as being analytic.

Finally, a function that is analytic for any *finite* z is called an **entire function** (it can have singularities at infinity). A typical example of the entire function is a polynomial. If an entire function is bounded, then it is a constant (according to the Liouville theorem).

2. Branch point

Draw a closed path C around a point z_0, if the value of a function $f(z)$ changes after moving around C and coming back to the starting point, then z_0 is **branch point** of $f(z)$. For example $z_0 = 0$ is the branch point of $f(z) = z^{1/2}$. As explained in I.A, to prevent the function from having multiple values, one can draw a line emanating from $z_0 = 0$. As long as this line is not crossed, $f(z)$ would remain single-valued. Such a line attached to z_0 is called a **branch cut** of $f(z)$.

Alternatively, if a Riemann surface is employed to deal with a multi-valued function, then a branch cut joins different sheets (or branches) of the Riemann surface (see Sec. I.A). The function $f(z)$ enters a different branch when a branch cut is crossed. Note that since $f(z)$ is single-valued at z_0, but multi-valued for the points nearby, thus $f(z)$ cannot be differentiable at z_0.

The location of a branch cut is arbitrary, as long as it can prevent $f(z)$ from being multi-valued. As long as a contour C avoids the branch cut, then Cauchy's integral theorem would remain valid.

Ex: Find out the branch points of $f(z) = (z^2 - 1)^{1/2}$, and draw out its branch cuts.
Sol'n: Since

$$f(z) = (z+1)^{1/2}(z-1)^{1/2}, \tag{1.131}$$

so there are branch points at $z = -1$ and $+1$. As shown in Fig. 8, let

$$\begin{aligned} z+1 &= re^{i\theta}, & (1.132)\\ z-1 &= \rho e^{i\phi}, & (1.133)\end{aligned}$$

then

$$f(z) = \sqrt{r}\sqrt{\rho}e^{i(\theta+\phi)/2}. \tag{1.134}$$

We can choose its branch cut to be the one in Fig. 8(b). Follow the contour in the figure, one would have the following arguments of $f(z)$:

	θ	ϕ	$(\theta+\phi)/2$
a	0	0	0
b	0	π	$\pi/2$
c	0	π	$\pi/2$
d	π	π	π
e	2π	π	$3\pi/2$
f	2π	π	$3\pi/2$
a	2π	2π	2π

When one comes back to a, the function is also back to its initial value.

It is possible to draw the branch cuts to be on the outside of the two branch points, instead of the one above on the inside. You may check that the function would remain single-valued as long as you walk along a contour that avoids the branch cuts.

G. Analytic continuation

Recall that if $f(z)$ is analytic within a circle of convergence C centered at z_0, then it can be expanded as a Taylor series,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!}(z-z_0)^n \tag{1.135}$$

We now prove an interesting property.
Lemma 1: If $f(z)$ is analytic in a region M, and $f(z) = 0$ along an arc PQ in M, then $f(z) = 0$ throughout M.
Pf: Expand $f(z)$ with respect to z_0 (both z and z_0 are on PQ), then

$$\begin{aligned} f(z) &= f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots \\ &= 0. \end{aligned} \tag{1.136}$$

This implies that

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = 0. \tag{1.137}$$

Since the same set of coefficients applies to the Taylor expansion of other points z in M, hence $f(z) = 0$ throughout M. QED.

Lemma 2: If both $f(z)$ and $g(z)$ are analytic in M, and $f(z) = g(z)$ along an arc PQ in M, then $f(z) = g(z)$ throughout M.
Pf: To prove it, just let $F(z) = f(z) - g(z)$, which is zero along PQ, and apply Lemma 1 to the function $F(z)$. QED.

Now, suppose $f(z) = \sum_n a_n(z-z_0)^n$ within a circle of convergence C_1. We may expand $f(z)$ with respect to a

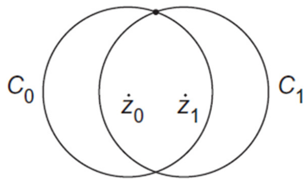

FIG. 9 Analytic continuation from one circle of convergence to another one.

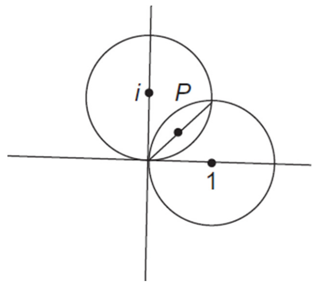

FIG. 10 Two circles of convergence in Example 1.

different point z_1 inside C_1, such that

$$\tilde{f}(z) = \sum_{n=0}^{\infty} b_n (z - z_1)^n \tag{1.138}$$

within another circle of convergence C_2 (Fig. 9). Since $\tilde{f}(z) = f(z)$ within C_1, they are the same function according to Lemma 2 above. The function $\tilde{f}(z)$ is called an **analytic continuation** of $f(z)$ since its domain extends beyond the region confined by C_1. Note that such an extension is unique.

Ex 1: Consider the following two series expansions,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} (-1)^n (z-1)^n, \text{ converges if } |z-1| < 1, \\ g(z) &= \sum_{n=0}^{\infty} i^{n-1} (z-i)^n, \text{ converges if } |z-i| < 1. \end{aligned}$$

You may check that $f(z) = g(z)$ along the diagonal line in Fig. 10, so they are the same function in the overlapped region. However, one series is convergent within C_1, while the other is convergent within C_2.

In fact, they are expansions of $h(z) = 1/z$ with respect to $z_0 = 1$ and $z_0 = i$ respectively,

$$\frac{1}{z} = \frac{1}{1 + (z-1)} = \frac{1}{i + (z-i)}. \tag{1.139}$$

However, $h(z)$ is analytic everywhere on the complex plane except $z = 0$. So the same function can appear in different forms, with different domains of convergence.

Ex 2: The series

$$f(z) = z - z^2 + z^3 - \cdots \tag{1.140}$$

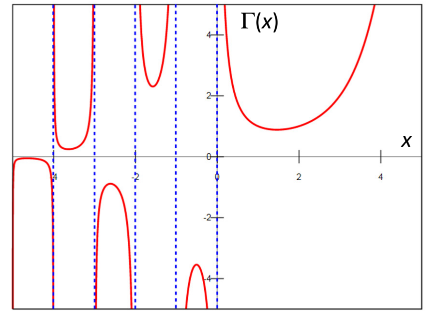

FIG. 11 The Gamma function.

is analytic and convergent within $|z| < 1$. Find an analytic continuation of $f(z)$ beyond $|z| = 1$.
Sol'n: The function

$$g(z) = \frac{z}{1+z} \tag{1.141}$$

is equal to $f(z)$ when $|z| < 1$ and is analytic everywhere except at $z = -1$, so (z) can be an analytic continuation of $f(z)$.

Note: if one uses the series expansion of $f(z)$ outside its circle of convergence, e.g. $z = 2$, then one gets

$$2 - 2^2 + 2^3 - \cdots \text{"} = \text{"} \frac{2}{3}. \tag{1.142}$$

Similar ridiculous expression is used by some physicists for its shock value. The most famous example is related to the following **Riemann zeta function**,

$$\zeta(z) \equiv 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots. \tag{1.143}$$

This series diverges when Re $z \leq 1$. It has an analytic continuation beyond this range with a different form that converges. Its value is $-1/12$ when $z = -1$, for example. However, if the series above is applied to $z = -1$, then one has

$$1 + 2 + 3 + \cdots \text{"} = \text{"} -\frac{1}{12}. \tag{1.144}$$

Ex 3: The **Gamma function** is defined as,

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad x \in R, \tag{1.145}$$

which is convergent if $x > 0$. One can show that

$$\Gamma(x+1) = x\Gamma(x). \tag{1.146}$$

Pf:

$$\begin{aligned} \Gamma(x+1) &= \int_0^{\infty} t^x e^{-t} dt \qquad (1.147) \\ &= -\, t^x e^{-t}\Big|_0^{\infty} + \int_0^{\infty} x t^{x-1} e^{-t} dt \\ &= x\Gamma(x). \qquad (1.148) \end{aligned}$$

Starting from $\Gamma(1) = 1$, one gets $\Gamma(n+1) = n!$ for a positive integer n. Thus the Gamma function is also known as the **factorial function**. When x is not an integer, one has, for example,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad (1.149)$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}. \quad (1.150)$$

We now extend x to a complex number z,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad z \in C. \quad (1.151)$$

Since

$$t^z = t^{x+iy} = t^x e^{iy\ln t}, \text{ and } |e^{iy\ln t}| = 1, \quad (1.152)$$

the integral is convergent if $Re\ z = x > 0$. Similar to the derivation above, one can also show that

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad (1.153)$$

$$= -\frac{1}{z}\,t^z e^{-t}|_0^\infty + \int_0^\infty \frac{1}{z}t^z e^{-t}dt \quad (1.154)$$

$$= \frac{\Gamma(z+1)}{z}, \quad (1.155)$$

which is convergent if $x > -1$. As a result, we can extend the domain of the Gamma function from $x > 0$ to $x > -1$ (Fig. 11). Also, it is not difficult to see that $\Gamma(z)$ has a simple pole at $z = 0$.

Further iteration leads to

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \cdots \quad (1.156)$$

In this way, stripe by stripe, we can extend the domain of the Gamma function to the whole complex plane. Also, one can see that $\Gamma(z)$ has simple poles at $z = 0, -1, -2, \cdots$. For more discussion of the Gamma function, you may watch Prof. Balakrishnan's lecture "Analytic continuation and the gamma function" on youtube. In fact, all of his lectures on various subjects are good and worth watching.

H. Calculus of residues

Recall that $\oint_C (z-z_0)^n dz \neq 0$ only if $n = -1$. From the Laurent expansion,

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n, \quad (1.157)$$

we know that z_0 is a singularity if $n < 0$ terms exist. Integrates both sides along a closed contour surrounding C, from the equation above, one has

$$\oint_C f(z)dz = 2\pi i a_{-1}. \quad (1.158)$$

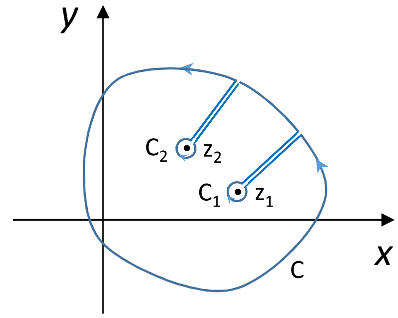

FIG. 12 A closed contour that avoids isolated singularities.

All of the terms in the series vanish except the term with $n = -1$. The coefficient a_{-1} is called the **residue** of $f(z)$ at $z = z_0$.

Residue theorem:
Suppose $f(z)$ has singularities at $z_1, z_2, \cdots$, and C encloses these singularities, then

$$\oint_C f(z)dz = 2\pi i(a_{-1,1} + a_{-1,2} + \cdots), \quad (1.159)$$

in which $a_{-1,i}$ are the residues of $f(z)$ at z_i.
Pf: Choose a contour $\mathcal{C}$ that's similar to C but avoids enclosing these singularities, as shown in Fig. 12 According to the Cauchy integral theorem,

$$\oint_{\mathcal{C}} = \oint_C + \oint_{-C_1} + \oint_{-C_2} + \cdots = 0. \quad (1.160)$$

Thus,

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots \quad (1.161)$$

$$= 2\pi i(a_{-1,1} + a_{-1,2} + \cdots) \quad \text{QED.}$$

If $f(z)$ has a simple pole at z_0, then

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots \quad (1.162)$$

Hence

$$a_{-1} = \lim_{z\to z_0}(z-z_0)f(z). \quad (1.163)$$

If $f(z)$ has a pole of order $n > 1$ at z_0, then

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + \cdots. \quad (1.164)$$

It follows that

$$a_{-1} = \frac{1}{(n-1)!}\lim_{z\to z_0}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]. \quad (1.165)$$

However, it is often easier to find a_{-1} with the Laurent expansion.

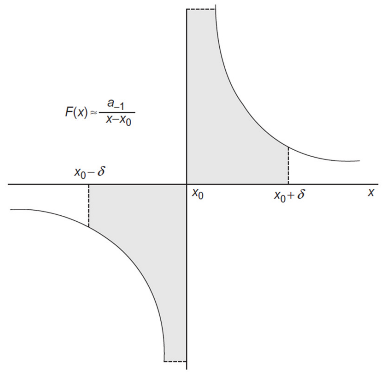

FIG. 13 Taking the Cauchy principal value of the integral of $1/x$.

Examples:

1. Residue of $\dfrac{1}{4z+1}$ at $z=-\dfrac{1}{4}$: $a_{-1}=\dfrac{1}{4}$.
2. Residue of $\dfrac{1}{\sin z}$ at $z=0$: $a_{-1}=1$.
3. Residue of $\dfrac{\ln z}{z^2+4}$ at $z=\pm 2i$: $a_{-1}=\dfrac{\pi}{8}-i\dfrac{\ln 2}{4}$.
4. Residue of $\dfrac{z}{\sin^2 z}$ at $z=\pi$: $a_{-1}=1$.
   Write $z=\pi+w$ and expand the function w.r.t. w.
5. Residue of $\dfrac{\cot \pi z}{z(z+2)}$ at $z=0$: $a_{-1}=-\dfrac{1}{4\pi}$.
6. Residue of $e^{-1/z}$ at $z=0$: $a_{-1}=-1$.

When the functions above are integrated along a contour C, the integrals would pick up the residues inside C, according to the residue theorem.

1. The argument theorem

Suppose $f(z)$ has P poles and N zeros in a region M. Near a pole or a zero at z_0, write

$$f(z)=(z-z_0)^{\mu}g(z), \tag{1.166}$$

where μ is the order of the pole or the multiplicity of the zero, and $g(z)$ is finite and nonzero at z_0. Thus,

$$f(z\to z_0)\simeq (z-z_0)^{\mu}. \tag{1.167}$$

Near z_0,

$$\frac{f'(z)}{f(z)}=\frac{\mu}{z-z_0}+\frac{g'(z)}{g(z)}. \tag{1.168}$$

That is, $f'(z)/f(z)$ has a simple pole at $z=z_0$ with a residue μ.

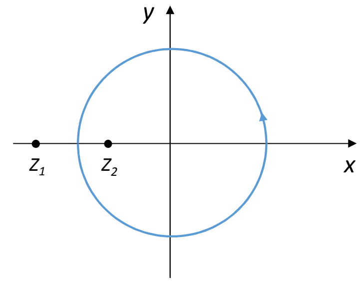

FIG. 14 The singularities and the contour C in Example 1.

If C encloses P poles and N zeros, then

$$\oint_C \frac{f'(z)}{f(z)}dz = \oint_C \frac{d}{dz}\ln f\, dz \tag{1.169}$$
$$= 2\pi i(N-P). \tag{1.170}$$

The second integral evaluates the change of the argument of $f(z)$ around C. Note that this theorem might be difficult to use. For example, $f(z)=z^3-2z+11$ has 3 zeros, but it's not easy to integrate $f'(z)/f(z)$.

I. Evaluation of integrals

Some real integrals can be evaluated with the help of complex integrals. We will study the following types of real integrals:

$$I_1 = \int_0^{2\pi} f(\sin\theta,\cos\theta)d\theta, \tag{1.171}$$
$$I_2 = \int_{-\infty}^{\infty} f(x)dx, \tag{1.172}$$
$$I_3 = \int_{-\infty}^{\infty} f(x)e^{iax}dx. \tag{1.173}$$

Before doing this, let's introduce a terminology: Suppose a function $f(x)$ has a singularity at $x=x_0$. When it is integrated over an interval $[b,a]$ that includes x_0, one can take

$$P\int_b^a f(x)dx = \lim_{\delta\to 0}\left(\int_b^{x_0-\delta} f(x)dx + \int_{x_0+\delta}^a f(x)dx\right). \tag{1.174}$$

That is, the point x_0 is approached from two sides with the same rate (Fig. 13). This is called the **Cauchy principle value** of the integral.

For example,

$$P\int_{-1}^{1}\frac{dx}{x}=\lim_{\delta\to 0}\left(\int_{-1}^{-\delta}\frac{dx}{x}+\int_{\delta}^{1}\frac{dx}{x}\right)=0. \tag{1.175}$$

Note that if the singularity is approached from two sides with different rates, then the result could be different,

$$\lim_{\delta\to 0}\left(\int_{-1}^{-2\delta}\frac{dx}{x}+\int_{\delta}^{1}\frac{dx}{x}\right)=\ln 2. \tag{1.176}$$

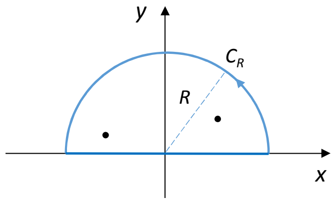

FIG. 15 A contour C closed by a large semicircle C_R in the upper half-plane.

$$I_1 = \int_0^{2\pi} f(\sin\theta, \cos\theta) d\theta, \tag{1.177}$$

in which f is a rational function of $\sin\theta$ and $\cos\theta$.

Recipe: Introduce

$$z = e^{i\theta}, \; dz = id\theta e^{i\theta}, \tag{1.178}$$

and substitute $d\theta = dz/iz$, as well as

$$\sin\theta = \frac{z - z^{-1}}{2i}, \; \cos\theta = \frac{z + z^{-1}}{2}. \tag{1.179}$$

The integral above would become an integral of $f(z)$. We can then use the residue theorem to evaluate it.

Ex 1:

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \quad |a| < 1 & (1.180)\\ &= \frac{2}{ia} \oint_C \frac{dz}{z^2 + \frac{2}{a}z + 1}, \quad C \text{ is a unit circle} & (1.181)\\ &= \frac{2}{ia} \underbrace{\oint_C \frac{dz}{(z - z_1)(z - z_2)}}_{=2\pi i \frac{1}{z_2 - z_1}}, \; z_{1/2} = -\frac{1 \pm \sqrt{1 - a^2}}{a} & \\ &= \frac{2\pi}{\sqrt{1 - a^2}}. & (1.182) \end{aligned}$$

Note that only z_2 is inside C (Fig. 14).

Ex 2:

$$\begin{aligned} I &= \int_0^{2\pi} \frac{\cos 2\theta}{5 - 4\cos\theta} d\theta & (1.183)\\ &= \frac{i}{4} \oint_C \frac{z^4 + 1}{z^2 \left(z - \frac{1}{2}\right)(z - 2)} dz. & (1.184) \end{aligned}$$

The integrand has a simple pole at $z = 1/2$ and a double pole at $z = 0$ within C. Its residue at $z = 1/2$ is $-17/6$; while the residue at z=0 is $5/2$. Therefore,

$$I = \frac{i}{4} 2\pi i \left(\frac{5}{2} - \frac{17}{6}\right) = \frac{\pi}{6}. \tag{1.185}$$

$$I_2 = \int_{-\infty}^{\infty} f(x) dx \tag{1.186}$$

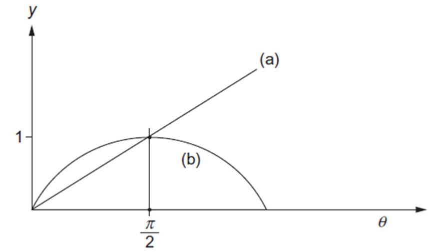

FIG. 16 (a) $y = 2\theta/\pi$. (b) $y = \sin\theta$.

With the assumptions,
1. $f(z)$ is analytic in the upper-half (or lower-half) complex plane, except a finite number of poles.
2. $f(z)z \to 0$ as $z \to \infty$.

Recipe: Consider a contour integral along the infinite closed loop C = x-axis+C_R shown in Fig. 15, C_R is a semi-circle with radius $R \to \infty$. Then

$$\begin{aligned} \oint_C f(z)dz &= \int_{-\infty}^{\infty} f(x)dx + \int_{C_R} f(z)dz \\ &= 2\pi i \sum [Res \; of \; f(z)]. & (1.187) \end{aligned}$$

Because of Assumption 2, the second integral

$$\left| \int_{C_R} f(z)dz \right| \le (\text{max of } |f(z)| \text{ on } C_R) \times 2\pi R \to 0. \tag{1.188}$$

Therefore,

$$I = \int_{-\infty}^{\infty} f(x)dx = \oint_C f(z)dz. \tag{1.189}$$

Ex 3:

$$\begin{aligned} I &= \int_0^{\infty} \frac{dx}{1 + x^2} & (1.190)\\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} & (1.191)\\ &= \frac{1}{2} \oint_C \frac{dz}{1 + z^2} & (1.192)\\ &= \frac{1}{2} 2\pi i \left. \frac{1}{x + i} \right|_{x=i} = \frac{\pi}{2}. & (1.193) \end{aligned}$$

Note:
1. This integral can also be calculated by conventional method, and

$$I = \tan^{-1} x \Big|_0^{\infty} = \frac{\pi}{2}. \tag{1.194}$$

2. One can also take a semi-circle around the lower-half complex plane. The result will be the same.

$$I_3 = \int_{-\infty}^{\infty} f(x) e^{iax} dx, \; a > 0. \tag{1.195}$$

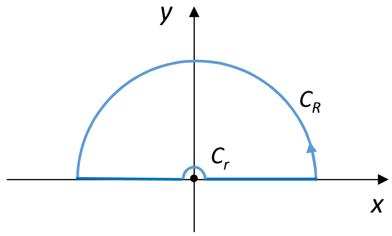

FIG. 17 A contour C that avoids the singularity at the origin.

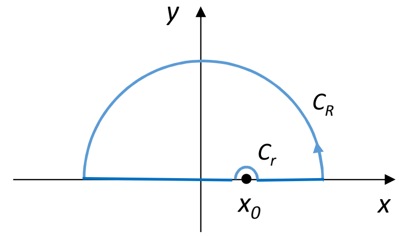

FIG. 18 A contour C that avoids the singularity at x_0.

With the assumptions,
1. $f(z)$ is analytic in the upper-half (or lower-half) complex plane, except a finite number of poles.
2. $f(z) \to 0$ as $|z| \to \infty$.

Consider a contour integral along the infinite closed loop C shown in Fig. 15, then

$$\oint_C f(z)e^{iaz}dz = \int_{-\infty}^{\infty} f(x)e^{iax}dx + \int_{C_R} f(z)e^{iaz}dz$$
$$= 2\pi i \sum [Res\ of\ f(z)e^{iaz}]. \quad (1.196)$$

Because of Assumption 2, the second integral is zero. This is known as **Jordan's lemma**,

$$I_R \equiv \left|\int_{C_R} f(z)e^{iaz}dz\right| \to 0, \text{ as } R \to \infty. \quad (1.197)$$

Pf: Along the semi-circle C_R, $z = Re^{i\theta}$, hence

$$I_R = \int_{C_R} f(z)e^{iaR\cos\theta}e^{-aR\sin\theta}id\theta Re^{i\theta}. \quad (1.198)$$

Suppose $|f| < f_0$ along C_R, then

$$|I_R| \le \int_0^{\pi} |f(z)| \left|e^{iaR\cos\theta}e^{-aR\sin\theta}id\theta Re^{i\theta}\right| \quad (1.199)$$
$$\le f_0 R \int_0^{\pi} e^{-aR\sin\theta}d\theta \quad (1.200)$$
$$= 2f_0 R \int_0^{\pi/2} e^{-aR\sin\theta}d\theta. \quad (1.201)$$

Within $\theta \in [0, \pi/2]$ (see Fig. 16), $\frac{2}{\pi}\theta \le \sin\theta$, hence

$$|I_R| \le 2f_0 R \int_0^{\pi/2} e^{-2aR\theta/\pi}d\theta \quad (1.202)$$
$$= 2f_0 R \frac{\pi}{2aR}\left(1 - e^{-aR}\right) \quad (1.203)$$
$$\le \frac{\pi}{a} f_0, \quad f_0 \to 0 \text{ as } R \to \infty. \quad (1.204)$$

From Eq. (1.196), we then have

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i \sum \left[Res \text{ of } f(z)e^{iaz}\right]. \quad (1.205)$$

Ex 4:

$$I = \int_0^{\infty} \frac{\cos x}{x^2+1}dx \quad (1.206)$$
$$= \frac{1}{2}\int_0^{\infty} \frac{e^{ix}}{x^2+1}dx + \frac{1}{2}\int_0^{\infty} \frac{e^{-ix}}{x^2+1}dx \quad (1.207)$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}dx \quad (1.208)$$
$$= \frac{1}{2}\oint_C \frac{e^{iz}}{z^2+1}dz \quad (1.209)$$
$$= \frac{1}{2} 2\pi i \left.\frac{e^{iz}}{z+i}\right|_{z=i} = \frac{\pi}{2e}. \quad (1.210)$$

Ex 5:

$$I = \int_0^{\infty} \frac{\sin x}{x}dx \quad (1.211)$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}dx \quad (1.212)$$
$$= \frac{1}{2i} P \int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx. \quad (1.213)$$

There is a simple pole at $x = 0$. Choose an infinite closed loop to avoid the pole at $z = 0$, as shown in Fig. 17, then

$$0 = \oint_C \frac{e^{iz}}{z}dz = P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx \quad (1.214)$$
$$+ \int_{C_r} \frac{e^{iz}}{z}dz + \int_{C_R} \frac{e^{iz}}{z}dz, \quad (1.215)$$

in which C_r is clockwise. The last integral would vanish as $R \to \infty$. For the second integral along C_r, $z = re^{i\theta}$ ($r \ll 1$), and

$$\int_{C_r} \frac{e^{iz}}{z}dz = \int_{\pi}^{0} \frac{id\theta re^{i\theta}}{re^{i\theta}} = -\pi i. \quad (1.216)$$

Thus, finally $I = \frac{\pi}{2}$. You will get the same result if C_r goes under the origin, that is if the contour C encloses the singularity.

J. Some applications

1. Plemelj formula

Suppose $f(x)$ has no singularity, and we'd like to evaluate the following integral,

$$P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx. \tag{1.217}$$

Consider the infinite closed loop C shown in Fig. 18, then

$$\oint_C = P\int_{-\infty}^{\infty} + \int_{C_r} + \int_{C_R}, \tag{1.218}$$

in which C_r is clockwise. If $f(z)\to 0$ as $z\to\infty$, then for $z_0 = x_0$ outside C,

$$\begin{aligned} 0 &= \oint_C \frac{f(z)}{z-x_0}dz && (1.219)\\ &= P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx - i\pi f(x_0) + 0. && (1.220)\end{aligned}$$

Therefore,

$$P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx = i\pi f(x_0). \tag{1.221}$$

The same result can be obtained if the small semi-circle C_r goes under x_0. You may apply this result to Eq. (1.213) to get the same result above.

Instead of detouring around x_0, we can displace the singularity slightly off the x-axis, $z_0 = x_0 - i\varepsilon$. It follows that

$$\begin{aligned} 0 &= \oint_C \frac{f(z)}{z-x_0+i\varepsilon}dz && (1.222)\\ &= \int_{-\infty}^{\infty}\frac{f(x)}{x-x_0+i\varepsilon}dx. && (1.223)\end{aligned}$$

This should agree with the result in Eq. (1.220), hence

$$\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0+i\varepsilon}dx = P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx - i\pi f(x_0). \tag{1.224}$$

Alternatively, we can let $z_0 = x_0 + i\varepsilon$ and get

$$\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0-i\varepsilon}dx = P\int_{-\infty}^{\infty}\frac{f(x)}{x-x_0}dx + i\pi f(x_0). \tag{1.225}$$

Eqs. (1.224) and (1.225) are valid for any analytic $f(x)$, therefore we can simply write

$$\lim_{\varepsilon\to 0}\frac{1}{x-x_0\pm i\varepsilon} = P\frac{1}{x-x_0} \mp i\pi\delta(x-x_0). \tag{1.226}$$

This is called the **Plemelj formula**.

One can also use

$$\frac{1}{x-x_0\pm i\varepsilon} = \frac{x-x_0}{(x-x_0)^2+\varepsilon^2} \mp i\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2}. \tag{1.227}$$

It is known that

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\frac{\varepsilon}{(x-x_0)^2+\varepsilon^2} = \delta(x-x_0), \tag{1.228}$$

and then we can also reach Eq. (1.226).

2. Kramers-Kronig relations

Recall that in electromagnetism,

$$D(\omega) = \varepsilon(\omega)E(\omega) = \varepsilon_0[1+\chi(\omega)]E(\omega). \tag{1.229}$$

The electric susceptibility $\chi(z)$ is analytic in the upper-half plane (including the x-axis), and $\chi(z)\to 0$ as $z\to\infty$. See Jackson for a detailed explanation. Note that for a conductor, its $\chi(\omega)$ can have a simple pole at $\omega=0$. Such a case is excluded in the discussion below.

With an infinite semi-circle C, one has

$$\begin{aligned} \chi(z) &= \frac{1}{2\pi i}\oint_C \frac{\chi(\omega')}{\omega'-z}d\omega' && (1.230)\\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\chi(\omega')}{\omega'-z}d\omega', && (1.231)\end{aligned}$$

in which z is within the upper-half plane. Choose $z = \omega + i\epsilon$ $(\omega\in R)$, and use the Plemelj formula above, then

$$\begin{aligned} \chi(\omega) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\chi(\omega')}{\omega'-\omega-i\epsilon}d\omega' && (1.232)\\ &= \frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{\chi(\omega')}{\omega'-\omega}d\omega' + \frac{1}{2}\chi(\omega), \\ \to\quad \chi(\omega) &= \frac{1}{\pi i}P\int_{-\infty}^{\infty}\frac{\chi(\omega')}{\omega'-\omega}d\omega'. && (1.233)\end{aligned}$$

We have used $\chi(\omega+i\epsilon) = \omega(\omega)$ as $\epsilon\to 0$, since it is analytic at ω.

Decompose χ into real part and imaginary part,

$$\chi = \chi_1 + i\chi_2, \tag{1.234}$$

then

$$\begin{aligned} \chi_1(\omega) &= +\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\chi_2(\omega')}{\omega'-\omega}d\omega', && (1.235)\\ \chi_2(\omega) &= -\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\chi_1(\omega')}{\omega'-\omega}d\omega'. && (1.236)\end{aligned}$$

They are known as **Kramers-Kronig relations**.

Let n be the **index of refraction**. We know that

$$n^2 = \frac{\varepsilon}{\varepsilon_0} = 1+\chi, \tag{1.237}$$

thus $\chi = n^2 - 1$. Using the Kramers-Kronig (KK) relations, we can have relations connecting the real part and the imaginary part of $n = n_1 + in_2$.

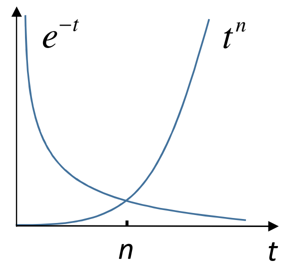

FIG. 19 The integrand of the Gamma-function integral is composed of e^{-t} and t^n.

Physically, the frequency ω is positive, hence it helps to rewrite the KK relations in the following way: First, $\chi(\omega)$ is the Fourier transform of $\chi(t)$,

$$\chi(\omega) = \int_{-\infty}^{\infty} \chi(t) e^{i\omega t} dt. \tag{1.238}$$

Since electric susceptibility $\chi(t) \in R$, hence $\chi(-\omega) = \chi^*(\omega)$, which means

$$\begin{aligned} \chi_1(-\omega) &= +\chi_1(\omega), & (1.239) \\ \chi_2(-\omega) &= -\chi_2(\omega). & (1.240) \end{aligned}$$

As a result,

$$\begin{aligned} \chi_1(\omega) &= +\frac{2}{\pi}\int_0^{\infty} \frac{\omega' \chi_2(\omega')}{\omega'^2 - \omega^2} d\omega', & (1.241) \\ \chi_2(\omega) &= -\frac{2\omega}{\pi}\int_0^{\infty} \frac{\chi_1(\omega')}{\omega'^2 - \omega^2} d\omega'. & (1.242) \end{aligned}$$

For example, if there is a sharp absorption peak at ω_0,

$$\chi_2(\omega) \simeq \frac{\alpha}{\omega_0}\delta(\omega - \omega_0), \tag{1.243}$$

then one gets from the KK relations,

$$\chi_1(\omega) \simeq \frac{2\alpha}{\pi}\frac{1}{\omega_0^2 - \omega^2}. \tag{1.244}$$

K. Methods of approximation

1. Gamma function

Recall that the Gamma function,

$$\begin{aligned} \Gamma(n+1) = n! &= \int_0^{\infty} t^n e^{-t} dt & (1.245) \\ &= \int_0^{\infty} e^{n \ln t - t} dt. & (1.246) \end{aligned}$$

The integrand is a product of an increasing function t^n and a decreasing function e^{-t}, as shown in Fig. 19. Therefore, the integrand, and hence its exponent $f(t) = n \ln t - t$ has a maximum. Also

$$f'(t) = 0, \text{ at } t = n, \text{ and } f''(n) = -\frac{1}{n}. \tag{1.247}$$

The value of $f(t)$ near the extrema can be approximated as

$$f(t) \simeq f(n) + 0 + \frac{1}{2} f''(n)(t-n)^2. \tag{1.248}$$

Therefore,

$$n! \simeq e^{n \ln n - n} \underbrace{\int_0^{\infty} e^{-\frac{1}{2n}(t-n)^2} dt}_{\equiv I}. \tag{1.249}$$

Since the peak of the exponential in the integrand is located (supposedly far) away from the origin, one can extend the range of integration without affecting the integral much,

$$I \simeq \int_{-\infty}^{\infty} e^{-\frac{1}{2n}(t-n)^2} dt = \sqrt{2\pi n}. \tag{1.250}$$

It follows that

$$n! \simeq n^n e^{-n} \sqrt{2\pi n}. \tag{1.251}$$

This is known as **Stirling's formula**. This approximation gets better when n gets larger. However, even if $n = 1$, which gives $e \simeq \sqrt{2\pi}$, the error is only about 8%.

2. Method of stationary phase

Consider the following integral,

$$I = \int_b^a f(\omega) e^{i\phi(\omega)} d\omega. \tag{1.252}$$

Suppose the phase $\phi(\omega) \in R$ oscillates rapidly compared to the variation of $f(\omega)$, but is stationary at ω_0.

Near the stationary point,

$$\phi(\omega) \simeq \phi(\omega_0) + \frac{1}{2}\phi''(\omega_0)(\omega - \omega_0)^2, \tag{1.253}$$

hence

$$I \simeq e^{i\phi(\omega_0)} \int_b^a f(\omega) e^{\frac{i}{2}\phi''(\omega-\omega_0)^2} d\omega. \tag{1.254}$$

The contribution to this integral is mainly from the region near the stationary point. Far away from ω_0, the integrand oscillates rapidly so that part contributes little to the integral.

Since $f(\omega)$ varies slowly near ω_0,

$$\begin{aligned} I &\simeq e^{i\phi(\omega_0)} f(\omega_0) \int_{-\infty}^{\infty} e^{\frac{i}{2}\phi''(\omega-\omega_0)^2} d\omega & (1.255) \\ &= e^{i\phi(\omega_0)} f(\omega_0) \sqrt{\frac{2\pi i}{\phi''(\omega_0)}}, & (1.256) \end{aligned}$$

in which the following equation has been used,

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}, \ \alpha \in C. \tag{1.257}$$

For example, the integral representation of the Bessel function is

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(nt - x\sin t)dt, \tag{1.258}$$

which can be rewritten as

$$J_n(x) = Re\frac{1}{\pi}\int_0^{\pi} e^{int}e^{-ix\sin t}dt. \tag{1.259}$$

Suppose the argument $|x| \gg 1$, then the second exponential oscillates rapidly with respect to t. In comparison, the first exponent varies slowly and can be taken as the function f in Eq. (1.252).

Within $t \in [0, \pi]$, $\phi(t) = -x\sin t$ is stationary at $t = \pi/2$, and

$$\phi\left(\frac{\pi}{2}\right) = -x,\ \phi''\left(\frac{\pi}{2}\right) = x. \tag{1.260}$$

Applying Eq. (1.256), we then have

$$\begin{aligned} J_n(x) &\simeq Re\left(e^{-ix}e^{in\frac{\pi}{2}}\sqrt{\frac{2i}{\pi x}}\right) \qquad (1.261) \\ &= \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right),\ \text{as } |x| \gg 1. \end{aligned}$$