Lecture notes on classical electrodynamics

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Contents

I. RADIATING SYSTEMS AND MULTIPOLE FIELDS

A. Radiation of a localized oscillating source

In this chapter we study the electromagnetic radiation generated from oscillating charge and current. In Chap 16, we have learned that under the Lorenz gauge, the scalar and vector potentials satisfy

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho_0}{\varepsilon_0}, \quad (1.1)$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}. \quad (1.2)$$

Their solutions are (see the Appendix at the end)

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho(\mathbf{r}',t_R)}{|\mathbf{r}-\mathbf{r}'|}, \quad (1.3)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int dv'\frac{\mathbf{J}(\mathbf{r}',t_R)}{|\mathbf{r}-\mathbf{r}'|}, \quad (1.4)$$

in which $t_R \equiv t - R/c$ is the **retarded time**, and $R = |\mathbf{r}-\mathbf{r}'|$.

Suppose the source is oscillating with frequency ω,

$$\rho(\mathbf{r},t) = \rho(\mathbf{r})e^{-i\omega t}, \quad (1.5)$$

$$\mathbf{J}(\mathbf{r},t) = \mathbf{J}(\mathbf{r})e^{-i\omega t}, \quad (1.6)$$

then the potentials are also oscillating with the same frequency,

$$\phi(\mathbf{r},t) = \phi(\mathbf{r})e^{-i\omega t}, \quad (1.7)$$

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}(\mathbf{r})e^{-i\omega t}, \quad (1.8)$$

where

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\rho(\mathbf{r}')\frac{e^{ikR}}{R}, \quad (1.9)$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int dv'\mathbf{J}(\mathbf{r}')\frac{e^{ikR}}{R}. \quad (1.10)$$

The electromagnetic fields can be calculated using $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ and $\mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A}$. Since they all have the same $e^{-i\omega t}$-dependence, so we'll only focus on the spatial part.

In fact, you don't need to calculate the electric field and magnetic field separately, since one field determines the other. In vacuum with $\mathbf{J} = 0$,

$$\nabla\times\mathbf{H}(\mathbf{r},t) = \varepsilon_0\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t). \quad (1.11)$$

Therefore,

$$\mathbf{E}(\mathbf{r}) = \frac{i}{\varepsilon_0\omega}\nabla\times\mathbf{H}(\mathbf{r}). \quad (1.12)$$

The electric field can be determined from this equation, without the need to know $\phi(\mathbf{r},t)$.

For a radiating system, there are three important length scales: the size of the source d, the wave length λ of the radiation, and the distance R between the source and an observation point. For the rest of this chapter, we only deal with the cases with

$$\lambda, R \gg d. \quad (1.13)$$

If d is larger, then numerical calculation might be required. Furthermore, the location of observation can be divided into 3 regimes:

$$\text{near zone}: \quad R \ll \lambda, \quad (1.14)$$

$$\text{intermediate zone}: \quad R \simeq \lambda, \quad (1.15)$$

$$\text{far (or radiation) zone}: \quad R \gg \lambda. \quad (1.16)$$

In the *near zone* with $kR \ll 1$,

$$\frac{e^{ikR}}{R} \simeq \frac{1}{R}. \quad (1.17)$$

Thus, the potentials calculated from Eqs. (1.9), (1.10) are simply static potentials multiplied by $e^{-i\omega t}$ (quasi-static case).

On the other hand, in the *far zone* with $r \gg r'$,

$$R = |\mathbf{r}-\mathbf{r}'| \simeq r - \hat{\mathbf{r}}\cdot\mathbf{r}'. \quad (1.18)$$

Thus,

$$\frac{e^{ikR}}{R} \simeq \frac{e^{ikr}e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}}{r-\hat{\mathbf{r}}\cdot\mathbf{r}'} \quad (1.19)$$

$$\simeq \frac{e^{ikr}}{r}e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} + O\left(\frac{1}{r^2}\right) \quad (1.20)$$

$$= \frac{e^{ikr}}{r}\left(1 - ik\hat{\mathbf{r}}\cdot\mathbf{r}' + \cdots\right). \quad (1.21)$$

The first term gives *electric dipole* radiation, while the second term gives *magnetic dipole* radiation and electric quadrupole radiation. In the intermediate zone, the approximations above cannot be applied and the calculation would be more difficult.

B. Electric dipole radiation

Keep the first term of Eq. (1.21) in the far-zone expansion, then

$$\frac{e^{ikR}}{R} \simeq \frac{e^{ikr}}{r}, \quad (1.22)$$

and

$$\mathbf{A}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int dv'\mathbf{J}(\mathbf{r}'). \quad (1.23)$$

With a trick, we can relate the integral with electric dipole moment. Note that

$$\nabla\cdot(r_i\mathbf{J}) = J_i + r_i\nabla\cdot\mathbf{J}, \quad (1.24)$$

and the integral of $\nabla\cdot(r_i\mathbf{J})$ is zero. Also, Eq. of continuity gives

$$\nabla\cdot\mathbf{J}(\mathbf{r}) = i\omega\rho(\mathbf{r}). \quad (1.25)$$

Therefore,

$$\int dv'\mathbf{J}(\mathbf{r}') = -\int dv'\mathbf{r}'(\nabla'\cdot\mathbf{J}) \quad (1.26)$$

$$= -i\omega\int dv'\mathbf{r}'\rho(\mathbf{r}') \quad (1.27)$$

$$= -i\omega\mathbf{p}, \quad (1.28)$$

where $\mathbf{p}$ is the electric dipole moment. Thus,

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}i\omega\mathbf{p}. \quad (1.29)$$

It follows that,

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0}\nabla\times\mathbf{A} \quad (1.30)$$

$$= -\frac{i\omega}{4\pi}\underbrace{\nabla\times\left(\frac{e^{ikr}}{r}\mathbf{p}\right)}_{=\nabla\left(\frac{e^{ikr}}{r}\right)\times\mathbf{p}} \quad (1.31)$$

$$= \frac{ck^2}{4\pi}\hat{\mathbf{r}}\times\mathbf{p}\frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right). \quad (1.32)$$

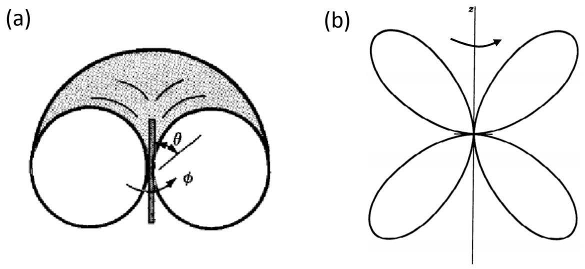

FIG. 1 The patterns of electric dipole radiation (a) and electric quadrupole radiation (b).

Also,

$$\mathbf{E}(\mathbf{r}) = \frac{i}{\varepsilon_0\omega}\nabla\times\mathbf{H}(\mathbf{r}) \quad (1.33)$$

$$= -\frac{k^2}{4\pi\varepsilon_0}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{p})\frac{e^{ikr}}{r} + O\left(\frac{1}{r^2}\right) \quad (1.34)$$

$$= \underbrace{\sqrt{\frac{\mu_0}{\varepsilon_0}}}_{=Z_0}\mathbf{H}(\mathbf{r})\times\hat{\mathbf{r}}, \quad (1.35)$$

where $Z_0 \simeq 376.7$ ohm is the **wave impedance** of vacuum.

Furthermore, we can calculate the power of radiating field. Recall that the Poynting vector is the energy current density (energy/time·area). Using the complex notation, after time-average,

$$\langle\mathbf{S}\rangle_T = \frac{1}{2}Re\left(\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})\right). \quad (1.36)$$

The time-averaged power radiated toward solid angle $d\Omega$ is,

$$dP = \langle\mathbf{S}\rangle_T\cdot\hat{\mathbf{r}}\,r^2d\Omega. \quad (1.37)$$

The angular distribution of radiated power is (Fig. 1(a)),

$$\frac{dP}{d\Omega} = \frac{r^2}{2}Re\left(\mathbf{E}\times\mathbf{H}^*\cdot\hat{\mathbf{r}}\right) \quad (1.38)$$

$$= \frac{r^2}{2}Z_0|\mathbf{H}\times\hat{\mathbf{r}}|^2 \quad (1.39)$$

$$= \frac{c}{2\varepsilon_0}\left(\frac{k^2}{4\pi}\right)^2\underbrace{|\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{p})|^2}_{=|\mathbf{p}-\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p})|^2} \quad (1.40)$$

$$= \frac{Z_0}{2}\left(\frac{k^2}{4\pi}\right)^2c^2p^2\sin^2\theta. \quad (1.41)$$

It can be integrated to get the total power,

$$P = \int d\Omega\left(\frac{dP}{d\Omega}\right) \quad (1.42)$$

$$= \frac{Z_0}{12\pi}c^2p^2k^4 \quad \propto k^4. \quad (1.43)$$

C. Magnetic dipole radiation and electric quadrupole radiation

We now consider the second term of Eq. (1.21) in the far-zone expansion,

$$\mathbf{A}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int dv'\mathbf{J}(\mathbf{r}')(-ik\hat{\mathbf{r}}\cdot\mathbf{r}'). \tag{1.44}$$

The following equation can be used to separate magnetic dipole radiation from electric quadrupole radiation,

$$\begin{aligned}(\hat{\mathbf{r}}\cdot\mathbf{r}')\mathbf{J} &= \frac{1}{2}\left[(\hat{\mathbf{r}}\cdot\mathbf{r}')\mathbf{J}+(\hat{\mathbf{r}}\cdot\mathbf{J})\mathbf{r}'\right] \rightarrow EQ \\ &- \frac{1}{2}\hat{\mathbf{r}}\times(\mathbf{r}'\times\mathbf{J}) \rightarrow MD. \end{aligned}\tag{1.45}$$

To check its validity, just apply the BAC-CAB rule to the second term.

1. Magnetic dipole field

Let's focus on the second term of Eq. (1.45). Recall that the magnetic moment is

$$\mathbf{m} = \frac{1}{2}\int dv'\mathbf{r}'\times\mathbf{J}. \tag{1.46}$$

Thus, the *magnetic dipole* part of the vector potential in Eq. (1.44) gives,

$$\begin{aligned}\mathbf{A}_{MD}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\frac{ik}{2}\int dv'\hat{\mathbf{r}}\times(\mathbf{r}'\times\mathbf{J}) &(1.47)\\ &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}ik(\hat{\mathbf{r}}\times\mathbf{m}). &(1.48)\end{aligned}$$

It follows that,

$$\begin{aligned}\mathbf{H}(\mathbf{r}) &= \frac{1}{\mu_0}\nabla\times\mathbf{A}(\mathbf{r}) &(1.49)\\ &= -\frac{k^2}{4\pi}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{m})\frac{e^{ikr}}{r}+O\left(\frac{1}{r^2}\right). &(1.50)\end{aligned}$$

This is the same as the $\mathbf{E}/Z_0$ of electric dipole radiation, except that $\mathbf{p}$ is replaced by $\mathbf{m}/c$. Also,

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= \frac{i}{\varepsilon_0\omega}\nabla\times\mathbf{H}(\mathbf{r}) &(1.51)\\ &= -\frac{i}{\varepsilon_0\omega}\frac{k^2}{4\pi}\nabla\left(\frac{e^{ikr}}{r}\right)\times\left[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{m})\right] &(1.52)\\ &= -Z_0\frac{k^2}{4\pi}(\hat{\mathbf{r}}\times\mathbf{m})\frac{e^{ikr}}{r}+O\left(\frac{1}{r^2}\right). &(1.53)\end{aligned}$$

Thus,

$$\mathbf{H} = \frac{1}{Z_0}\hat{\mathbf{r}}\times\mathbf{E} \text{ or } \mathbf{E} = Z_0\mathbf{H}\times\hat{\mathbf{r}}. \tag{1.54}$$

The $\mathbf{E}(\mathbf{r})$ in Eq. (1.53) is the same as the $-Z_0\mathbf{H}$ of electric dipole radiation, except that $c\mathbf{p}$ is replaced by $\mathbf{m}$ (Fig. 2).

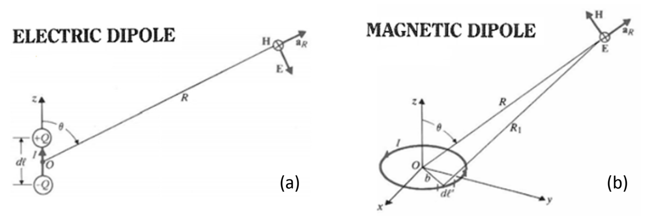

FIG. 2 A comparison between electric dipole radiation (a) and magnetic dipole radiation (b).

Compare electric dipole radiation with magnetic dipole radiation, we find that if $\mathbf{p}\rightarrow\mathbf{m}/c$, then

$$\begin{aligned}\mathbf{H}_{MD} &= \frac{1}{Z_0}\mathbf{E}_{ED}, \text{ or } \mathbf{B}_{MD} = \frac{1}{c}\mathbf{E}_{ED}, &(1.55)\\ \mathbf{E}_{MD} &= -Z_0\mathbf{H}_{ED}, \text{ or } \mathbf{E}_{MD} = -c\mathbf{B}_{ED}, &(1.56)\end{aligned}$$

thus

$$\begin{aligned}\langle\mathbf{S}\rangle_T^{MD} &= \frac{1}{2}Re\left(\mathbf{E}_{MD}\times\mathbf{H}^*_{MD}\right) &(1.57)\\ &= \frac{1}{2}Re\left(\mathbf{E}_{ED}\times\mathbf{H}^*_{ED}\right) = \langle\mathbf{S}\rangle_T^{ED}. &(1.58)\end{aligned}$$

The patterns of the two radiations are the same. For the magnetic dipole radiation, one has

$$\frac{dP}{d\Omega} = \frac{Z_0}{2}\left(\frac{k^2}{4\pi}\right)^2 m^2\sin^2\theta. \tag{1.59}$$

The total power of radiation is,

$$P = \frac{Z_0}{12\pi}m^2k^4 \quad \propto k^4. \tag{1.60}$$

Comparing the powers of radiation, we have

$$\frac{P_{MD}}{P_{ED}} \sim \left(\frac{m}{cp}\right)^2 \sim \left(\frac{J}{c\rho}\right)^2 \sim \left(\frac{v}{c}\right)^2. \tag{1.61}$$

Or, if d is the size of the system, then since

$$\frac{J}{d} \sim \omega\rho, \tag{1.62}$$

we have

$$\frac{P_{MD}}{P_{ED}} \sim \left(\frac{J}{c\rho}\right)^2 \sim \left(\frac{d}{\lambda}\right)^2. \tag{1.63}$$

Thus, $P_{MD} \ll P_{ED}$ when $v \ll c$ or $d \ll \lambda$.

2. Duality symmetry

The relations above are the result of a symmetry of Maxwell equations. In the absence of source, the Maxwell equations are,

$$\nabla\cdot\mathbf{E} = 0, \tag{1.64}$$
$$\nabla\cdot\mathbf{B} = 0 \tag{1.65}$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{1.66}$$
$$\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}. \tag{1.67}$$

It is not difficult to see that the equations are invariant under the following replacement,

$$(\mathbf{E}, c\mathbf{B}) \leftrightarrow \pm(c\mathbf{B}, -\mathbf{E}). \tag{1.68}$$

That is, if $(\mathbf{E}, c\mathbf{B})$ is a solution of the Maxwell equations, then $(\mathbf{E}', c\mathbf{B}') = \pm\alpha(c\mathbf{B}, -\mathbf{E})$ is also a solution (α is a constant). This is called the **duality symmetry** of Maxwell equations, and is the reason why we have the relations in Eqs. (1.55) and (1.56). In the presence of source, the duality symmetry is broken since there are electric charges but no magnetic charges. The symmetry could be restored if magnetic monopoles do exist.

3. Electric quadrupole field

We now focus on the first term in Eq. (1.45). It is left as an exercise for you to show that,

$$\frac{1}{2}\int dv'\,[(\hat{\mathbf{r}}\cdot\mathbf{r}')\mathbf{J} + (\hat{\mathbf{r}}\cdot\mathbf{J})\mathbf{r}'] = -\frac{i\omega}{2}\int dv'\mathbf{r}'(\mathbf{r}'\cdot\hat{\mathbf{r}})\rho(\mathbf{r}'). \tag{1.69}$$

Therefore, the *electric quadrupole* part of the vector potential is,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left(-\frac{k\omega}{2}\right)\int dv'\mathbf{r}'(\mathbf{r}'\cdot\hat{\mathbf{r}})\rho(\mathbf{r}'). \tag{1.70}$$

The integral is related to the electric quadrupole moment,

$$\begin{aligned}\int dv'\mathbf{r}'\mathbf{r}'\cdot\hat{\mathbf{r}}\rho(\mathbf{r}') &= \int dv'\left(\mathbf{r}'\mathbf{r}'\cdot\hat{\mathbf{r}} - \frac{r'^2}{3}\hat{\mathbf{r}}\right)\rho(\mathbf{r}') \\ &\quad + \int dv'\frac{r'^2}{3}\hat{\mathbf{r}}\rho(\mathbf{r}') \qquad (1.71)\\ &= \frac{1}{3}\mathsf{Q}\cdot\hat{\mathbf{r}} + \frac{\hat{\mathbf{r}}}{3}\int dv' r'^2\rho(\mathbf{r}'), \qquad (1.72)\end{aligned}$$

in which Q is the electric quadrupole moment,

$$Q_{ij} \equiv \int dv'\left(3r'_ir'_j - r'^2\delta_{ij}\right)\rho(\mathbf{r}'). \tag{1.73}$$

One can also write $\mathsf{Q}\cdot\hat{\mathbf{r}} = \mathbf{Q}(\hat{\mathbf{r}})$, which is a vector $\mathbf{Q}$ that depends on $\hat{\mathbf{r}}$. Thus,

$$\mathbf{A}_{EQ}(\mathbf{r}) = -\frac{\mu_0 ck^2}{8\pi}\frac{e^{ikr}}{r}\left[\frac{1}{3}\mathbf{Q}(\hat{\mathbf{r}}) + \frac{\hat{\mathbf{r}}}{3}\int dv' r'^2\rho\right]. \tag{1.74}$$

It follows that,

$$\begin{aligned}\mathbf{H}(\mathbf{r}) &= \frac{1}{\mu_0}\nabla\times\mathbf{A}(\mathbf{r}) &(1.75)\\ &= -\frac{ck^2}{8\pi}\frac{e^{ikr}}{r}ik\hat{\mathbf{r}}\times\frac{1}{3}\mathbf{Q}(\hat{\mathbf{r}}) + O\left(\frac{1}{r^2}\right) &(1.76)\\ &= -i\frac{ck^3}{24\pi}\frac{e^{ikr}}{r}\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}}) + O\left(\frac{1}{r^2}\right), &(1.77)\end{aligned}$$

and

$$\mathbf{E}(\mathbf{r}) = Z_0\mathbf{H}(\mathbf{r})\times\hat{\mathbf{r}}. \tag{1.78}$$

Thus, the angular distribution of radiated power is,

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{r^2}{2}Re\left(\mathbf{E}\times\mathbf{H}^*\cdot\hat{\mathbf{r}}\right) &(1.79)\\ &= \frac{r^2}{2}Z_0|\mathbf{H}\times\hat{\mathbf{r}}|^2 &(1.80)\\ &= \frac{Z_0}{2}\left(\frac{ck^3}{24\pi}\right)^2|\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times\mathbf{Q}(\hat{\mathbf{r}})]|^2. &(1.81)\end{aligned}$$

For a diagonal electric quadrupole matrix with

$$Q_{11} = Q_{22} = -\frac{Q_0}{2},\ Q_{33} = Q_0, \tag{1.82}$$

we have (Fig. 1(b))

$$\frac{dP}{d\Omega} = Z_0\left(\frac{ck^3}{24\pi}\right)^2 Q_0^2\underbrace{\sin^2\theta\cos^2\theta}_{=\frac{1}{4}\sin^2 2\theta}. \tag{1.83}$$

To calculate the total power $P = \int d\Omega\frac{dP}{d\Omega}$, first write $(\hat{\mathbf{r}}\to\hat{\mathbf{n}})$

$$\begin{aligned}&|\hat{\mathbf{n}}\times[\hat{\mathbf{n}}\times\mathbf{Q}(\hat{\mathbf{n}})]|^2\\ &= \mathbf{Q}^*\cdot\mathbf{Q} - |\hat{\mathbf{n}}\cdot\mathbf{Q}|^2 &(1.84)\\ &= \sum_{ijk}Q^*_{ij}Q_{ik}n_jn_k - \sum_{ijkl}Q^*_{ij}Q_{kl}n_in_jn_kn_l. &(1.85)\end{aligned}$$

Two identities are required to calculate the integral over solid angle:

$$\int d\Omega\ n_jn_k = \frac{4\pi}{3}\delta_{jk}, \tag{1.86}$$
$$\int d\Omega\ n_in_jn_kn_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{1.87}$$

Pf. First,

$$\int d\Omega\ n_jn_k = 0,\quad \text{if } j\neq k. \tag{1.88}$$

Also,

$$\begin{aligned}\int d\Omega\ n_x^2 &= \int d\Omega\ n_y^2 = \int d\Omega\ n_z^2 &(1.89)\\ &= \frac{1}{3}\int d\Omega\ |\hat{\mathbf{n}}|^2 = \frac{4\pi}{3}. &(1.90)\end{aligned}$$

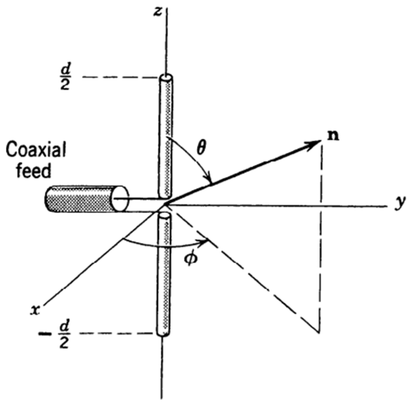

FIG. 3 A center-fed linear antenna. Fig. from Jackson, 1998.

Thus we have Eq. (1.86)

Second, for the integral with 4 n's, at least two of the subscripts must be the same. If the other two n's have different subscripts, then the integral is zero, similar to Eq. (1.88) above. So the subscript must form two pairs for the integral to be non-zero,

$$\int d\Omega\ n_i n_j n_k n_l = C(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad (1.91)$$

where C is a constant. The constant $C = 4\pi/15$ can be determined by choosing, e.g., $i = j = k = l = z$. Q.E.D.

It follows that,

$$\int d\Omega\ |\hat{\mathbf{n}} \times [\hat{\mathbf{n}} \times \mathbf{Q}(\hat{\mathbf{n}})]|^2 \quad (1.92)$$

$$= \frac{4\pi}{3}\sum_{ij}|Q_{ij}|^2 - \frac{4\pi}{15}\left(\sum_i Q_{ii}^* \sum_k Q_{kk} + 2\sum_{ij}|Q_{ij}|^2\right)$$

$$= \frac{4\pi}{5}\sum_{ij}|Q_{ij}|^2. \quad (1.93)$$

The second term above is zero because the electric quadrupole matrix is traceless. Finally,

$$\begin{aligned} P &= \int d\Omega \frac{dP}{d\Omega} \\ &= \frac{Z_0}{60\cdot 24\pi} c^2 k^6 \sum_{ij}|Q_{ij}|^2 \quad \propto k^6. \end{aligned} \quad (1.94)$$

D. Center-fed linear antenna

Consider an antenna made of a thin, straight wire with length d (Fig. 3). Suppose the current distribution in such a center-fed antenna is sinusoidal in space. The current density is,

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r})e^{-i\omega t}, \quad (1.95)$$

$$\mathbf{J}(\mathbf{r}) = I_0 \sin\left[k\left(\frac{d}{2} - |z|\right)\right]\delta(x)\delta(y)\hat{\mathbf{z}}. \quad (1.96)$$

The current distribution is symmetric with respect to the origin and vanishes at two ends. In the far zone, the vector potential in Eq. (1.10) can be calculated with the approximation in Eq. (1.20),

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int dv'\mathbf{J}(\mathbf{r}')e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'} \quad (1.97)$$

$$= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r} I_0 \underbrace{\int_{-d/2}^{d/2} dz \sin\left(\frac{kd}{2} - k|z|\right) e^{-ikz\cos\theta}}_{=\frac{2}{k}\{\cos[(kd/2)\sin\theta] - \cos(kd/2)\}/\sin^2\theta}\hat{\mathbf{z}}.$$

The magnetic field is,

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu_0} ik\hat{\mathbf{r}} \times \mathbf{A}(\mathbf{r}) + O\left(\frac{1}{r^2}\right). \quad (1.98)$$

Also,

$$\frac{dP}{d\Omega} = \frac{r^2}{2} Re\left(\mathbf{E} \times \mathbf{H}^* \cdot \hat{\mathbf{r}}\right) \quad (1.99)$$

$$= \frac{r^2}{2} Z_0 |\mathbf{H} \times \hat{\mathbf{r}}|^2 \quad (1.100)$$

$$= \frac{Z_0}{2}\left(\frac{I_0}{2\pi}\right)^2 \left|\frac{\cos\left(\frac{kd}{2}\sin\theta\right) - \cos\frac{kd}{2}}{\sin\theta}\right|^2 \quad (1.101)$$

For a half-wave antenna with $kd = \pi$, and a full-wave antenna with $kd = 2\pi$, one has

$$\frac{dP}{d\Omega} = \frac{Z_0}{2}\left(\frac{I_0}{2\pi}\right)^2 \begin{cases} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}, & kd = \pi \\ \frac{4\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}, & kd = 2\pi \end{cases} \quad (1.102)$$

Their patterns of radiation are shown in Fig. 4.

If the length of the antenna is much shorter than the wavelength ($kd \ll 1$), then

$$\frac{dP}{d\Omega} \simeq \frac{Z_0}{8}\left(\frac{I_0}{2\pi}\right)^2\left(\frac{kd}{2}\right)^4 \sin^2\theta, \quad (1.103)$$

and

$$P = Z_0 \frac{I_0^2}{12\pi}\left(\frac{kd}{2}\right)^4. \quad (1.104)$$

The input current $I_{in} = I(z=0) \sim I_0 kd/2$ for a short linear antenna (Garg, 2012). If we define the **radiation resistance** from

$$P = \frac{1}{2} I_{in}^2 R_{rad}, \quad (1.105)$$

then

$$R_{rad} = Z_0 \frac{\pi}{6}\left(\frac{d}{\lambda}\right)^2. \quad (1.106)$$

For example, for a short linear antenna with $d = \lambda/10$, $R_{rad} \simeq 2\ \Omega$. To produce EM wave with 1 KW power, one needs $I_0 \simeq 30$ A.

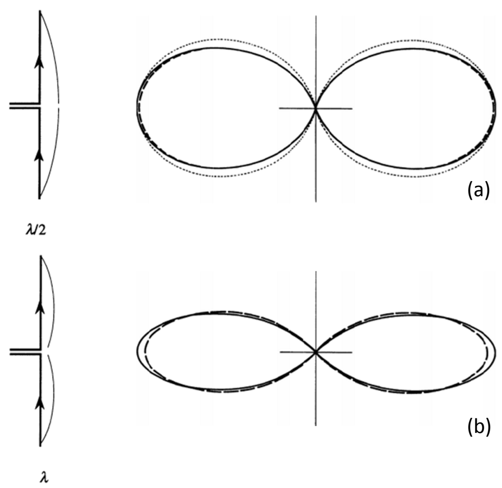

FIG. 4 Comparison of the radiations from (a) half-wave antenna and (b) full-wave antenna. The latter has narrower angular distribution. Solid lines are the results calculated without making any approximation. Dashed lines and dotted line are the results from two-term spherical multipole expansion and dipole approximation respectively. Figs from Jackson, 1998.

An antenna can be used to transmit or to receive signals. An **antenna reciprocity theorem** tells us that the radiation pattern of transmitting antenna a, which transmits to the receiving antenna b is equal to the radiation pattern of antenna b, if it transmits and antenna a receives the signal. See, for example, Prob. 20.14 of Zangwill, 2013, or McDonald, 2010 for more information.

E. Revisiting electric dipole radiation

We now study the radiation generated from a dipole with general time-dependence (not necessarily $e^{-i\omega t}$). Start from

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int dv' \frac{\rho(\mathbf{r}',t_R)}{|\mathbf{r}-\mathbf{r}'|}, \quad (1.107)$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int dv' \frac{\mathbf{J}(\mathbf{r}',t_R)}{|\mathbf{r}-\mathbf{r}'|}, \quad (1.108)$$

where $t_R \equiv t - R/c$ and $R = |\mathbf{r}-\mathbf{r}'|$. Focus on the far zone with $r \gg r'$, such that

$$t_R \simeq t - \frac{1}{c}(r - \hat{\mathbf{r}}\cdot\mathbf{r}') = t_r + \frac{1}{c}\hat{\mathbf{r}}\cdot\mathbf{r}', \quad (1.109)$$

$$\text{where } t_r \equiv t - \frac{r}{c}. \quad (1.110)$$

Taylor-expand the retarded charge density and current density to the first order (dipole approximation) to get,

$$\rho\left(\mathbf{r}',t-\frac{R}{c}\right) \simeq \rho(\mathbf{r}',t_r) + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\frac{d\rho}{dt}\Big|_{t=t_r}, \quad (1.111)$$

$$\mathbf{J}\left(\mathbf{r}',t-\frac{R}{c}\right) \simeq \mathbf{J}(\mathbf{r}',t_r) + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\frac{d\mathbf{J}}{dt}\Big|_{t=t_r}. \quad (1.112)$$

The second term is much smaller than the first term (and higher order terms can be ignored) if $\omega r'/c \ll 1$, or $r' \ll \lambda$, where ω is the characteristic frequency of the radiating source.

In the *far zone*, we have

$$\begin{aligned}\phi(\mathbf{r},t) &\simeq \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\left[\int dv'\rho(\mathbf{r}',t_r) + \frac{\hat{\mathbf{r}}}{c}\cdot\frac{d}{dt}\int dv'\mathbf{r}'\rho(\mathbf{r}',t_r)\right] \\ &= \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\left[Q + \frac{\hat{\mathbf{r}}}{c}\cdot\frac{d}{dt}\mathbf{p}(t_r)\right] + O\left(\frac{1}{r^2}\right). \quad (1.113)\end{aligned}$$

The monopole term will be dropped, since Q is a constant and does not generate radiation.

One can do the same to the vector potential. Before doing that, it helps to know that

$$\int dv\, \mathbf{J} = \frac{d\mathbf{p}}{dt}. \quad (1.114)$$

This is a generalization of the relation in Eq. (1.28). It can be proved by using

$$\begin{aligned}\nabla\cdot(r_i\mathbf{J}) &= J_i + r_i\nabla\cdot\mathbf{J} \quad (1.115)\\ &= J_i - r_i\frac{\partial\rho}{\partial t} \quad (1.116)\\ &= J_i - \frac{\partial}{\partial t}(r_i\rho),\ r_i \text{ is fixed.} \quad (1.117)\end{aligned}$$

We have relied the equation of continuity to get the second equality. Eq. (1.114) follows after the equation above is integrated over space. As a result,

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\frac{1}{r}\int dv'\mathbf{J}(\mathbf{r}',t_r) + O\left(\frac{1}{r^2}\right), \quad (1.118)\\ &\simeq \frac{\mu_0}{4\pi}\frac{1}{r}\frac{d}{dt}\mathbf{p}(t_r). \quad (1.119)\end{aligned}$$

With the scalar potential and vector potential at hand, we can calculate the electromagnetic field. First,

$$\nabla\phi = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}(t_r)}{cr}\nabla t_r + O\left(\frac{1}{r^2}\right), \quad (1.120)$$

in which

$$\nabla t_r = -\frac{1}{c}\nabla r = -\frac{1}{c}\hat{\mathbf{r}}. \quad (1.121)$$

Also,

$$\begin{aligned}\frac{\partial\mathbf{A}}{\partial t} &= \frac{\mu_0}{4\pi}\frac{\ddot{\mathbf{p}}(t_r)}{r}, \quad (1.122)\\ \nabla\times\mathbf{A} &= \frac{\mu_0}{4\pi r}\nabla t_r\times\ddot{\mathbf{p}}(t_r) + O\left(\frac{1}{r^2}\right). \quad (1.123)\end{aligned}$$

This gives

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \quad (1.124)\\ &= \frac{\mu_0}{4\pi r}[\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}) - \ddot{\mathbf{p}}] = \frac{\mu_0}{4\pi r}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}}),\end{aligned}$$

and

$$\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}=-\frac{\mu_0}{4\pi r}\frac{1}{c}\hat{\mathbf{r}}\times\ddot{\mathbf{p}}. \tag{1.125}$$

Thus,

$$\mathbf{E}=c\mathbf{B}\times\hat{\mathbf{r}}. \tag{1.126}$$

It follows that,

$$\begin{aligned}\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} &(1.127)\\ &= \frac{\mu_0}{16\pi^2 c}\frac{1}{r^2}\left|\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}})\right|^2\hat{\mathbf{r}} &(1.128)\\ &= \frac{\mu_0}{16\pi^2 c}\frac{1}{r^2}\left|\ddot{\mathbf{p}}_\perp(t_r)\right|^2\hat{\mathbf{r}}, &(1.129)\end{aligned}$$

Note that given a vector $\mathbf{u}$, one has

$$\begin{aligned}\left|\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{u})\right| &= \left|\mathbf{u}-\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{u})\right| &(1.130)\\ &= |\mathbf{u}_\perp|, &(1.131)\end{aligned}$$

in which $\mathbf{u}_\perp$ is the component of $\mathbf{u}$ perpendicular to $\hat{\mathbf{r}}$.

The angular distribution of radiated power is,

$$\begin{aligned}\frac{dP}{d\Omega} &= \mathbf{S}\cdot\hat{\mathbf{r}}r^2 &(1.132)\\ &= \frac{\mu_0}{16\pi^2 c}\sin^2\theta\left|\ddot{\mathbf{p}}(t_r)\right|^2, &(1.133)\end{aligned}$$

where θ is the angle between $\mathbf{r}$ and $\ddot{\mathbf{p}}$. This gives the total power of radiation,

$$P(t)=\int d\Omega\frac{dP}{d\Omega}=\frac{\mu_0}{6\pi c}\left|\ddot{\mathbf{p}}(t_r)\right|^2. \tag{1.134}$$

Note that this is not time-averaged.

Because of the duality symmetry, to get the angular distribution for magnetic dipole radiation, just replace the $\mathbf{p}$ in Eq. (1.133) with $\mathbf{m}/c$,

$$\frac{dP}{d\Omega}=\frac{\mu_0}{16\pi^2 c^3}\sin^2\theta\left|\ddot{\mathbf{m}}(t_r)\right|^2. \tag{1.135}$$

Similarly, the total power of radiation is,

$$P(t)=\frac{\mu_0}{6\pi c^3}\left|\ddot{\mathbf{m}}(t_r)\right|^2. \tag{1.136}$$

1. Radiation from accelerated charge

We will study the radiation from a point charge in a later chapter. Here we use the formula of dipole radiation as a short-cut to get the result. A point charge located at $\mathbf{r}(t)$ near the origin (with an extent $d \ll \lambda$) can be considered as a dipole with,

$$\begin{aligned}\mathbf{p}(t) &= q\mathbf{r}(t), &(1.137)\\ \rightarrow\ \ddot{\mathbf{p}}(t) &= q\mathbf{a}(t),\ \mathbf{a}\equiv\ddot{\mathbf{r}}. &(1.138)\end{aligned}$$

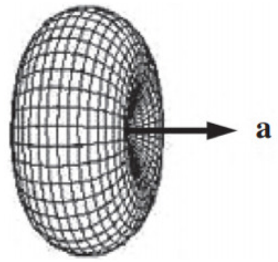

FIG. 5 The pattern of radiation from a point charge with linear acceleration $\mathbf{a}$.

According to Eq. (1.129), one has

$$\frac{dP}{d\Omega}=\frac{\mu_0}{16\pi^2 c}q^2\left|\mathbf{a}_\perp(t_r)\right|^2. \tag{1.139}$$

A moving charge can radiate only if it is accelerating. For linear motion along $\mathbf{a}$, the radiation is strongest when $\mathbf{r}$ is perpendicular to $\mathbf{a}$. There is no radiation along the direction of $\mathbf{a}$. For example, the pattern of radiation for a uniformly accelerated charge is shown in Fig. 5. It is similar to the pattern of an oscillating dipole radiation.

The total power of radiation is,

$$P(t)=\frac{\mu_0 q^2}{6\pi c}\left|\mathbf{a}(t_r)\right|^2\ \propto\ a^2. \tag{1.140}$$

This is the **Larmor formula** (for $v \ll c$). We will discuss the general case for arbitrary velocity v in Chap 22. Note: With this short-cut approach, one can't rule out more radiations beyond the dipole one, but in fact there is no more.

2. The birth of dipole radiation

Consider the case when $\mathbf{p}(t)=p(t)\hat{\mathbf{z}}$. Then,

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\frac{1}{r}\dot{p}(t_r)\hat{\mathbf{z}}. \tag{1.141}$$

Hence,

$$\begin{aligned}\frac{\partial\mathbf{E}}{\partial t}=c^2\nabla\times\mathbf{B} &= c^2\nabla(\nabla\cdot\mathbf{A})-c^2\nabla^2\mathbf{A} &(1.142)\\ &= \frac{1}{4\pi\varepsilon}\left[\nabla\frac{\partial}{\partial z}\left(\frac{\dot{p}}{r}\right)-\hat{\mathbf{z}}\nabla^2\left(\frac{\dot{p}}{r}\right)\right]_{t_r}.\end{aligned}$$

Integrate over t, and use the cylindrical coordinate to get,

$$\mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\left[\hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho}\frac{\partial}{\partial z}\left(\frac{p}{r}\right)-\hat{\mathbf{z}}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\frac{p}{r}\right)\right]_{t_r}, \tag{1.143}$$

where $r=\sqrt{\rho^2+z^2}$. If we define

$$W(\mathbf{r},t)=-\rho\frac{\partial}{\partial\rho}\frac{p(t_r)}{r}, \tag{1.144}$$

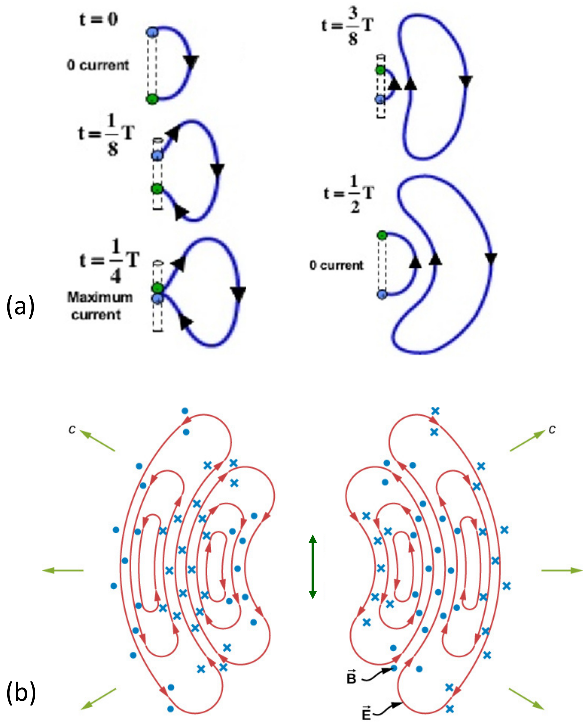

FIG. 6 (a) Field lines from an oscillating electric dipole. A field line would reconnet to form a closed loop and expand outward. Note: The space-time coordinate of the connection point is not necessarily at $(0, T/4)$. (b) The distribution of magnetic field is shown with blue dots and blue crosses.

then

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{1}{\rho}\left(-\frac{\partial W}{\partial z}\hat{\boldsymbol{\rho}} + \frac{\partial W}{\partial \rho}\hat{\mathbf{z}}\right). \tag{1.145}$$

On the other hand,

$$\nabla W = \frac{\partial W}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{\partial W}{\partial z}\hat{\mathbf{z}}. \tag{1.146}$$

Thus,

$$\mathbf{E}\cdot\nabla W = 0. \tag{1.147}$$

That is, $\mathbf{E}$ is perpendicular to the gradient of W, or is parallel to the tangent of the contour lines of W. Hence, the contours of W are electric field lines.

For example, given

$$p(t_r) = p_0\cos(\omega t_r + \phi) = p_0\cos(kr - \omega t - \phi), \tag{1.148}$$

one should be able to produce plots similar to the sequence in Fig. 6.

3. Stability of atom

In the Rutherfold model of hydrogen atom. The electron is circulating around the nucleus. Thus, the atom can be seen as a rotating electric dipole:

$$\mathbf{r}(t) = r_0(\cos\omega t\hat{\mathbf{x}} + \sin\omega t\hat{\mathbf{y}}), \tag{1.149}$$

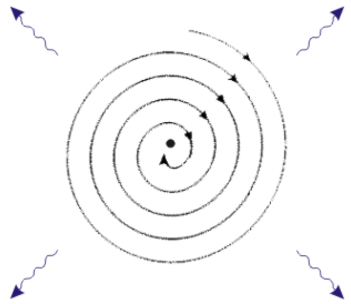

FIG. 7 A circulating electron emits radiation and loses energy.

and $\mathbf{p}(t) = q\mathbf{r}(t), q = -e$. According to the Larmor formula (Eq. (1.140)),

$$P = \frac{\mu_0}{6\pi c}q^2a^2. \tag{1.150}$$

If the radius of the electron orbit is r, then its energy is

$$\begin{aligned} E &= \frac{1}{2}mv^2 - \frac{q^2}{4\pi\varepsilon_0}\frac{1}{r} && (1.151)\\ &= -\frac{q^2}{8\pi\varepsilon_0}\frac{1}{r}. && (1.152)\end{aligned}$$

When the electron radiates and loses energy, the radius shrinks (Fig. 7), the change of energy is,

$$dE = \frac{q^2}{8\pi\varepsilon_0}\frac{1}{r^2}dr, \quad (dr < 0) \tag{1.153}$$

Thus,

$$\begin{aligned} Pdt &= |dE| && (1.154)\\ \rightarrow \frac{\mu_0}{6\pi c}q^2a^2dt &= -\frac{q^2}{8\pi\varepsilon_0}\frac{1}{r^2}dr, && (1.155)\end{aligned}$$

Since

$$a = \omega^2 r = \frac{v^2}{r} = \frac{\mu_0 q^2 c^2}{4\pi m r^2}, \tag{1.156}$$

this leads to,

$$dt = -\frac{3c^3}{4a^2}\frac{dr}{r^2} = -\gamma r^2 dr, \text{ where } \gamma = \frac{3}{4c}\frac{1}{r_e^2}, \tag{1.157}$$

in which r_e is the **classical electron radius** that satisfies

$$mc^2 = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{r_e}. \tag{1.158}$$

It is related to the Bohr radius a_0 via $r_e = \alpha^2 a_0$, where $\alpha \simeq 1/137$ is the **fine-structure constant**. The time for the radius of the electron to shrink from a_0 to zero is,

$$\tau = -\gamma\int_{a_0}^{0} r^2 dr = \frac{a_0}{4c}\frac{1}{\alpha^8}. \tag{1.159}$$

You can plug in numbers to get $\tau \simeq 1.31 \times 10^{-11}$ s. That is, according to classical theory, the Rutherford atom should collapse immediately because of the radiation energy loss. The stability of the atom used to be a mystery before the discovery of quantum mechanics.

Problems:

1. Show that

$$\frac{1}{2}\int dv'\,[(\hat{\mathbf{r}}\cdot\mathbf{r}')\mathbf{J} + (\hat{\mathbf{r}}\cdot\mathbf{J})\mathbf{r}'] = -\frac{i\omega}{2}\int dv'\mathbf{r}'(\mathbf{r}'\cdot\hat{\mathbf{r}})\rho(\mathbf{r}').$$

Appendix: Green's function for wave equation

The wave equation,

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho_0}{\varepsilon_0}, \quad (1.160)$$

is of the following form,

$$\begin{aligned} \mathcal{L}(\mathbf{r},t)\phi(\mathbf{r},t) &= h(\mathbf{r},t), & (1.161)\\ \mathcal{L}(\mathbf{r},t) &\equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. & (1.162) \end{aligned}$$

Note that $\mathcal{L}$ is a linear differential operator. To solve it, one can first solve the potential from a point source (in space-time),

$$\mathcal{L}(\mathbf{r},t)G(\mathbf{r},t;\mathbf{r}',t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \quad (1.163)$$

then add up the potentials from a collection of point sources to get ϕ.

Before doing that, let's introduce the following **4-vector notation**,

$$\begin{aligned} x &= (\mathbf{r},t), & (1.164)\\ k &= (\mathbf{k},\omega), & (1.165)\\ k\cdot x &= \mathbf{k}\cdot\mathbf{r} - \omega t, & (1.166) \end{aligned}$$

and

$$\begin{aligned} \delta(x-x') &= \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), & (1.167)\\ d^4x &= d^3v dt, & (1.168)\\ d^4k &= d^3k d\omega. & (1.169) \end{aligned}$$

With this new notation, the equations above become,

$$\begin{aligned} \mathcal{L}(x)\phi(x) &= h(x), & (1.170)\\ \mathcal{L}(x)G(x,x') &= \delta(x-x'). & (1.171) \end{aligned}$$

The general solution of Eq. (1.170) can be written as,

$$\phi(x) = \phi_0(x) + \int d^4x' G(x,x')h(x'), \quad (1.172)$$

where ϕ_0 is a solution of $\mathcal{L}\phi = 0$. This can be checked as follows,

$$\begin{aligned} \mathcal{L}\phi(x) &= \mathcal{L}\phi_0 + \int d^4x' \underbrace{\mathcal{L}(x)G(x,x')}_{=\delta(x-x')} h(x') & (1.173)\\ &= h(x). & (1.174) \end{aligned}$$

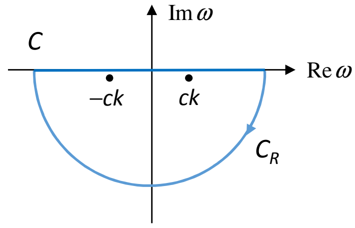

FIG. 8 A closed path C consists of x-axis and a semi-circle C_R.

To solve for $G(x,x')$, first note that it should depend only on $x-x'$, that is $G(x,x') = G(x-x')$. Next, perform the Fourier transformation,

$$\begin{aligned} G(x-x') &= \int \frac{d^4k}{(2\pi)^4}\tilde{G}(k)e^{ik\cdot(x-x')}, & (1.175)\\ \delta(x-x') &= \int \frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-x')}. & (1.176) \end{aligned}$$

Then Eq. (1.171) becomes

$$\begin{aligned} \mathcal{L}(x)G(x,x') &= \int \frac{d^4k}{(2\pi)^4}\tilde{G}(k)\mathcal{L}e^{ik\cdot(x-x')} & (1.177)\\ &= \int \frac{d^4k}{(2\pi)^4}\tilde{G}(k)\left(-|\mathbf{k}|^2 + \frac{\omega^2}{c^2}\right)e^{ik\cdot(x-x')} \\ &= \int \frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-x')}. & (1.178) \end{aligned}$$

It follows that

$$\tilde{G}(k) = \frac{1}{\left(\frac{\omega}{c}\right)^2 - |\mathbf{k}|^2}. \quad (1.179)$$

Thus,

$$G(x,x') = \int \frac{d^4k}{(2\pi)^4}\frac{1}{\left(\frac{\omega}{c}\right)^2 - |\mathbf{k}|^2}e^{ik\cdot(x-x')}. \quad (1.180)$$

We will first integrate over ω,

$$I = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{v^2}{\omega^2 - c^2|\mathbf{k}|^2}e^{-i\omega(t-t')}, \quad (1.181)$$

Consider the following integral,

$$\oint_C = \int_{-\infty}^{\infty} + \int_{C_R}, \quad (1.182)$$

where the path C is shown in Fig. 8, and C_R is a semi-circle with radius $R \to \infty$. Physically, one expects

$$G(\mathbf{r}-\mathbf{r}', t-t')\begin{cases} = 0 \text{ if } t < t', \\ \neq 0 \text{ if } t > t'. \end{cases} \quad (1.183)$$

Decompose the frequency to real part and imaginary part, $\omega = \omega' + i\omega''$, then

$$e^{-i\omega(t-t')} = e^{-i\omega'(t-t')}e^{\omega''(t-t')}. \quad (1.184)$$

The integral over C_R would vanish if it goes under the lower-half complex plane ($\omega'' < 0$).

Along the x-axis, the integrand of I has singularities at $\omega = \pm vk$. One can bypass the singularities from above or from below. According to **Cauchy's residue theorem**, if it goes under, then $I = 0$ since there is no singularity in C. To get a nonzero result, it has to go above the singularities. Alternatively, we can shift the poles down a little to $\pm vk - i\varepsilon$, as shown in Fig. (8). It follows that

$$\begin{aligned} I &= \oint_C \frac{d\omega}{2\pi} \frac{c^2}{\omega^2 - c^2|\mathbf{k}|^2} e^{-i\omega(t-t')} && (1.185) \\ &= -2\pi i \frac{c^2}{2\pi} \left[\frac{e^{ic|\mathbf{k}|(t-t')}}{-2c|\mathbf{k}|} + \frac{e^{-ic|\mathbf{k}|(t-t')}}{+2c|\mathbf{k}|} \right] && (1.186) \\ &= \frac{i}{2} \frac{c}{|\mathbf{k}|} \left[e^{ic|\mathbf{k}|(t-t')} - e^{-ic|\mathbf{k}|(t-t')} \right]. && (1.187) \end{aligned}$$

Finally, integrate over $\mathbf{k}$ to get

$$\begin{aligned} & G(x,x') \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{ic}{2k} \left[e^{ick(t-t')} - e^{-ick(t-t')} \right] e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \\ &= \frac{2\pi}{(2\pi)^3} \int k^2 dk \frac{ic}{2k} \left[e^{ick(t-t')} - e^{-ick(t-t')} \right] \\ &\qquad \times \int_{-1}^{1} d\cos\theta \; e^{ik|\mathbf{r}-\mathbf{r}'|\cos\theta} \qquad (1.188) \\ &= \frac{1}{(2\pi)^2} \frac{c}{2} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \int_0^\infty dk \left[e^{ick(t-t')} - e^{-ick(t-t')} \right] \\ &\qquad \times \left[e^{ik|\mathbf{r}-\mathbf{r}'|} - e^{-ik|\mathbf{r}-\mathbf{r}'|} \right]. \end{aligned}$$

The product gives 4 terms. Flip the signs of k in two of the terms, we then get the integral,

$$\begin{aligned} & \int_{-\infty}^{\infty} dk \left[e^{i\omega(t-t')+ik|\mathbf{r}-\mathbf{r}'|} - e^{-i\omega(t-t')+ik|\mathbf{r}-\mathbf{r}'|} \right] \\ &= 2\pi\delta[c(t-t') + |\mathbf{r}-\mathbf{r}'|] - 2\pi\delta[c(t-t') - |\mathbf{r}-\mathbf{r}'|]. \end{aligned}$$

The first delta function makes no contribution since its argument cannot be zero. Therefore,

$$G(x,x') = -\frac{1}{4\pi} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c} \right). \qquad (1.189)$$

We now come back to the original wave equation in Eq. (1.160). According to Eq. (1.172),

$$\begin{aligned} \phi(\mathbf{r},t) &= \int d^4x' G(x,x') \left[-\frac{\rho(x')}{\varepsilon_0} \right] && (1.190) \\ &= \frac{1}{4\pi\varepsilon_0} \int dv' dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c} \right) \\ &= \frac{1}{4\pi\varepsilon_0} \int dv' \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}. && (1.191) \end{aligned}$$

This is Eq. (1.3). The same method can be applied to solve Eq. (1.2) and get the solution in Eq. (1.4).

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