Lecture notes on classical electrodynamics

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Contents

I. SCATTERING OF EM WAVE

An EM wave would be scattered by an obstacle, large or small. In general three length scales are involved in this process: wavelength λ, size d of the obstacle, and distance of detection R. If we assume $R \gg d, \lambda$ and just compare d with λ, then we have:
1. $d \ll \lambda$, which is called Rayleigh scattering
2. $d \simeq \lambda$, which called Mie scattering
3. $d \gg \lambda$, which is the scattering of optical rays
We'll investigate case 1 in this chapter.

A. Single scatterer

Let's start with the scattering of a single scatterer. Consider the **Rayleigh scattering** with $d \ll \lambda$ (or $kd \ll 1$). The process of scattering follows three steps:

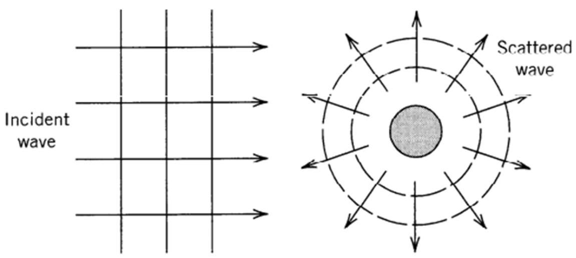

FIG. 1 Incident wave and scattered wave

1. There is an incoming plane wave with the electric field,

$$\begin{aligned} \mathbf{E}_i(\mathbf{r},t) &= \mathbf{E}_0 e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega t)} \equiv \mathbf{E}_i(\mathbf{r})e^{-i\omega t}, \quad (1.1)\\ \text{where } \mathbf{E}_i(\mathbf{r}) &= E_0\hat{\epsilon}_0 e^{i\mathbf{k}_0\cdot\mathbf{r}}. \quad (1.2)\end{aligned}$$

The magnetic field of the EM wave is

$$\begin{aligned} \mathbf{H}_i(\mathbf{r},t) &= \mathbf{H}_i(\mathbf{r})e^{-i\omega t}, \quad (1.3)\\ \text{where } \mathbf{H}_i(\mathbf{r}) &= \frac{1}{Z_0}\hat{\mathbf{k}}_0\times\mathbf{E}_i(\mathbf{r}). \quad (1.4)\end{aligned}$$

2. The oscillating fields interact with the scatterer and induce dipole moments $\mathbf{p}$, $\mathbf{m}$.
3. The oscillating dipoles re-radiate EM waves in all directions (Fig. 1)

In Chap 20, we have studied the EM radiation due to oscillating dipoles. Quoting the results there, the electromagnetic field from a scattered EM wave would be,

$$\begin{aligned} \mathbf{E}_s(\mathbf{r},t) &= \mathbf{E}_s(\mathbf{r})e^{-i\omega t}, \quad (1.5)\\ \text{where } \mathbf{E}_s(\mathbf{r}) &= \frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\left[-\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{p})-\hat{\mathbf{k}}\times\frac{\mathbf{m}}{c}\right],\end{aligned}$$

and

$$\begin{aligned} \mathbf{H}_s(\mathbf{r},t) &= \mathbf{H}_s(\mathbf{r})e^{-i\omega t}, \quad (1.6)\\ \text{where } \mathbf{H}_s(\mathbf{r}) &= \frac{1}{Z_0}\hat{\mathbf{k}}\times\mathbf{E}_s(\mathbf{r}). \quad (1.7)\end{aligned}$$

What's left to be determined is that, given a scatterer, how do the induced dipoles relate to incoming fields? We will come back to this later.

In general, the total electric field can be written in the following form,

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \mathbf{E}_i(\mathbf{r})+\mathbf{E}_s(\mathbf{r}) \quad (1.8)\\ &= E_0\left(\hat{\epsilon}_0 e^{i\mathbf{k}_0\cdot\mathbf{r}}+\mathbf{f}(\mathbf{k})\frac{e^{ikr}}{r}\right), \quad (1.9)\end{aligned}$$

where $\mathbf{f}(\mathbf{k})$ is called **scattering amplitude**. For example, for the scattered field in Eq. (1.5),

$$\mathbf{f}(\mathbf{k}) = -\frac{k^2}{4\pi\varepsilon_0 E_0}\left[\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{p})+\hat{\mathbf{k}}\times\frac{\mathbf{m}}{c}\right]. \quad (1.10)$$

Suppose the power (or energy flux) of the radiation scattered through solid angle $d\Omega$ is dP, then

$$dP = \mathbf{S}\cdot\hat{\mathbf{k}}\, r^2 d\Omega, \quad (1.11)$$

where $\mathbf{S}$ is the Poynting vector. Evidently, it is proportional to the power (or energy flux) of the incident EM wave. The energy flux of an incident wave passing through an area A perpendicular to $\mathbf{k}_0$ is,

$$P_i = \mathbf{S}_i \cdot \hat{\mathbf{k}}_0 A. \tag{1.12}$$

Thus, the angular distribution of the scattered radiation can be characterized by the ratio,

$$\begin{aligned} d\sigma &= \frac{\text{energy flux scattered through } d\Omega}{\text{incident energy flux/area}} && (1.13) \\ &= \frac{dP(\text{within } d\Omega)}{P_i/A} = \frac{\mathbf{S}\cdot\hat{\mathbf{k}}\, r^2 d\Omega}{\mathbf{S}_i\cdot\hat{\mathbf{k}}_0}. && (1.14) \end{aligned}$$

It has the dimension of an area. Thus, define the **differential cross-section** as,

$$\frac{d\sigma}{d\Omega} = \frac{\mathbf{S}\cdot\hat{\mathbf{k}}}{\mathbf{S}_i\cdot\hat{\mathbf{k}}_0}\, r^2, \tag{1.15}$$

For a time-harmonic EM wave with $e^{-i\omega t}$-dependence, after averaging over time (see Chap 16),

$$\mathbf{S}\cdot\hat{\mathbf{k}} = \frac{1}{2} Re\left[\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})\cdot\hat{\mathbf{k}}\right] = \frac{1}{2Z_0}|\mathbf{E}(\mathbf{r})|^2. \tag{1.16}$$

For brevity, the bracket $\langle\ \rangle_T$ has been dropped. The differential cross-section is,

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{|\mathbf{E}_s(\mathbf{r})|^2}{|\mathbf{E}_i(\mathbf{r})|^2} r^2 = |\mathbf{f}(\mathbf{k})|^2 && (1.17) \\ &= \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{p}) + \hat{\mathbf{k}}\times\frac{\mathbf{m}}{c}\right|^2. && (1.18) \end{aligned}$$

If the scattered EM wave is a mixture with different polarizations, and we are only interested in the power with certain polarization $\hat{\boldsymbol{\epsilon}}$, then

$$\mathbf{S}\cdot\hat{\mathbf{k}} = \frac{1}{2Z_0}|\hat{\boldsymbol{\epsilon}}^*\cdot\mathbf{E}(\mathbf{r})|^2. \tag{1.19}$$

The complex conjugate $\hat{\boldsymbol{\epsilon}}^*$ is needed for circular polarization. The differential cross-section for polarization $\hat{\boldsymbol{\epsilon}}$ is,

$$\begin{aligned} \frac{d\sigma_{\hat{\boldsymbol{\epsilon}}}}{d\Omega} &= \frac{|\hat{\boldsymbol{\epsilon}}^*\cdot\mathbf{E}_s(\mathbf{r})|^2}{|\hat{\boldsymbol{\epsilon}}_0\cdot\mathbf{E}_i(\mathbf{r})|^2} r^2 = |\hat{\boldsymbol{\epsilon}}^*\cdot\mathbf{f}(\mathbf{k})|^2 && (1.20) \\ &= \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\hat{\boldsymbol{\epsilon}}^*\cdot\left(\mathbf{p} - \hat{\mathbf{k}}\times\frac{\mathbf{m}}{c}\right)\right|^2. && (1.21) \end{aligned}$$

Note that the scattered power is proportional to k^4, or λ^{-4}. That is, an EM wave with a shorter wavelength is scattered more. This is **Rayleigh** k^4 (or ω^4)**-law**.

Rayleigh first obtained this k^4-law with the following dimensional analysis (Bohren and Fraser, 1985):
First, the strength of scattered electric field should be proportional to the incident field, $E_s \propto E_0$.
Second, energy of the scattered wave is conserved during propagation. So its *intensity* should be proportional to $1/r^2$. Thus, $E_s \propto 1/r$.
Third, if the size of a particle scatterer (composed of many molecules) is much smaller than wavelength, then the response of these molecules are coherent. Thus, the strength of the scattered field should be proportional to the volume v of the particle.
From these three points, we have

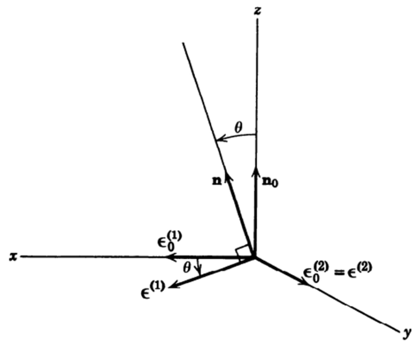

FIG. 2 Polarizations of incident wave along $\hat{\mathbf{n}}_0$ and scattered wave along $\hat{\mathbf{n}}$.

$$E_s \propto E_0\frac{v}{r}. \tag{1.22}$$

This is not dimensionally balanced yet. In addition to $v^{1/3}$ and r, there is one more length scale in this problem: wavelength λ of the EM wave. To balance the dimension, write

$$E_s \sim \alpha E_0 \frac{v}{r}\frac{1}{\lambda^2}, \tag{1.23}$$

where α is dimensionless. Therefore, the *intensity* of scattered radiation should be proportional to $1/\lambda^4$, or k^4.

1. Calculation of induced dipole

We now study the electric dipole of a scatterer induced by the incident field $\mathbf{E}_i$. Since the magnetic dipole radiation is weaker than the electric dipole radiation by a factor of d/λ (see Chap 20), so the magnetic dipole will be ignored when $d \ll \lambda$. Suppose the scatterer is a small dielectric sphere with radius a. Near the scatterer,

$$\begin{aligned} \mathbf{E}_i(\mathbf{r}) &= E_0\hat{\boldsymbol{\epsilon}}_0 e^{i\mathbf{k}_0\cdot\mathbf{r}} && (1.24) \\ &\simeq E_0\hat{\boldsymbol{\epsilon}}_0 \quad (kd \ll 1). && (1.25) \end{aligned}$$

Thus the field is nearly uniform. Since the *quasi-electrostatic approximation* is valid in the near zone (Chap 15), we can rely on the result of electrostatics. If the the relative permittivity of the sphere is ε_r, then the induced dipole $\mathbf{p}(t) = \mathbf{p}e^{-i\omega t}$, with (see Chap 6)

$$\mathbf{p} = 4\pi\varepsilon_0\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)a^3\mathbf{E}_0. \tag{1.26}$$

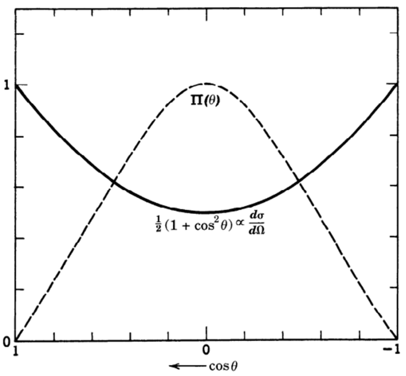

FIG. 3 Differential cross-section (solid line) and degree of polarization (dashed line).

Therefore,

$$\frac{d\sigma_{\hat{\boldsymbol{\epsilon}}}}{d\Omega} = k^4 a^6 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)^2 |\hat{\boldsymbol{\epsilon}}^* \cdot \hat{\boldsymbol{\epsilon}}_0|^2. \tag{1.27}$$

Suppose the incident wave is *not* polarized (like natural light). We are interested in the differential cross-section $\langle d\sigma_{\hat{\boldsymbol{\epsilon}}}/d\Omega\rangle_{\hat{\boldsymbol{\epsilon}}_0}$ of certain $\hat{\boldsymbol{\epsilon}}$ (averaged over initial polarizations $\hat{\boldsymbol{\epsilon}}_0$). The polarizations of incident wave and scattered wave are shown in Fig. 2. Assume $\mathbf{k}_0 \parallel \hat{\mathbf{z}}$, $\mathbf{k}$ lies on the $x - z$ plane, and

$$\hat{\boldsymbol{\epsilon}}_0 = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}, \tag{1.28}$$

$$\hat{\boldsymbol{\epsilon}}_1 = \cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{z}}, \tag{1.29}$$

$$\hat{\boldsymbol{\epsilon}}_2 = \hat{\mathbf{y}}. \tag{1.30}$$

Both $\hat{\boldsymbol{\epsilon}}_1$ and $\hat{\boldsymbol{\epsilon}}_2$ are perpendicular to $\mathbf{k}$, and $\hat{\boldsymbol{\epsilon}}_1 \perp \hat{\boldsymbol{\epsilon}}_2$. They are two special directions of the polarization of scattered wave: $\hat{\boldsymbol{\epsilon}}_1$ lies on the $\mathbf{k}_0$-$\mathbf{k}$ plane, and $\hat{\boldsymbol{\epsilon}}_2$ is perpendicular to the $\mathbf{k}_0$-$\mathbf{k}$ plane.

Case 1: $\mathbf{E}_s \parallel \mathbf{k}_0$-$\mathbf{k}$ plane

The polarization factor $|\hat{\boldsymbol{\epsilon}}^* \cdot \hat{\boldsymbol{\epsilon}}_0|^2 = |\hat{\boldsymbol{\epsilon}}_1 \cdot \hat{\boldsymbol{\epsilon}}_0|^2$. Thus, after averaging over initial polarizations,

$$\langle|\hat{\boldsymbol{\epsilon}}_1 \cdot \hat{\boldsymbol{\epsilon}}_0|^2\rangle_\phi \equiv \frac{1}{2\pi}\int_0^{2\pi} |\hat{\boldsymbol{\epsilon}}_1 \cdot \hat{\boldsymbol{\epsilon}}_0|^2 d\phi \tag{1.31}$$

$$= \frac{1}{2}\cos^2\theta. \tag{1.32}$$

Case 2: $\mathbf{E}_s \perp \mathbf{k}_0$-$\mathbf{k}$ plane

After averaging over initial polarizations,

$$\langle|\hat{\boldsymbol{\epsilon}}_2 \cdot \hat{\boldsymbol{\epsilon}}_0|^2\rangle_\phi \equiv \frac{1}{2\pi}\int_0^{2\pi} |\hat{\boldsymbol{\epsilon}}_2 \cdot \hat{\boldsymbol{\epsilon}}_0|^2 d\phi \tag{1.33}$$

$$= \frac{1}{2}. \tag{1.34}$$

It follows that,

$$\frac{d\sigma_\parallel}{d\Omega} = \frac{a^2}{2} k^4 a^4 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)^2 \cos^2\theta, \tag{1.35}$$

$$\frac{d\sigma_\perp}{d\Omega} = \frac{a^2}{2} k^4 a^4 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)^2 \cdot 1, \tag{1.36}$$

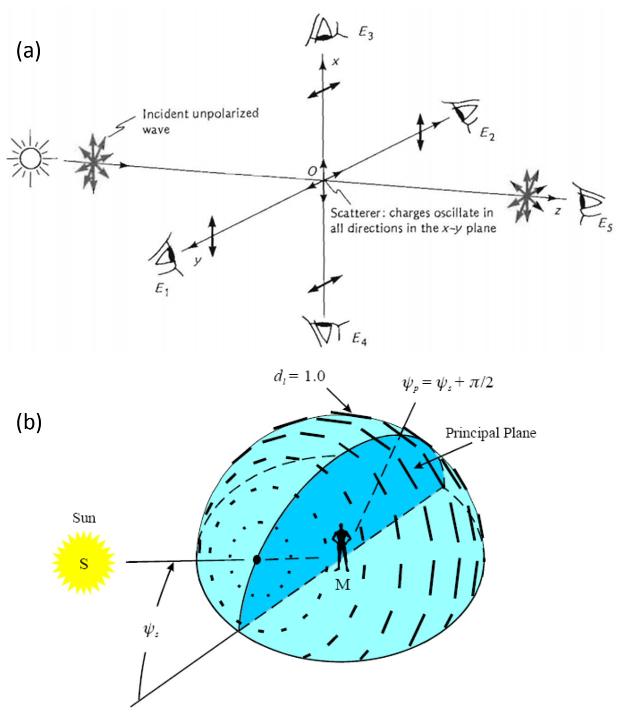

FIG. 4 (a) When viewed along a plane (containing the scatterer) perpendicular to the incident light, the direction of the polarization of scattered wave is the same as the direction of dipole oscillation. (b) Sunlight is most polarized in the plane (passing through the observer) perpendicular to the line of sun-observer. Fig. (b) from Smith, 2007.

and the total differential cross-section is,

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_\parallel}{d\Omega} + \frac{d\sigma_\perp}{d\Omega}. \tag{1.37}$$

The degree of polarization of scattered wave can be quantified with

$$\Pi \equiv \frac{\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega}}{\frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega}} \tag{1.38}$$

$$= \frac{\sin^2\theta}{1 + \cos^2\theta} \in [0, 1]. \tag{1.39}$$

Its angular dependence is shown in Fig. 3. It is maximum at $\theta = \pi/2$, when $\mathbf{k}$ is perpendicular to $\mathbf{k}_0$ (Fig. 4(a)).

Sunlight would be scattered by molecules in the air. As a result, the scattered sunlight is polarized. If you face the sun directly (Fig. 4(b)), and swivel your head around looking at the sky, then you will find out that the blue sky is most strongly polarized in a plane (you are in) perpendicular to the line connecting you and the sun.

2. Scattering from charged particle

Suppose a monochromatic plane wave passes through a point charge at the origin, the charge would oscillate and re-radiate an EM wave. This is called **Thomson scattering**. The equation of motion for the charge is,

$$m\ddot{\mathbf{r}} = qE_0\hat{\boldsymbol{\epsilon}}_0 e^{i(\mathbf{k}_0\cdot\mathbf{r}-\omega t)}, \tag{1.40}$$

and $\ddot{\mathbf{r}} = -\omega^2\mathbf{r}$. The dipole moment associated with the charge is $\mathbf{p} = q\mathbf{r} = e\ddot{\mathbf{r}}/\omega^2$ $(q = -e)$. According to Eq. (1.18), the differential cross-section for the Thomson scattering is,

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{p})\right|^2 && (1.41)\\ &= \frac{e^4k^4}{(4\pi\varepsilon_0 m\omega^2)^2}|\mathbf{k}\times(\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}}_0)|^2 && (1.42)\\ &= r_e^2|\hat{\mathbf{k}}\times\hat{\boldsymbol{\epsilon}}_0|^2, && (1.43)\end{aligned}$$

in which r_e is the **classical electron radius**,

$$r_e = \frac{e^2}{4\pi\varepsilon_0 mc^2} \simeq 2.8 \text{ fm}. \tag{1.44}$$

After integration over Ω, we have the total cross-section,

$$\sigma_{Th} = \frac{8\pi}{3}r_e^2. \tag{1.45}$$

B. Collection of scatterers

Consider a collection of particles (molecules or other dipole scatterers) that maybe orderly or randomly spaced. The position of the detector is far away so that the far-zone approximation remains valid. Let's again focus only on the induced electric dipole radiation. A single scatterer at the origin generates (see Eq. (1.5))

$$\mathbf{E}_s(\mathbf{r}) = -\frac{k^2}{4\pi\varepsilon_0}\frac{e^{ikr}}{r}\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{p}). \tag{1.46}$$

It gives the differential cross-section (see Eq. (1.21)),

$$\frac{d\sigma_{\hat{\epsilon}}}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\hat{\boldsymbol{\epsilon}}^*\cdot\mathbf{p}\right|^2. \tag{1.47}$$

When there are many particles at locations $\mathbf{r}_j$, just superpose the scattered fields to get

$$\mathbf{E}_s(\mathbf{r}) = -\frac{k^2}{4\pi\varepsilon_0}\sum_j\frac{e^{ikR_j}}{R_j}\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{p}_j), \tag{1.48}$$

where $R_j = |\mathbf{r}-\mathbf{r}_j| \simeq r - \hat{\mathbf{r}}\cdot\mathbf{r}_j$. Thus,

$$\frac{e^{ikR_j}}{R_j} \simeq \frac{e^{ikr}}{r}e^{-i\mathbf{k}\cdot\mathbf{r}_j}. \tag{1.49}$$

It follows that,

$$\frac{d\sigma_{\hat{\epsilon}}}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\sum_j\hat{\boldsymbol{\epsilon}}^*\cdot\mathbf{p}_je^{-i\mathbf{k}\cdot\mathbf{r}_j}\right|^2. \tag{1.50}$$

The induced dipole is proportional to the incoming field,

$$\begin{aligned}\mathbf{p}_j &= \varepsilon_0\gamma_m\mathbf{E}_i(\mathbf{r}_j), && (1.51)\\ \text{where}\quad \mathbf{E}_i(\mathbf{r}_j) &= E_0\hat{\boldsymbol{\epsilon}}_0e^{i\mathbf{k}_0\cdot\mathbf{r}_j}, && (1.52)\end{aligned}$$

and γ_m is the **molecular polarizability** (for a molecular scatterer). Therefore,

$$\frac{d\sigma_{\hat{\epsilon}}}{d\Omega} = \frac{k^4}{(4\pi)^2}\gamma_m^2|\hat{\boldsymbol{\epsilon}}^*\cdot\hat{\boldsymbol{\epsilon}}_0|^2\left|\sum_j e^{-i\mathbf{q}\cdot\mathbf{r}_j}\right|^2, \quad q \equiv \mathbf{k}-\mathbf{k}_0. \tag{1.53}$$

The summation over j is called the **form factor** of the scatterers,

$$\mathcal{F}(\mathbf{q}) \equiv \left|\sum_{j=1}^N e^{-i\mathbf{q}\cdot\mathbf{r}_j}\right|^2. \tag{1.54}$$

For a random distribution of particles,

$$|\mathcal{F}(\mathbf{q})|^2 = \sum_{j=\ell}1 + \sum_{j,\ell(j\neq\ell)}e^{i\mathbf{q}\cdot(\mathbf{r}_\ell-\mathbf{r}_j)} \simeq N. \tag{1.55}$$

The second summation is nearly zero due to the random distribution of phases. Therefore,

$$\frac{d\sigma_{\hat{\epsilon}}}{d\Omega} = \frac{k^4}{(4\pi)^2}\gamma_m^2|\hat{\boldsymbol{\epsilon}}^*\cdot\hat{\boldsymbol{\epsilon}}_0|^2N. \tag{1.56}$$

Averaging over incident polarizations with Eqs. (1.32), (1.34), we get

$$\left\langle\frac{d\sigma_{\hat{\epsilon}}}{d\Omega}\right\rangle_\phi = \frac{k^4}{(4\pi)^2}N\gamma_m^2\frac{1}{2}\left(1+\cos^2\theta\right). \tag{1.57}$$

The total scattering cross-section is,

$$\sigma_T = \int d\Omega\left\langle\frac{d\sigma_{\hat{\epsilon}}}{d\Omega}\right\rangle = \frac{k^4}{(4\pi)^2}N\gamma_m^2\frac{8\pi}{3}. \tag{1.58}$$

On the other hand, consider a periodic array of particles (like the atoms in a crystal) at locations,

$$\mathbf{r}_n = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3, \tag{1.59}$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are **primitive lattice vectors**, and $n_{1,2,3} = 0, \cdots, N_{1,2,3}-1$. It is left as an exercise to show that,

$$|\mathcal{F}(\mathbf{q})|^2 = \frac{\sin^2\frac{1}{2}N_1(\mathbf{a}_1\cdot\mathbf{q})}{\sin^2\frac{1}{2}(\mathbf{a}_1\cdot\mathbf{q})}\frac{\sin^2\frac{1}{2}N_2(\mathbf{a}_2\cdot\mathbf{q})}{\sin^2\frac{1}{2}(\mathbf{a}_2\cdot\mathbf{q})}\frac{\sin^2\frac{1}{2}N_3(\mathbf{a}_3\cdot\mathbf{q})}{\sin^2\frac{1}{2}(\mathbf{a}_3\cdot\mathbf{q})}.$$

It is known that

$$\lim_{N\to\infty}\frac{1}{2\pi}\frac{\sin\left(N\frac{x}{2}\right)}{\sin\frac{x}{2}} = \sum_{m=-\infty}^{\infty}\delta(x-2\pi m). \tag{1.60}$$

Therefore, given a crystal, its form factor is non-zero when

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{q} &= 2\pi m, && (1.61) \\ \mathbf{a}_2 \cdot \mathbf{q} &= 2\pi n, && (1.62) \\ \mathbf{a}_3 \cdot \mathbf{q} &= 2\pi \ell, && (1.63) \end{aligned}$$

where m, n, ℓ are any integers. This is called the **Laue condition of diffraction**. Solving it for possible $\mathbf{q}$'s, then one can find diffraction spots along directions $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$ (Kittel, 2004).

If the wavelength is much longer than the distance between neighboring atoms, or $k_0 a_i \ll 1$, then the Laue diffraction condition can be satisfied only with $m = n = \ell = 0$ in 3D. Thus, $\mathbf{q} = 0$ and the scattered radiation is confined only to the forward direction. A crystal would be transparent if there is no other scattering or absorption involved.

1. Scattering cross-section and refraction index

The scattering cross-section σ_T is related to the molecular polarizability γ_m in Eq. (1.58). On the other hand, γ_m is related to electric polarization $\mathbf{P}$, which in turn relates to relative permittivity ε_r and index of refraction n. Thus, we are able to link the macroscopic n with the microscopic γ_m, as shown below.

First, for a given molecule.

$$\mathbf{p} = \varepsilon_0 \gamma_m \mathbf{E}_i. \quad (1.64)$$

For a homogeneous material with n_d moleculars per unit volume, the electric polarization

$$\mathbf{P} = n_d \mathbf{p}. \quad (1.65)$$

Thus,

$$\begin{aligned} \mathbf{D} &= \varepsilon_0 \mathbf{E}_i + \mathbf{P} && (1.66) \\ &= \varepsilon_0 (1 + n_d \gamma_m) \mathbf{E}_i. && (1.67) \\ \to \quad \varepsilon_r &= 1 + n_d \gamma_m. && (1.68) \end{aligned}$$

The scattering cross-section per molecule is (Eq. (1.58)),

$$\begin{aligned} \sigma = \frac{\sigma_T}{N} &= \frac{k^4}{6\pi n_d^2} (\varepsilon_r - 1)^2 && (1.69) \\ &= \frac{2k^4}{3\pi n_d^2} (n-1)^2. && (1.70) \end{aligned}$$

We have approximated $\varepsilon_r - 1 \simeq 2(n-1)$, assuming that ε_r is close to 1.

Now, suppose an EM wave is passing through a thin slab of material, then its intensity I changes according to

$$\frac{dI}{I} = -n_d \sigma dx. \quad (1.71)$$

The coefficient $n_d \sigma$ is called **extinction coefficient**. Hence,

$$I(x) = I_0 e^{-n_d \sigma x} = I_0 e^{-x/\Lambda}. \quad (1.72)$$

The **attenuation length** is defined as,

$$\Lambda = \frac{1}{n_d \sigma} = \frac{3\pi n_d}{2k^4} \frac{1}{(n-1)^2}. \quad (1.73)$$

For example, for the air around you under standard condition (20 °C and 1 atm), $n_d \simeq 2.69 \times 10^{19}/\mathrm{cm}^3$, and $n = 1.00028$. We have, for visible light,

$$\begin{aligned} \text{red} : \lambda = 6500\ \mathring{A} &\ \to\ \Lambda = 188\ \text{km}, \\ \text{green} : \lambda = 5200\ \mathring{A} &\ \to\ \Lambda = 77\ \text{km}, \\ \text{violet} : \lambda = 4100\ \mathring{A} &\ \to\ \Lambda = 30\ \text{km}. \end{aligned} \quad (1.74)$$

Take into account the fact that the density of air diminishes exponentially with height. Assume the intensity of light on top of the atmosphere is 1, then on the surface of earth, we have (Jackson, 2002)

color	zenith	sunrise − sunset
red	0.96	0.21
blue	0.90	0.024
violet	0.76	0.000065

(1.75)

The sunlight is dimmer in sunrise-sunset because the atmospheric path is longer (Fig. 5(a)). Also, because of their shorter wavelengths, there is little blue/violet light reaching earth's surface during sunrise-sunset. That's why the sky is red during sunrise-sunset (Fig. 5(b)).

At other hours, a clear sky is blue. The reason is that when you look *away* from the direction of the sun, the deflected sun light undergoes more scatterings with molecules. Since a red light is less scattered compared to a blue/violet light, only the latter can survive large-angle deflection and reach your eyes (Fig. 5(a)). This is also the reason why the smoke of a cigarette is bluish (Fig. 5(c)). Around 1500 AC, Da Vinci had already surmised that the sky being blue could be the result of the sunlight scattered by small particles, based on the empirical fact that the smoke from dry wood appears blue (Kerker, 1969).

Note: Fog or smog are composed of liquid droplets, dust, smoke, or other pollutant particles roughly the size of 1 μm or less. Many of them are better described by the Mie scattering, instead of the Rayleigh scattering.

Furthermore, in Eq. (1.73), we see that $\Lambda \propto n_d$. Therefore, if the medium is continuous without atoms ($n_d \to \infty$), then Λ is infinite. Based on this result, Maxwell suggested that one can estimate n_d, hence the Avogadro constant by measuring the attenuation length (Jackson, 2002).

A remark: the average distance between neighboring molecules in the air is about 3 nm, which is much shorter

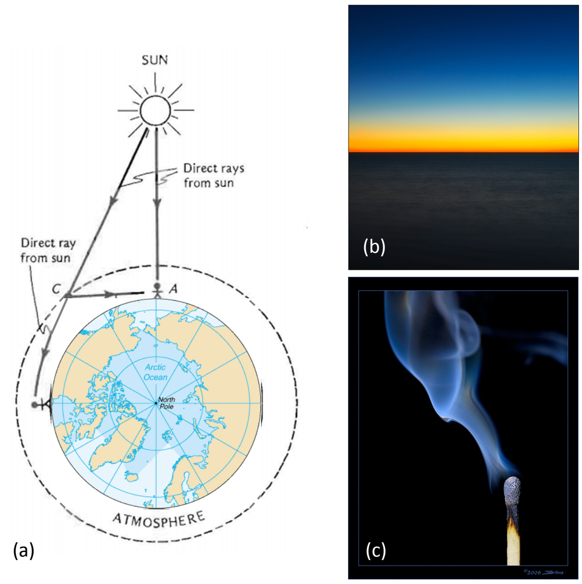

FIG. 5 (a) The paths of sunlight at high noon and near dawn/dusk. (b) The sky during sunrise. (c) A smoke looks bluish.

than the wavelength of visible light ($\lambda \sim 500$ nm). Thus, for the molecules within the range of λ, the responses could be coherent, instead of random. This shortcoming of the analysis that is based on incoherent scatterers can be remedied in the next subsection.

2. Scattering from density fluctuation

Instead of the molecular point of view, we can also study the scattering of EM wave by density fluctuation. Due to density fluctuation,

$$\begin{aligned} \mathbf{D}(\mathbf{r}) &= [\varepsilon_0 + \delta\varepsilon(\mathbf{r})]\mathbf{E}(\mathbf{r}) = \varepsilon_0 \mathbf{E}(\mathbf{r}) + \delta\mathbf{P}(\mathbf{r}), \qquad (1.76) \\ \mathbf{B}(\mathbf{r}) &= [\mu_0 + \delta\mu(\mathbf{r})]\mathbf{H}(\mathbf{r}) = \mu_0 \mathbf{H}(\mathbf{r}) + \mu_0 \delta\mathbf{M}(\mathbf{r})(1.77) \end{aligned}$$

where

$$\begin{aligned} \delta\mathbf{P}(\mathbf{r}) &= \delta\varepsilon \mathbf{E}(\mathbf{r}) \simeq \delta\varepsilon \mathbf{E}_i(\mathbf{r}), \;\; \delta\varepsilon \ll \varepsilon_0 \qquad (1.78) \\ \delta\mathbf{M}(\mathbf{r}) &= \frac{\delta\mu}{\mu_0}\mathbf{H}(\mathbf{r}) \simeq \frac{\delta\mu}{\mu_0}\mathbf{H}_i(\mathbf{r}), \;\; \delta\mu \ll \mu_0. \quad (1.79) \end{aligned}$$

In order to take advantage of earlier analysis, let's discretize the space with cells. Each cell has volume dV, and dipole moments,

$$\mathbf{p}_j = \delta\mathbf{P}(\mathbf{r}_j)dV, \;\; \mathbf{m}_j = \delta\mathbf{M}(\mathbf{r}_j)dV. \qquad (1.80)$$

According to Eq. (1.50) (now with magnetic dipoles added),

$$\frac{d\sigma_{\hat{\epsilon}}}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\sum_j \hat{\boldsymbol{\epsilon}}^* \cdot \left(\mathbf{p}_j - \hat{\mathbf{k}} \times \frac{\mathbf{m}_j}{c}\right) e^{-i\mathbf{k}\cdot\mathbf{r}_j}\right|^2, \qquad (1.81)$$

in which

$$\begin{aligned} \mathbf{p}_j = \delta\varepsilon \mathbf{E}_i(\mathbf{r}_j) &= \delta\varepsilon \mathbf{E}_0 e^{i\mathbf{k}_0\cdot\mathbf{r}_j} dV, \qquad (1.82) \\ \mathbf{m}_j = \frac{\delta\mu}{\mu_0}\mathbf{H}_i(\mathbf{r}_j) &= \frac{\delta\mu}{\mu_0}\mathbf{H}_0 e^{i\mathbf{k}_0\cdot\mathbf{r}_j} dV, \qquad (1.83) \end{aligned}$$

with

$$\mathbf{E}_0 = E_0 \hat{\boldsymbol{\epsilon}}_0, \;\; \mathbf{H}_0 = \frac{1}{Z_0}\hat{\mathbf{k}}_0 \times \mathbf{E}_0. \qquad (1.84)$$

Thus,

$$\begin{aligned} \frac{d\sigma_{\hat{\epsilon}}}{d\Omega} &= \frac{k^4}{(4\pi)^2}\Bigg|\sum_j dV \Bigg[\frac{\delta\varepsilon(\mathbf{r}_j)}{\varepsilon_0}\hat{\boldsymbol{\epsilon}}^* \cdot \hat{\boldsymbol{\epsilon}}_0 \qquad (1.85) \\ &+ \frac{\delta\mu(\mathbf{r}_j)}{\mu_0}(\hat{\mathbf{k}} \times \hat{\boldsymbol{\epsilon}}^*) \cdot (\hat{\mathbf{k}}_0 \times \hat{\boldsymbol{\epsilon}}_0)\Bigg] e^{-i\mathbf{q}\cdot\mathbf{r}_j}\Bigg|^2, \end{aligned}$$

Suppose cell-j with volume dV has $\bar{N}_j$ particles, then

$$\bar{N}_j = \bar{n}_j dV, \qquad (1.86)$$

where $\bar{n}_j$ is the density of particles. Write the fluctuation of particle density as $\Delta\bar{n}_j$, then

$$\frac{\delta\varepsilon_j}{\varepsilon_0} = \frac{\partial\varepsilon_r}{\partial\bar{n}}\Delta\bar{n}_j. \qquad (1.87)$$

Recall that $\mathbf{p}_j = \varepsilon_0\gamma_m \mathbf{E}_i(\mathbf{r}_j)$ (Eq. (1.51)). The **Clausius-Mossotti relation** (Chap 6) gives,

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \gamma_m \frac{\bar{n}}{3\varepsilon_0}, \qquad (1.88)$$

where $\bar{n}$ is the average particle density. This leads to

$$\frac{\partial\varepsilon_r}{\partial\bar{n}} = \frac{(\varepsilon_r - 1)(\varepsilon_r + 2)}{3\bar{n}}. \qquad (1.89)$$

Hence,

$$\frac{\delta\varepsilon_j}{\varepsilon_0} = \frac{(\varepsilon_r - 1)(\varepsilon_r + 2)}{3\bar{n}}\Delta\bar{n}_j. \qquad (1.90)$$

According to Eq. (1.85) (now without $\mathbf{m}_j$), one has

$$\begin{aligned} \frac{d\sigma_{\hat{\epsilon}}}{d\Omega} &= \frac{k^4}{(4\pi)^2}\left|\sum_j dV \frac{\delta\varepsilon(\mathbf{r}_j)}{\varepsilon_0}\hat{\boldsymbol{\epsilon}}^* \cdot \hat{\boldsymbol{\epsilon}}_0 e^{-i\mathbf{q}\cdot\mathbf{r}_j}\right|^2 \qquad (1.91) \\ &= \frac{k^4}{(4\pi)^2}\left[\frac{(\varepsilon_r - 1)(\varepsilon_r + 2)}{3\bar{n}}\right]^2 |\hat{\boldsymbol{\epsilon}}^* \cdot \hat{\boldsymbol{\epsilon}}_0|^2 \mathcal{F}(\mathbf{q}), \end{aligned}$$

with the form factor,

$$\mathcal{F}(\mathbf{q}) = \left|\sum_j \Delta\bar{N}_j e^{-i\mathbf{q}\cdot\mathbf{r}_j}\right|^2. \qquad (1.92)$$

If the correlation length between cells is small compared with λ, then the oscillating terms in the form factor can be ignored. Define the fluctuation of particle number as,

$$\Delta\bar{N} = \sqrt{\langle\bar{N}^2\rangle - \langle\bar{N}\rangle^2}, \tag{1.93}$$

then

$$\mathcal{F}(\mathbf{q}) = \sum_{j,k} \Delta\bar{N}_j \Delta\bar{N}_k = \Delta\bar{N}^2. \tag{1.94}$$

As a result, integration over solid angle involves only the angular part $|\hat{\boldsymbol{\epsilon}}^* \cdot \hat{\boldsymbol{\epsilon}}_0|^2$, which leads to a factor of $8\pi/3$ (see Eq. (1.56) and below). Thus, the total cross-section is,

$$\sigma_T = \frac{k^4}{(4\pi)^2}\left[\frac{(\varepsilon_r - 1)(\varepsilon_r + 2)}{3\bar{n}}\right]^2 \frac{8\pi}{3}\Delta\bar{N}^2. \tag{1.95}$$

From it we get the extinction coefficient,

$$\alpha = \bar{n}\sigma_T = \frac{k^4}{(4\pi)^2}\left[\frac{(\varepsilon_r - 1)(\varepsilon_r + 2)}{3}\right]^2 \frac{8\pi}{3\bar{n}}\Delta\bar{N}^2. \tag{1.96}$$

The fluctuation of particles can be related to **isothermal compressibility** β_T (Kubo, 1965),

$$\Delta\bar{N}^2 = \bar{n}k_BT\beta_T, \;\; \beta_T \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T. \tag{1.97}$$

Thus,

$$\alpha \simeq \frac{k^4}{6\pi}\left[\frac{(\varepsilon_r - 1)(\varepsilon_r + 2)}{3}\right]^2 k_BT\beta_T. \tag{1.98}$$

This is called the **Einstein-Smoluchowski relation**.

For a dilute gas, $\varepsilon_r \simeq 1$ and $\bar{n}k_BT\beta_T \simeq 1$, thus

$$\alpha = \frac{2k^4}{3\pi\bar{n}}(n-1)^2 \tag{1.99}$$

This is the same as the result in Eq. (1.73). Note that the Einstein-Smoluchowski relation deals with density fluctuation. It does *not* require the fluid to be composed of particles.

When a gas or liquid undergoes a phase transition, then near the critical point, the fluctuation of particle number becomes very large. In principle, $\beta_T \to \infty$ at the critical point. Thus, an EM wave would be strongly scattered and the fluid would become opaque. This is called **critical opalescence**. Note that near criticality, the coherence length of density fluctuation becomes longer than the wavelength and the treatment here is no longer accurate. See Sec. 10.2 of Jackson, 1998 for more details.

Problems:

1. Starting from Eq. (1.43), show that the total cross-section of the Thomson scattering is

$$\sigma_{Th} = \frac{8\pi}{3}r_e^2.$$

2. The atoms in a lattice are located at

$$\mathbf{r}_n = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3,$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are **primitive lattice vectors**, and $n_{1,2,3} = 0, \cdots, N_{1,2,3} - 1$. Show that the form factor squared of the lattice is

$$|\mathcal{F}(\mathbf{q})|^2 = \frac{\sin^2\frac{1}{2}N_1(\mathbf{a}_1\cdot\mathbf{q})}{\sin^2\frac{1}{2}(\mathbf{a}_1\cdot\mathbf{q})}\frac{\sin^2\frac{1}{2}N_2(\mathbf{a}_2\cdot\mathbf{q})}{\sin^2\frac{1}{2}(\mathbf{a}_2\cdot\mathbf{q})}\frac{\sin^2\frac{1}{2}N_3(\mathbf{a}_3\cdot\mathbf{q})}{\sin^2\frac{1}{2}(\mathbf{a}_3\cdot\mathbf{q})}.$$

II. DIFFRACTION OF EM WAVE

Diffraction can be considered as a type of scattering, when the size of an obstacle is much larger than the wavelength. The distinction between scattering and diffraction is not always clear. Some would say that "scattering" occurs from an object with smooth boundary, while "diffraction" is from an object with sharp edge (Zangwill, 2013). No matter what, for diffraction the boundary condition for fields is relatively more important, and this makes its study difficult. We will start with the diffraction of scalar wave, then investigate the diffraction of vector wave later.

A. Diffraction of scalar wave

Recall that the electromagnetic fields in free space follow the wave equations,

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0, \tag{2.1}$$

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = 0. \tag{2.2}$$

In the discussion below, we assume everything oscillates with the factor $e^{-i\omega t}$ (time-harmonic fields), e.g., $\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})e^{-i\omega t}$. Therefore, the wave equation becomes the Helmholtz equation,

$$\left(\nabla^2 + k^2\right)\mathbf{E}(\mathbf{r}) = 0, \;\; k = \omega/c. \tag{2.3}$$

We first focus on the equation for scalar wave,

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{r}) = 0, \;\; k = \omega/c. \tag{2.4}$$

It can be solved with the method of Green function. The **Green function** is the solution for a point source. For the Helmholtz equation,

$$\left(\nabla^2 + k^2\right)G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'), \tag{2.5}$$

the solution is,

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.6}$$

This means that, for a general Helmholtz equation,

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}, \tag{2.7}$$

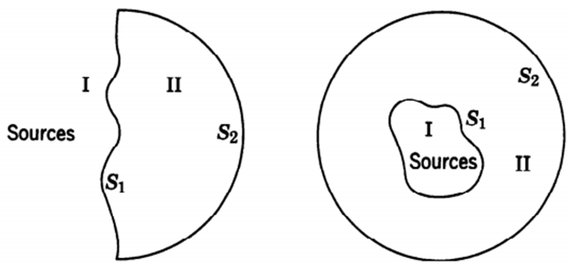

FIG. 6 The region of interest (II) is abounded by surfaces S_1 and S_2.

the solution is,

$$\psi(\mathbf{r}) = \int dv' G(\mathbf{r},\mathbf{r}')\frac{\rho(\mathbf{r}')}{\varepsilon_0} \tag{2.8}$$

$$= \frac{1}{4\pi\varepsilon_0}\int dv' \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}'). \tag{2.9}$$

The discussion above is for a system without physical boundary. If there are boundaries, then we need **Green's identity** below:

$$\int_V dv(\phi\nabla^2\psi - \psi\nabla^2\phi) = \int_S d\mathbf{a}\cdot(\phi\nabla\psi - \psi\nabla\phi), \tag{2.10}$$

in which volume V is bounded by surface S, $d\mathbf{a} = da\hat{\mathbf{n}}$, and $\hat{\mathbf{n}}$ points *out of* V.
Pf: First,

$$\nabla\cdot(\phi\nabla\psi) = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi, \tag{2.11}$$

$$\nabla\cdot(\psi\nabla\phi) = \nabla\psi\cdot\nabla\phi + \psi\nabla^2\phi. \tag{2.12}$$

Subtract the second equation from the first, one gets

$$\nabla\cdot(\phi\nabla\psi - \psi\nabla\phi) = \phi\nabla^2\psi - \psi\nabla^2\phi. \tag{2.13}$$

Integration over space and use the divergence theorem to write the RHS as a surface integral, then we get Eq. (2.10). QED.

Suppose $\psi(\mathbf{r})$ and $\phi(\mathbf{r})$ satisfy the Helmholtz equation,

$$\left(\nabla^2 + k^2\right)\psi(\mathbf{r}) = -\frac{\rho}{\varepsilon_0}, \tag{2.14}$$

$$\left(\nabla^2 + k^2\right)\phi_{\mathbf{r}'}(\mathbf{r}) = -\delta(\mathbf{r}-\mathbf{r}'), \tag{2.15}$$

in which $\phi_{\mathbf{r}'}(\mathbf{r})$ is obviously the Green function $G(\mathbf{r},\mathbf{r}')$. According to Green's identity,

$$\psi(\mathbf{r}) = \int_V dv' G(\mathbf{r},\mathbf{r}')\frac{\rho(\mathbf{r}')}{\varepsilon_0} + \int_S d\mathbf{a}'\cdot\left[G(\mathbf{r},\mathbf{r}')\nabla'\psi(\mathbf{r}') - \psi(\mathbf{r}')\nabla' G(\mathbf{r},\mathbf{r}')\right]. \tag{2.16}$$

If the region of interest V is not bounded by any surface, then the surface integral vanishes and we are back to Eq. (2.8).

In Fig. 6, there are two possible scenarios: In the first, The space is divided by a surface S_1 into two parts. The region of interest V is bounded by S_1 and another one S_2 at infinity. In the second, V is external to a source and is bounded by an inner surface S_1 and an outer surface S_2 at infinity. The surface integral above is over $S = S_1 + S_2$, but the integral over S_2 at infinity usually makes no contribution and can be ignored.

There are two major types of boundary condition (BC), either the value of ψ on S is given (**Dirichlet** BC), or the normal derivative, $\hat{\mathbf{n}}\cdot\nabla\psi = \partial\psi/\partial n$, on S is known (**Neumann** BC). In addition, there can be **mixed** BC: ψ is known in part of S, while $\partial\psi/\partial n$ is known for the rest of S.

For simplification, suppose there is no source ρ inside V and the first term of Eq. (2.16) can be dropped. For the Dirichlet BC, we can choose

$$G_D(\mathbf{r},\mathbf{r}') = 0 \text{ for } \mathbf{r}' \text{ on } S, \tag{2.17}$$

so that

$$\psi(\mathbf{r}) = -\int_S da'\psi(\mathbf{r}')\frac{\partial G_D}{\partial n'}(\mathbf{r},\mathbf{r}'). \tag{2.18}$$

On the other hand, for the Neumann BC, we can choose

$$\frac{\partial G_N}{\partial n'}(\mathbf{r},\mathbf{r}') = 0 \text{ for } \mathbf{r}' \text{ on } S, \tag{2.19}$$

so that

$$\psi(\mathbf{r}) = \int_S da' G_N(\mathbf{r},\mathbf{r}')\frac{\partial\psi(\mathbf{r}')}{\partial n'}. \tag{2.20}$$

For example, if S is the $x-y$ plane and V is the semi-infinite space above the plane, then

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} - \frac{e^{ik\bar{R}}}{\bar{R}}\right), \tag{2.21}$$

$$G_N(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} + \frac{e^{ik\bar{R}}}{\bar{R}}\right), \tag{2.22}$$

where $R = |\mathbf{r}-\mathbf{r}'|$, $\bar{R} = |\mathbf{r}-\bar{\mathbf{r}'}|$, $\bar{\mathbf{r}'}$ is the image point of $\mathbf{r}'$ with respect to the plane. Note that $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$, so that $\partial G_0/\partial n' = -\partial G_0/\partial z'$. For the Dirichlet BC, on the surface with $z = 0$,

$$\left.\frac{\partial G_D}{\partial z'}\right|_{z'=0} = 2\frac{\partial}{\partial z'}\left.\frac{e^{ikR}}{R}\right|_{z'=0}, \tag{2.23}$$

and

$$\psi(\mathbf{r}) = \frac{1}{2\pi}\int_S da'\psi(\mathbf{r}')\frac{\partial}{\partial z'}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.24}$$

Except for an infinite plane or a sphere ... etc, it's difficult to find the G_D or G_N that could vanish on S.

Since we are dealing with a dynamic field interacting with an obstacle, it's difficult to know the precise value of field on S. There are several ways to move ahead from Eq. (2.24). For a plane with an aperture, we can make

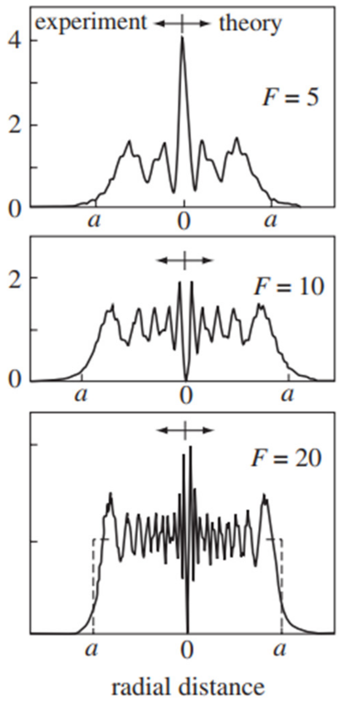

FIG. 7 Left: Intensity of an EM wave diffracted by a circular aperture with radius a. $F = a^2/\lambda z$, where z is the distance between aperture and screen. The theoretical results are based on Eq. (2.24) with the Kirchhoff approximation. Figs from Zangwill, 2013

the **Kirchhoff approximation**:
1. The amplitude of wave ψ is zero on S, except at the aperture.
2. At the aperture, the value of ψ is close to the value of the incident wave ψ_i. This can be valid only if the size of the aperture d is much larger than wavelength λ.

In Fig. 7, we see that the *scalar* diffraction theory with the *Kirchhoff approximation* can produce satisfactory result (see the explanation below Eq. (2.34)). The diffraction from a circular aperture will be studied in details later using *vector* diffraction theory.

In the far-field regime,

$$k|\mathbf{r}-\mathbf{r}'| \simeq kr - \underbrace{k\hat{\mathbf{r}}\cdot\mathbf{r}'}_{\sim\, kd} + \underbrace{\frac{k}{2r}\left[r'^2 - (\hat{\mathbf{r}}\cdot\mathbf{r}')^2\right]}_{\sim\, kd^2/r}. \tag{2.25}$$

If $r \gg kd^2$, then the quadratic terms can be neglected, and

$$\frac{\partial}{\partial z'}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \simeq -ik_z\frac{e^{ikr}}{r}e^{-i\mathbf{k}\cdot\mathbf{r}'} + O\left(\frac{1}{r^2}\right). \tag{2.26}$$

In terms of the **Fresnel number** $F \equiv d^2/\lambda z$, the regions of observation can be divided into (see Fig. 8(a))
1. $F < 1/2$ (and $r \gg d$), called **Fraunhofer diffraction**.
2. $F \simeq 1$ (and $r \gg d$), called **Fresnel diffraction**.

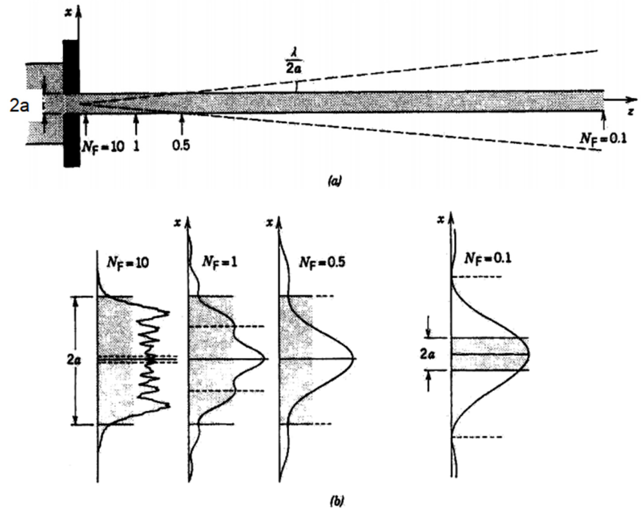

FIG. 8 (a) The diffraction of scalar wave from an aperture with size $2a$. Dashed lines show the width of the diffraction beam. It is larger than the geometric shadow when $F = 1/2$. (b) The evolution of the diffraction patterns as z increases. Figs from unknown source.

3. $F \gg 1$, called **near-field diffraction**.
The difference between their diffraction patterns can be seen in Fig. 8(b). For example, for green light with $\lambda = 0.5$ μm. If $d = 1$ mm (or 0.1 mm), then to be in the Fraunhofer regime, one needs $z > 2d^2/\lambda = 4$ m (or 4 cm).

For the Fraunhofer diffraction,

$$\begin{aligned}\psi(\mathbf{r}) &\simeq -\frac{ik_z}{2\pi}\int_{ap} da'\psi_i(\mathbf{r}')\frac{e^{ikr}}{r}e^{-i\mathbf{k}\cdot\mathbf{r}'} &(2.27)\\ &= -\frac{ik_z}{2\pi}\frac{e^{ikr}}{r}\int_{ap} da'\psi_i(\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}'} &(2.28)\end{aligned}$$

The first equation is consistent with the **Huygens principle**: Every area element of the aperture is a source of a spherical wave with amplitude $\psi(\mathbf{r}')$ and phase $e^{-i\mathbf{k}\cdot\mathbf{r}'}$. In the second equation, the integral is a 2D Fourier transform of the aperture field $\psi(x, y, 0)$. That is, the far field is proportional to the Fourier transform of the aperture field. Note: This does not apply to the diffraction from the edge of a plane, which has an infinite "aperture".

For example, for a single slit with width d on a plane, the aperture field is proportional to the function

$$f(x) = \begin{cases} 1 & |x| < d/2 \\ 0 & |x| > d/2 \end{cases}. \tag{2.29}$$

Its Fourier transform is,

$$F(k) = \frac{\sin[(kd/2)\sin\theta]}{(kd/2)\sin\theta},\ \sin\theta = \frac{x}{z}. \tag{2.30}$$

Thus, away from the central peak, the diffraction pattern has the first minimum at $\sin\theta_1 = \lambda/d$.

B. Diffraction of vector wave

For the diffraction of vectorial EM wave, one cannot simply apply Eq. (2.24) to the x, y, z components separately. A nice analysis of the diffraction of EM wave can be found in Sec. 21.8.2 of Zangwill, 2013. We will skip the derivation here and get to the result directly. The vector version of Eq. (2.24), valid for an infinite plane, should be,

$$\mathbf{E}(\mathbf{r}) = -2\int_{z'=0} da'\,[\hat{\mathbf{z}}\times\mathbf{E}(\mathbf{r}')]\times\nabla G_0(\mathbf{r},\mathbf{r}'), \tag{2.31}$$

where

$$G_0 = \frac{1}{4\pi}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.32}$$

Similarly,

$$\mathbf{B}(\mathbf{r}) = -2\int_{z'=0} da'\,[\hat{\mathbf{z}}\times\mathbf{B}(\mathbf{r}')]\times\nabla G_0(\mathbf{r},\mathbf{r}'). \tag{2.33}$$

These are referred to as **Smythe's formulas** (Smythe, 1989). Using the BAC-CAB rule for the cross product, one has

$$\mathbf{E}(\mathbf{r}) = -2\int_{z'=0} da'\mathbf{E}\;\hat{\mathbf{z}}\cdot\nabla G_0 + 2\hat{\mathbf{z}}\int_{z'=0} da'\mathbf{E}\cdot\nabla G_0. \tag{2.34}$$

It is not difficult to see that for $E_x(\mathbf{r})$ and $E_y(\mathbf{r})$, the equation agrees with Eq. (2.24) with the replacement $\psi(\mathbf{r})\to E_{x,y}(\mathbf{r})$. However, the equation for E_z component is different.

If the E_z component *can* be neglected, which is so when the incident direction is nearly normal and the observation point is close to the symmetry axis, then scalar diffraction theory gives good result. This explains the accuracy of the scalar diffraction in Fig. 7.

The *Kirchhoff approximation* for vector wave is,
1. $\hat{\mathbf{z}}\times\mathbf{E}\simeq 0$ on the screen.
2. $\hat{\mathbf{z}}\times\mathbf{E}\simeq\hat{\mathbf{z}}\times\mathbf{E}_i$ in the aperture.

For reference, for an infinite, thin, perfectly conducting plane, the boundary condition should be,
1. $\hat{\mathbf{z}}\times\mathbf{E}=0$ on the screen.
2. $\hat{\mathbf{z}}\times\mathbf{B}=\hat{\mathbf{z}}\times\mathbf{B}_i$ in the aperture.

Hence, the first Kirchhoff approximation is exact for a perfectly conducting plane.

Adopting the Kirchhoff approximation, one has

$$\mathbf{E}(\mathbf{r}) = 2\int_{ap} da'\nabla G_0(\mathbf{r},\mathbf{r}')\times[\hat{\mathbf{z}}\times\mathbf{E}_i(\mathbf{r}')]\,. \tag{2.35}$$

In the far zone,

$$\nabla G_0 \simeq ik\hat{\mathbf{r}}\frac{e^{ikr}}{4\pi r}e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}, \tag{2.36}$$

thus,

$$\mathbf{E}(\mathbf{r}) = \frac{e^{ikr}}{2\pi r}i\mathbf{k}\times\int_{ap} da' e^{-i\mathbf{k}\cdot\mathbf{r}'}\hat{\mathbf{z}}\times\mathbf{E}_i(\mathbf{r}') + O\left(\frac{1}{r^2}\right). \tag{2.37}$$

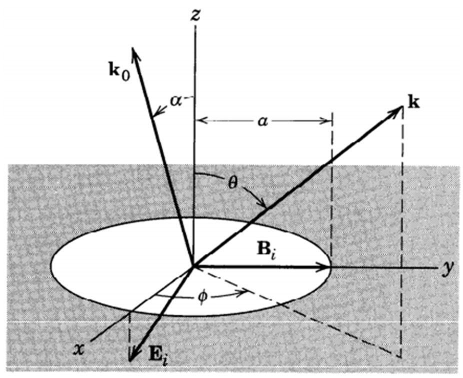

FIG. 9 The coordinates and variables related to a circular aperture.

The relation between $\mathbf{E}$ and $\mathbf{B}$ is simple in the far zone,

$$c\mathbf{B}(\mathbf{r}) = \hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r}). \tag{2.38}$$

1. Diffraction from circular aperture

We now study the diffraction due to a circular aperture in a conducting plane, under the assumption that $r\gg d$ and $r\gg d^2/\lambda$. As shown in Fig. 9, the incident wave vector $\mathbf{k}_0$ makes an angle α with the z-axis. Rotate the coordinate around z so that $\mathbf{k}_0$ and $\mathbf{E}_i$ lie on the $x-z$ plane. Hence,

$$\begin{aligned}\mathbf{k}_0 &= \hat{\mathbf{x}}\sin\alpha+\hat{\mathbf{z}}\cos\alpha, &(2.39)\\ \mathbf{E}_i(\mathbf{r}') &= E_0\left(\hat{\mathbf{x}}\cos\alpha-\hat{\mathbf{z}}\sin\alpha\right)e^{i\mathbf{k}_0\cdot\mathbf{r}'}. &(2.40)\end{aligned}$$

Also,

$$\begin{aligned}\hat{\mathbf{z}}\times\mathbf{E}_i(\mathbf{r}') &= E_0\hat{\mathbf{y}}\cos\alpha\;e^{i\mathbf{k}_0\cdot\mathbf{r}'}, &(2.41)\\ e^{i\mathbf{k}_0\cdot\mathbf{r}'} &= e^{ikx'\sin\alpha}, &(2.42)\\ e^{-i\mathbf{k}\cdot\mathbf{r}'} &= e^{-ik(x'\sin\theta\cos\phi+y'\sin\theta\sin\phi)}. &(2.43)\end{aligned}$$

Using the polar coordinate,

$$\mathbf{r}' = (\rho\cos\beta,\rho\sin\beta,0), \tag{2.44}$$

then,

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \frac{e^{ikr}}{2\pi r}i(\mathbf{k}\times\hat{\mathbf{y}})E_0\cos\alpha\int_0^a\rho d\rho\int_0^{2\pi}d\beta \qquad (2.45)\\ &\times\; e^{ik\rho(\sin\alpha\cos\beta-\sin\theta\cos\phi\cos\beta-\sin\theta\sin\phi\sin\beta)}.\end{aligned}$$

The integrand can be written as,

$$e^{ik\rho(\cdots)} = e^{ik\rho\xi\cos(\beta+\eta)}, \tag{2.46}$$

where η is some angle, and

$$\xi = \left(\sin^2\alpha+\sin^2\theta-2\sin\alpha\sin\theta\cos\phi\right)^{1/2}. \tag{2.47}$$

Then, with the help of

$$\frac{1}{2\pi}\int_0^{2\pi} d\beta e^{ik\rho\xi\cos\beta} = J_0(k\rho\xi), \qquad (2.48)$$

and $(x^2 J_n)' = x^n J_{n-1}$, we have

$$\int_0^a \rho d\rho \int_0^{2\pi} d\beta \cdots = 2\pi a^2 \frac{J_1(ka\xi)}{ka\xi}. \qquad (2.49)$$

Thus,

$$\mathbf{E}(\mathbf{r}) = \frac{e^{ikr}}{r} i\mathbf{k}\times\hat{\mathbf{y}}E_0 a^2 \cos\alpha \frac{J_1(ka\xi)}{ka\xi}. \qquad (2.50)$$

To calculate the power of diffracted EM wave, use

$$\frac{dP}{d\Omega} = \frac{1}{2} Re\left(\mathbf{E}\times\mathbf{H}^*\cdot\hat{\mathbf{r}}\right) r^2 = \frac{1}{2Z_0}|\mathbf{E}|^2 r^2. \qquad (2.51)$$

Then,

$$\frac{dP}{d\Omega} = P_{in}\cos\alpha |\hat{\mathbf{k}}\times\hat{\mathbf{y}}|^2 \frac{(ka)^2}{\pi}\left|\frac{J_1(ka\xi)}{ka\xi}\right|^2, \qquad (2.52)$$

in which

$$P_{in} = \frac{E_0^2}{2Z_0}\pi a^2 \cos\alpha, \qquad (2.53)$$

$$\text{and } |\hat{\mathbf{k}}\times\hat{\mathbf{y}}|^2 = \sin^2\theta\cos^2\phi + \cos^2\theta. \qquad (2.54)$$

For normal incidence, $\alpha = 0$, $\xi = \sin\theta$, and

$$\frac{dP}{d\Omega} = P_{in}\left(\sin^2\theta\cos^2\phi + \cos^2\theta\right)\frac{(ka)^2}{\pi}\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2. \qquad (2.55)$$

The Bessel function $J_1(x)$ has zeros at $x = 3.83, 7.01...$ (Fig. 10). Thus, the first dark ring in Fig. 10(b) is located at

$$\sin\theta_1 \simeq \frac{3.83}{\pi}\frac{\lambda}{2a} = 1.22\frac{\lambda}{2a}. \qquad (2.56)$$

By comparison, the first minimum of a single-slit diffraction occurs at $\sin\theta_1 = \lambda/d$, where d is the width of the slit.

Note: The result based on scalar diffraction has the same first minimum as the one above: The diffraction field is the Fourier transform of the circular aperture, which is also proportional to the same Bessel function $J_1(ka\sin\theta)$.

From Eq. (2.56), we have the **Rayleigh criterion** for the angular resolution of a telescope (or a pupil) with diameter d,

$$\theta \simeq 1.22\frac{\lambda}{d}. \qquad (2.57)$$

For example, for yellow light,

$$\theta \simeq \frac{15}{d\text{ (in cm)}}\text{arcsec} \quad (1\text{ arcsec} = 1/3600\text{ degree}). \qquad (2.58)$$

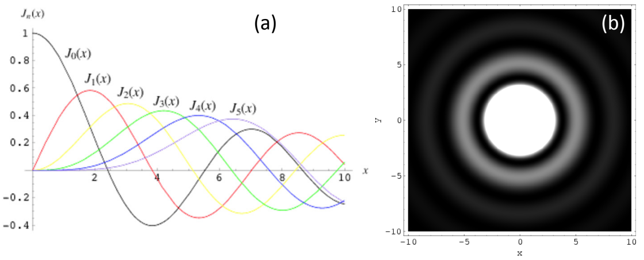

FIG. 10 (a) Bessel functions. (b) The pattern of diffraction from a circular aperture (aka an Airy disk).

A full moon spans roughly 1/2 degree. Suppose the diameter of a pupil is 1 cm, then it's hard to distinguish a feature smaller than 1/30 of the moon with naked eyes.

The Hubble telescope has a diameter of $d = 240$ cm, so the resolution power is about $\theta = 1.74\times 10^{-5}$ degrees. To distinguish two stars in the Andromeda galaxy two million light years away, their distance has to be larger than 0.6 light year.

An interesting review regarding the light in a tiny hole can be found in Genet and Ebbesen, 2007.

2. The Babinet principle

Let's start with the scalar diffraction. Suppose there is a flat plate S with an aperture, and a complementary flat plate S' that can fill the aperture precisely. Given the same incident wave ψ_i, the scattering wave of S is ψ, and scattering wave of S' is ψ'. Then, according to the *Huygens principle*, the sum of scattering waves,

$$\psi + \psi' = \psi_i, \qquad (2.59)$$

since the wavelets from $S + S'$ are the same as the wavelets from an infinite plane without any plate (Sommerfeld, 1954). This is called **Babinet principle** for scalar diffraction. It is valid (for as far as the Huygens principle is valid) for both Fresnel and Fraunhofer diffractions.

The intensity I for a wave is $|\psi|^2$. Therefore,

$$I + I' + \psi\psi'^* + \psi^*\psi' = I_i. \qquad (2.60)$$

Suppose that a point source is imaged by an error-free lens, so that ψ_i is a point on the image plane. Then, away from the image point, $\psi_i = 0$, and

$$\psi = -\psi' \quad \to \quad I = I'. \qquad (2.61)$$

That is, the two complementary plates produce the same diffraction pattern (and with the same intensity).

Now, for vector diffraction, consider a *thin, infinite, perfectly conducting* plate S with an aperture. Its complementary screen is S'. Unlike the scalar case, here we use slightly different incident plane waves for S and S' (Fig. 11),

$$\mathbf{E}'_i = -c\mathbf{B}_i, \qquad (2.62)$$

$$c\mathbf{B}'_i = \mathbf{E}_i. \qquad (2.63)$$

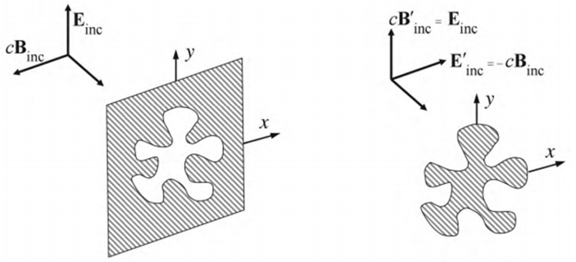

FIG. 11 A plane wave scattered by an aperture in a conducting screen (left). A dual plane wave scattered by a complementary conducting plate (right).

They propagate along the same direction $\mathbf{k}_0$ that may not be parallel to $\hat{\mathbf{z}}$. The $\mathbf{E}_i'$ and $\mathbf{B}_i'$ scattered from S' differ from the $\mathbf{E}_i$ and $\mathbf{B}_i$ scattered from S by a rotation around $\mathbf{k}_0$ by 90 degrees.

Suppose $\mathbf{E}$, $\mathbf{B}$ are the sum of incident and scattered fields,

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_i + \mathbf{E}_s, & (2.64)\\ \mathbf{B} &= \mathbf{B}_i + \mathbf{B}_s, & (2.65)\end{aligned}$$

with similar notation for the "primed" fields. Then the **Babinet principle** states that, on the side of the screen opposite to the source,

$$\begin{aligned} \mathbf{E}_s + c\mathbf{B}_s' &= -\mathbf{E}_i, & (2.66)\\ -c\mathbf{B}_s + \mathbf{E}_s' &= c\mathbf{B}_i. & (2.67)\end{aligned}$$

Alternatively speaking,

$$\begin{aligned} c\mathbf{B}_s' &= -\mathbf{E}, & (2.68)\\ \mathbf{E}_s' &= c\mathbf{B}. & (2.69)\end{aligned}$$

This applies to both near field and far field (without the need of Kirchhoff approximation).
Pf: To solve the original diffraction problem, we need

$$\begin{aligned} \nabla\times\mathbf{E}_s &= ik_0(c\mathbf{B}_s), & (2.70)\\ \nabla\times(c\mathbf{B}_s) &= -ik_0\mathbf{E}_s, & (2.71)\end{aligned}$$

together with the BC (P for plate, and A for aperture),

$$\begin{aligned} I \quad \hat{\mathbf{z}}\times\mathbf{E}_s &= -\hat{\mathbf{z}}\times\mathbf{E}_i \ \text{ on } P, & (2.72)\\ II \quad \hat{\mathbf{z}}\times c\mathbf{B}_s &= 0 \ \text{ on } A. & (2.73)\end{aligned}$$

In comparison, to solve the dual diffraction problem, we need

$$\begin{aligned} \nabla\times\mathbf{E}' &= ik_0(c\mathbf{B}'), & (2.74)\\ \nabla\times(c\mathbf{B}') &= -ik_0\mathbf{E}', & (2.75)\end{aligned}$$

together with the BC (P' and A' are the plate and aperture for the dual problem),

$$\begin{aligned} I' \quad \hat{\mathbf{z}}\times\mathbf{E}' &= 0 \ \text{ on } P'=A, & (2.76)\\ II' \quad \hat{\mathbf{z}}\times c\mathbf{B}' &= \hat{\mathbf{z}}\times c\mathbf{B}_i \ \text{ on } A'=P. & (2.77)\end{aligned}$$

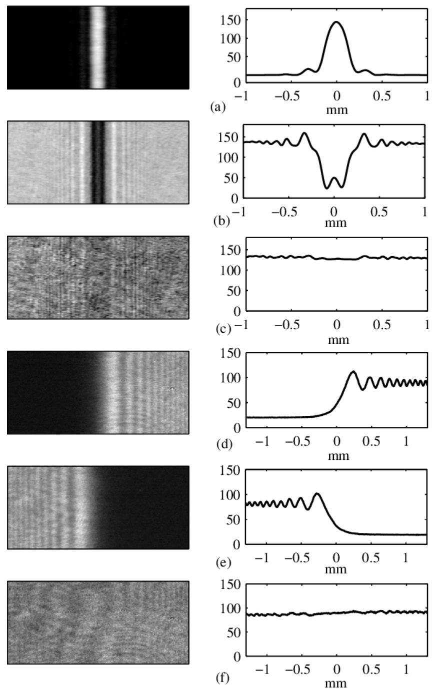

FIG. 12 Experimental realization of Babinet's principle (Tavassoly *et al.*, 2009). (a), (b) The diffraction patterns and intensity profiles of the light diffracted from a slit of 0.24 mm width and an opaque strip with the same width as the slit. (c) The pattern and intensity profile obtained by superimposing the diffracted fields in (a) and (b). (d), (e) The diffraction patterns and intensity profiles of the light diffracted from two complementary straight edges. (f) The pattern and intensity profile obtained by superimposing the diffraction fields in (d) and (e).

Close inspection shows that
$II' \to I$ if $c\mathbf{B}' \to -\mathbf{E}_s$, and
$I' \to II$ if $\mathbf{E}' \to c\mathbf{B}_s$.
Also, the two Maxwell equations remain the same under the dual transformation. Therefore, with the same initial condition in Eqs. (2.62) and (2.63), the solutions for $\mathbf{E}_s, c\mathbf{B}_s$ should be the same as those of $\mathbf{E}', -c\mathbf{B}'$ (Born and Wolf, 1980).

The diffraction patterns produced by an aperture and its dual problem can be seen in Fig. 12. Discussion of the Babinet principle from the viewpoint of sources on the plate can be found in Booker, 1946.

References

Bohren, C. F., and A. B. Fraser, 1985, The Physics Teacher **23**, 267.
Booker, H. G., 1946, Journal of the Institution of Electrical Engineers - Part IIIA: Radiolocation **93**(4), 620.
Born, M., and E. Wolf, 1980, in *Principles of Optics* (Pergamon), sixth edition, pp. 556 – 592.
Genet, C., and T. W. Ebbesen, 2007, Nature **445**, 39.

Jackson, J. D., 1998, *Classical Electrodynamics* (Wiley), 3rd edition.

Jackson, J. D., 2002, American Journal of Physics **70**(9), 917.

Kerker, M., 1969, in *The Scattering of Light and Other Electromagnetic Radiation*, edited by M. Kerker (Academic Press), volume 16 of *Physical Chemistry: A Series of Monographs*, pp. 1 – 7.

Kittel, C., 2004, *Introduction to Solid State Physics* (Wiley), 8th edition.

Kubo, R., 1965, *Statistical Mechanics: An Advanced Course with Problems and Solutions* (North-Holland Publishing Company).

Smith, G. S., 2007, American Journal of Physics **75**(1), 25.

Smythe, W. R., 1989, *Static And Dynamic Electricity* (CRC Press), 3rd edition.

Sommerfeld, A., 1954, *Optics*, volume IV of *Lectures on Theoretical Physics* (Academic Press).

Tavassoly, M., M. Amiri, A. Darudi, R. Aalipour, A. Saber, and A.-R. Moradi, 2009, Journal of the Optical Society of America. A, Optics, image science, and vision **26**, 540.

Zangwill, A., 2013, *Modern electrodynamics* (Cambridge Univ. Press, Cambridge).