Lecture notes on classical electrodynamics

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(Dated: June 6, 2023)

CONTENTS

I. ELECTROMAGNETIC WAVES IN DISPERSIVE MATTER

A. Dispersion relation

If the *phase velocity* of a wave in a material medium depends on its frequency (or wave length), then we call the medium a **dispersive matter**. For an EM wave,

$$v_p = \frac{\omega}{k} = \frac{c}{n(k)}, \tag{1.1}$$

where c is the velocity of light in vacuum, and n is the index of refraction. Thus, if n depends on frequency (or wave vector), then the medium is dispersive.

The group velocity of a wavepacket centered at $\mathbf{k}_0$ is,

$$\mathbf{v}_g = \frac{\partial \omega}{\partial \mathbf{k}_0} \tag{1.2}$$

$$= \frac{d\omega}{dk_0}\hat{\mathbf{k}} \text{ if } \omega(\mathbf{k}) = \omega(k). \tag{1.3}$$

Hence, in an *isotropic* medium ($\omega(\mathbf{k}) = \omega(k)$), $\mathbf{v}_p$ and $\mathbf{v}_g$ are both along the $\mathbf{k}$ direction.

The relation between ω and k is called **dispersion relation**. If $\omega = \alpha k$ is *linear* in k (α is a constant), then the medium is not dispersive, and $v_p = \alpha$. Also,

$$v_g = \frac{d\omega}{dk} = \alpha \ (= v_p). \tag{1.4}$$

There is no difference between v_p and v_g.

If $\omega(k)$ is *nonlinear*, then the medium is dispersive, and v_g is in general different from v_p. For $\omega(k) = \frac{c}{n(k)}k$,

$$v_g = \frac{d\omega}{dk} = \frac{c}{n} - \frac{ck}{n^2}\frac{dn}{dk}. \tag{1.5}$$

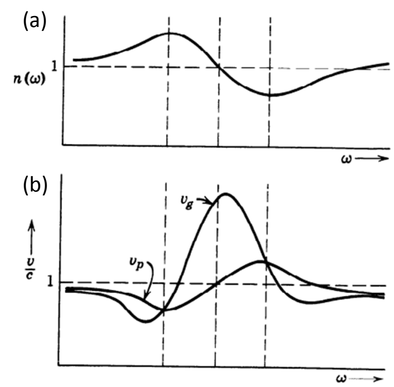

FIG. 1 Variation of refraction index (a) and velocities of wave propagation (b). The Fig. is from Jackson, 1998

One can rewrite $dn/dk = v_g dn/d\omega$, and get

$$v_g = \frac{c}{n(\omega) + \omega\frac{dn}{d\omega}}. \tag{1.6}$$

Therefore (see Fig. 1),

$$\text{if } \frac{dn}{d\omega} > 0, \text{ then } v_g < v_p; \tag{1.7}$$

$$\text{if } \frac{dn}{d\omega} < 0, \text{ then } v_g > v_p. \tag{1.8}$$

The former case is called **normal dispersion**; the latter **anomalous dispersion**. Note that if $\omega dn/d\omega$ is negative and large in magnitude, the group velocity could even be negative.

The discussions and terminologies above apply to other types of wave, such as sound wave, electron wave, and spin wave.

B. Frequency dispersion

For simplicity, suppose the material under consideration is homogeneous so that the conductivity does not depend on $\mathbf{r}$. When the electric field varies with time, we have the following Ohm's law,

$$\mathbf{J}(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \sigma(t - t')\mathbf{E}(\mathbf{r}, t'). \tag{1.9}$$

The current is driven by the field, and an effect follows its cause, so the current density $\mathbf{J}(\mathbf{r},t)$ only depends on the field at an earlier time $(t' < t)$. If we demand

$$\sigma(t-t') = 0 \text{ for } t' > t, \tag{1.10}$$

then we can write

$$\mathbf{J}(\mathbf{r},t) = \int_{-\infty}^{\infty} dt' \sigma(t-t')\mathbf{E}(\mathbf{r},t'). \tag{1.11}$$

The Ohm's law would look simpler in the frequency domain. Consider the following **Fourier transformation**,

$$\sigma(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\sigma(\omega)e^{-i\omega t}, \tag{1.12}$$

which has the inverse Fourier transformation,

$$\sigma(\omega) = \int_{-\infty}^{\infty} dt\sigma(t)e^{i\omega t}. \tag{1.13}$$

Similarly,

$$\begin{aligned} \mathbf{J}(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\mathbf{J}(\omega)e^{-i\omega t}, & (1.14)\\ \mathbf{E}(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\mathbf{E}(\omega)e^{-i\omega t}. & (1.15) \end{aligned}$$

Then, in the frequency domain, according to the **convolution theorem**, we have

$$\mathbf{J}(\mathbf{r},\omega) = \sigma(\omega)\mathbf{E}(\mathbf{r},\omega). \tag{1.16}$$

Since $\sigma^*(t) = \sigma(t)$, it follows that

$$\sigma^*(\omega) = \sigma(-\omega). \tag{1.17}$$

If $\sigma(\omega) = \sigma'(\omega) + i\sigma''(\omega)$, then

$$\begin{aligned} \sigma'(-\omega) &= \sigma'(\omega), & (1.18)\\ \sigma''(-\omega) &= -\sigma''(\omega). & (1.19) \end{aligned}$$

In the equations above, I have assumed that the current density at $\mathbf{r}$ depends only on the electric field at $\mathbf{r}$. In general, their relation can be *non-local*,

$$\mathbf{J}(\mathbf{r},t) = \int dv' \int_{-\infty}^{\infty} dt' \sigma(\mathbf{r}-\mathbf{r}',t-t')\mathbf{E}(\mathbf{r}',t'). \tag{1.20}$$

If so, then Eq. (1.16) is replaced by

$$\mathbf{J}(\mathbf{k},\omega) = \sigma(\mathbf{k},\omega)\mathbf{E}(\mathbf{k},\omega). \tag{1.21}$$

The conductivity σ is just one of the so-called **response functions**, which links response with external disturbance. Other examples of response function are χ_e and χ_m in

$$\begin{aligned} \mathbf{P}(\mathbf{k},\omega) &= \varepsilon_0\chi_e(\mathbf{k},\omega)\mathbf{E}(\mathbf{k},\omega), & (1.22)\\ \mathbf{M}(\mathbf{k},\omega) &= \chi_m(\mathbf{k},\omega)\mathbf{H}(\mathbf{k},\omega). & (1.23) \end{aligned}$$

Recall that

$$\begin{aligned} \mathbf{D} &= \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E}, & (1.24)\\ \mathbf{B} &= \mu_0(\mathbf{H} + \mathbf{M}) = \mu\mathbf{H}, & (1.25) \end{aligned}$$

thus

$$\begin{aligned} \varepsilon &= \varepsilon_0(1+\chi_e), & (1.26)\\ \mu &= \mu_0(1+\chi_m). & (1.27) \end{aligned}$$

In Chap 15, we show that the Ampere-Maxwell equation in matter involves effective currents related to polarization and magnetization,

$$\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J}_f + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}\right) + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{1.28}$$

or, equivalently,

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}. \tag{1.29}$$

The physics of various terms on the RHS of Eq. (1.28) is clear if the fields vary slowly: $\mathbf{J}_f$ is a free current, while $\mathbf{J}_P$ and $\mathbf{J}_M$ are bound currents. However, if the fields oscillate rapidly, then their distinction is blurred: all of the currents now oscillate back and forth locally. It is an intriguing fact that 1). We can dispense with $\mathbf{J}_f$ using an effective dielectric function. 2), We can employ $\mathbf{P}$ alone, or $\mathbf{M}$ alone, to describe *both* $\mathbf{J}_P$ and $\mathbf{J}_M$. This is explained below.

1). To simplify the discussion, suppose the non-local correlation mentioned in Eq. (1.20) can be ignored. Then, with $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{J}_f = \sigma\mathbf{E}$, we have

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \sigma\mathbf{E}(\mathbf{r},t) + \frac{\partial}{\partial t}[\varepsilon\mathbf{E}(\mathbf{r},t)]. \tag{1.30}$$

Consider a single Fourier component of the field, $\mathbf{E}(\mathbf{r},t){=}\mathbf{E}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, such that

$$\nabla \to i\mathbf{k}, \quad \frac{\partial}{\partial t} \to -i\omega. \tag{1.31}$$

It follows that,

$$\mathbf{k} \times \mathbf{H}(\mathbf{k},\omega) = \frac{1}{i}\sigma\mathbf{E}(\mathbf{k},\omega) - \varepsilon\omega\mathbf{E}(\mathbf{k},\omega). \tag{1.32}$$

Also, from

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t), \tag{1.33}$$

we have

$$\mathbf{k} \times \mathbf{E}(\mathbf{k},\omega) = \omega\mathbf{B}(\mathbf{k},\omega). \tag{1.34}$$

Multiply both sides of Eq. (1.32) from the left with $\mathbf{k}\times$, and *suppose* the EM wave is transverse, we get

$$\begin{aligned} k^2 &= \mu(\varepsilon\omega^2 + i\sigma\omega) & (1.35)\\ &= \mu\varepsilon_{eff}\omega^2, & (1.36) \end{aligned}$$

where

$$\varepsilon_{eff} = \varepsilon + i\frac{\sigma}{\omega}. \tag{1.37}$$

Thus, with this effective dielectric function, the $\mathbf{J}_f$ term can be dropped from the Ampere-Maxwell equation.

2). We now demonstrate that $\nabla\times\mathbf{M}$ can be included as part of $\partial\mathbf{P}/\partial t$. Recall that the Ampere-Maxwell law without $\mathbf{J}_f$ is,

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t}, \tag{1.38}$$

which has included the effective current,

$$\mathbf{J}_{eff} = \frac{\partial\mathbf{P}}{\partial t} + \nabla\times\mathbf{M}. \tag{1.39}$$

First, $\mathbf{M}$ can be written in terms of $\mathbf{H}$,

$$\nabla\times\mathbf{M} = \chi_m\nabla\times\mathbf{H}. \tag{1.40}$$

Again the nonlocal correlation is not considered. Second, $\mathbf{H}$ can be written in terms of $\mathbf{P}$,

$$\begin{aligned}\nabla\times\mathbf{H} &= \frac{\partial\mathbf{P}}{\partial t} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t} &(1.41)\\ &= \frac{\partial}{\partial t}\left(1+\frac{1}{\chi_e}\right)\mathbf{P}. &(1.42)\end{aligned}$$

Thus, using Eq. (1.40), one has

$$\nabla\times\mathbf{M} = \chi_m\left(1+\frac{1}{\chi_e}\right)\frac{\partial\mathbf{P}}{\partial t}. \tag{1.43}$$

Finally, we have

$$\mathbf{J}_{eff} = \left(1+\chi_m+\frac{\chi_m}{\chi_e}\right)\frac{\partial\mathbf{P}}{\partial t}. \tag{1.44}$$

Similarly, one can also write $\partial\mathbf{P}/\partial t$ in terms of $\nabla\times\mathbf{M}$. For details, see Sec. 18.2 of Zangwill, 2013.

C. Transverse wave and longitudinal wave

EM wave is a transverse wave in *vacuum*. However, it can have a longitudinal component in *matter*. For example, the plasma wave associated with charge oscillation in metals is a longitudinal EM wave (see next section). Let's go back to the fundamentals:

$$\begin{aligned}\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}, &(1.45)\\ \nabla\times\mathbf{H} &= \frac{\partial\mathbf{D}}{\partial t}. &(1.46)\end{aligned}$$

The $\mathbf{J}_f$ term is not listed, as its effect has been included in the effective ε_{eff} in Eq. (1.37) (now simply written as ε). For a single Fourier component with $(\mathbf{k},\omega)$ variables, we have

$$\begin{aligned}\mathbf{k}\times\mathbf{E} &= \omega\mu\mathbf{H}, &(1.47)\\ \mathbf{k}\times\mathbf{H} &= -\omega\varepsilon\mathbf{E}. &(1.48)\end{aligned}$$

Here $\mathbf{E}$ and $\mathbf{H}$ are shorthand notation for $\mathbf{E}(\mathbf{k},\omega)$ and $\mathbf{H}(\mathbf{k},\omega)$. Combine them to get

$$\begin{aligned}\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) &= \mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - k^2\mathbf{E} &(1.49)\\ &= -\omega^2\varepsilon\mu\mathbf{E}. &(1.50)\end{aligned}$$

Suppose we allow the possibility of a longitudinal component and write

$$\mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp, \tag{1.51}$$

then since $\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) = k^2\mathbf{E}_\parallel$, Eq. (1.50) gives

$$k^2\mathbf{E}_\parallel - k^2(\mathbf{E}_\parallel+\mathbf{E}_\perp) = -\omega^2\varepsilon\mu(\mathbf{E}_\parallel+\mathbf{E}_\perp). \tag{1.52}$$

The transverse part $\mathbf{E}_\perp$ gives a familiar result,

$$\begin{aligned}k^2 &= \omega^2\varepsilon\mu, &(1.53)\\ \rightarrow k &= \omega\frac{n}{c},\ n=\sqrt{\varepsilon_r\mu_r}. &(1.54)\end{aligned}$$

The longitudinal part $\mathbf{E}_\parallel$ gives

$$\varepsilon(\mathbf{k},\omega)\mu(\mathbf{k},\omega)\mathbf{E}_\parallel = 0. \tag{1.55}$$

As long as $\varepsilon\mu\neq 0$, then $\mathbf{E}_\parallel = 0$ and the EM wave is transverse, which is usually the case. However, *if* $\varepsilon(\omega)\mu(\omega) = 0$ (or $\varepsilon(\omega) = 0$ for non-magnetic material), then there can be a longitudinal component $\mathbf{E}_\parallel$. This is discussed in the next section.

D. Classical model for frequency dispersion

The electromagnetic properties of matter are all encapsulated in material parameters $\varepsilon(\mathbf{k},\omega), \mu(\mathbf{k},\omega)$, and $\sigma(\mathbf{k},\omega)$. In principle, they could be calculated theoretically using quantum manybody theory. Most of these calculations rely on *the theory of linear response*. Here we study these parameters using the much simpler classical mechanics, which nonetheless is able to capture the main physics in several cases.

1. Lorentz model of dielectric matter

Suppose the electrons in a matter are not interacting with each other, so that we can consider one electron at a time. The electrons are bound to atoms with elastic force $\mathbf{F} = -k\mathbf{r} = -m\omega_0^2\mathbf{r}$. Assume the material subjects to an oscillating electric field $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{-i\omega t}$. Let's focus on

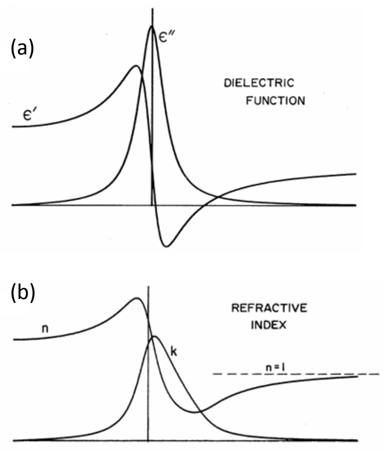

FIG. 2 Real part and imaginary part of (a) the dielectric function and (b) the index of refraction near resonance frequency.

one of the electron bound to an atom. The equation of motion for that electron with charge $q = -e$ is,

$$m\frac{d^2\mathbf{r}}{dt^2} = \underbrace{-m\omega_0^2\mathbf{r}}_{\text{elastic}}\underbrace{-\gamma m\frac{d\mathbf{r}}{dt}}_{\text{damping}}\underbrace{-e\mathbf{E}_0 e^{-i\omega t}}_{\text{driving}}. \quad (1.56)$$

We have added a damping force to mitigate the resonance from the driving force when $\omega \simeq \omega_0$.

In *steady state*, the electron would also oscillate at the driving frequency ω, thus (with the complex notation)

$$\mathbf{r}(t) = \mathbf{r}_0 e^{-i\omega t}. \quad (1.57)$$

A possible phase lag can be included in the so-far unknown amplitude $\mathbf{r}_0$. Substitute this into Eq. (1.56), it is straightforward to get

$$\mathbf{r}_0 = \frac{-e/m}{\omega_0^2 - \omega^2 - i\gamma\omega}\mathbf{E}_0. \quad (1.58)$$

Since the electron is displaced from its equilibrium position due to the external field, the material is electrically polarized. Its polarization $\mathbf{P}(t) = \mathbf{P}_0 e^{-i\omega t}$ is the electric dipole moments $\{\mathbf{p}_i(t) = \mathbf{p}_{i0} e^{-i\omega t}\}$ per unit volume V_0,

$$\begin{aligned}\mathbf{P}_0 = \frac{\sum_i \mathbf{p}_{i0}}{V_0} &= n\mathbf{p}_0 &(1.59)\\ &= n(-e)\mathbf{r}_0 &(1.60)\\ &= \frac{ne^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}\mathbf{E}_0 = \varepsilon_0\chi_e\mathbf{E}_0,\end{aligned}$$

in which n is the number of electrons per unit volume. It follows that

$$\begin{aligned}\frac{\varepsilon(\omega)}{\varepsilon_0} &= 1 + \chi_e &(1.61)\\ &= 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega}, &(1.62)\end{aligned}$$

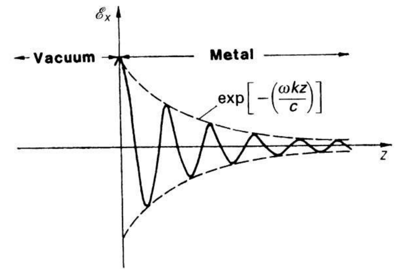

FIG. 3 Decay of a plane wave due to absorption.

where

$$\omega_p^2 \equiv \frac{ne^2}{\varepsilon_0 m} \quad (1.63)$$

is called **plasma frequency**.

Note that because of the damping, $\varepsilon(\omega)$ is a complex function,

$$\begin{aligned}\frac{\varepsilon}{\varepsilon_0} &= \varepsilon_r' + i\varepsilon_r'' &(1.64)\\ &= 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} + i\frac{\omega_p^2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}.\end{aligned}$$

Plots of its real part and imaginary part can be seen in Fig. 2.

For non-magnetic material ($\mu_r = 1$), the (complex) index of refraction

$$\mathcal{N}(\omega) = \sqrt{\varepsilon_r(\omega)} \equiv n + i\kappa, \quad (1.65)$$

where

$$\begin{aligned}n &= \frac{1}{\sqrt{2}}\sqrt{|\varepsilon_r| + \varepsilon_r'},\ |\varepsilon_r| = \sqrt{\varepsilon_r'^2 + \varepsilon_r''^2} &(1.66)\\ \kappa &= \frac{1}{\sqrt{2}}\sqrt{|\varepsilon_r| - \varepsilon_r'} \simeq \frac{\varepsilon_r''}{2\sqrt{\varepsilon_r'}}\ (\text{if } \varepsilon_r'' \ll \varepsilon_r'). &(1.67)\end{aligned}$$

Or, when the index of refraction becomes a complex function, a plane wave decays (Fig. 3),

$$\begin{aligned}e^{i\mathbf{k}\cdot\mathbf{r}} &= e^{i\mathcal{N}\frac{\omega}{c}\hat{\mathbf{k}}\cdot\mathbf{r}} &(1.68)\\ &= e^{in\frac{\omega}{c}\hat{\mathbf{k}}\cdot\mathbf{r}}e^{-\kappa\frac{\omega}{c}\hat{\mathbf{k}}\cdot\mathbf{r}}. &(1.69)\end{aligned}$$

When the imaginary part ε_r'' is non-zero, the EM field is attenuated. This happens near resonance when $\omega \simeq \omega_0$ (Fig. 2). That is, the attenuation is caused by **resonant absorption**.

If $n(\omega)$ increases with the frequency ω, then we have *normal* dispersion, and $v_p > v_g$. If it decreases with ω, then we have *anomalous* dispersion, and $v_g > v_p$. The latter occurs near resonance, and is always accompanied by a strong absorption (see Fig. 2(b)).

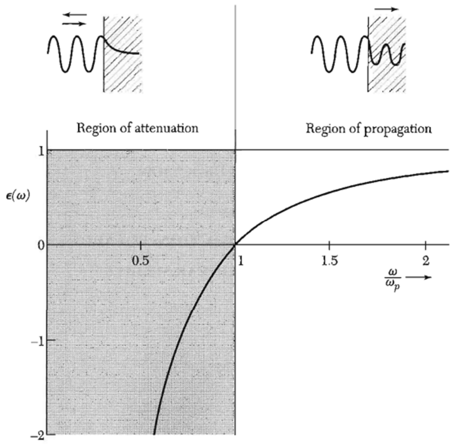

FIG. 4 EM wave with frequency $\omega < \omega_p$ ($\omega > \omega_p$) cannot (can) propagate in a plasma. The figure is from Kittel, 2004

2. Frequency dispersion in metal

In a metal, the valence electrons are not bound to the ions. This can be considered as a special case of the Lorentz model: set the elastic constant $k = m\omega_0^2$ as zero. From Eq. (1.62), one has

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega} \tag{1.70}$$

$$\gamma = \frac{1}{\tau} \rightarrow = 1 - \frac{(\omega_p\tau)^2}{(\omega\tau)^2 + i\omega\tau}, \tag{1.71}$$

where we have linked the damping coefficient γ with relaxation rate $1/\tau$, which is the inverse of the **relaxation time** τ. Microscopically, the relaxation of electrons is due to their scatterings with disorders in the solid. The dielectric function can also be written in the form of Eq. (1.37),

$$\varepsilon_{eff} = \varepsilon + i\frac{\sigma(\omega)}{\omega}, \tag{1.72}$$

with the AC conductivity,

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \ \sigma_0 = \frac{ne^2}{m}\tau. \tag{1.73}$$

For example, copper has $n = 8.5 \times 10^{28}$ atoms/m^3. If a sample has DC conductivity $\sigma_0 = 5.9 \times 10^7/\Omega$m, then $\tau \simeq 2.5 \times 10^{-14}$ sec. In the low-frequency limit, $\omega\tau \ll 1$,

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 - (\omega_p\tau)^2 + i\frac{\sigma_0}{\varepsilon_0\omega}. \tag{1.74}$$

On the other hand, if $\omega\tau \gg 1$, then

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 - \frac{\omega_p^2}{\omega^2}. \tag{1.75}$$

At low frequency $\omega < \omega_p$, $\varepsilon_r(\omega) < 0$ and the field could not propagate through (see Fig. 4). At high frequency

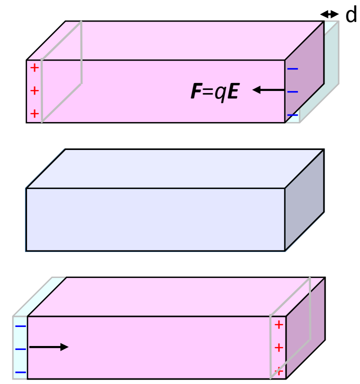

FIG. 5 Displace the electron gas from a rigid background of positive ions results in oscillations of the electron gas.

$\omega > \omega_p$, $\varepsilon_r(\omega) > 0$, and the EM wave can propagate as usual.

What happens if $\omega = \omega_p$? Since $\varepsilon(\omega_p) = 0$, according to the analysis below Eq. (1.55), there can be *longitudinal* oscillation of the electric field. This is called **plasma oscillation**. The simplest explanation of this oscillation is shown in Fig. 5: Suppose the electrons and ions in a metal are both uniform gases. If the electron gas as a whole is displaced form fixed positive ions with a displacement d, then it undergoes a strong restoring force $F = -qE$, where $q = (-e)nAd$ is the displaced charges on an end surface (n is the electron density, and A is the area of an end surface). The electric field $E = \sigma_s/\varepsilon_0$, where $\sigma_s = (-e)nd$ is the surface charge density. The mass of electrons inside a layer of thickness d should be $M = mnAd$, thus

$$F = M\ddot{d} = -(-e)ndA\frac{\sigma_s}{\varepsilon_0}, \tag{1.76}$$

$$\rightarrow \ddot{d} = -\frac{ne^2}{\varepsilon_0 m}d. \tag{1.77}$$

This is like the differential equation for a simple harmonic oscillator with oscillating frequency $\omega_p^2 = ne^2/\varepsilon_0 m$. Therefore, the charges would oscillate with the frequency ω_p. It is the resonant frequency for charge disturbance.

The analysis here is based on non-interacting electrons. However, even if the electrons are interacting (as in real metals), the plasma frequency is not changed. A simple rule of thumb gives,

$$f_p = \frac{\omega_p}{2\pi} \simeq 9.0\sqrt{n} \text{ Hz}, \tag{1.78}$$

where n is in units of 1/m^3. For a piece of copper with $n \simeq 8.5 \times 10^{28}$/m^3,

$$f_p \simeq 2.6 \times 10^{15} \text{ Hz, or } \lambda_p \simeq 0.12 \ \mu\text{m}. \tag{1.79}$$

This is slightly higher than the ultraviolet frequency. According to the study above, an EM wave with lower frequency would not propagate through, and the copper appears opaque. However, an ultraviolet light could

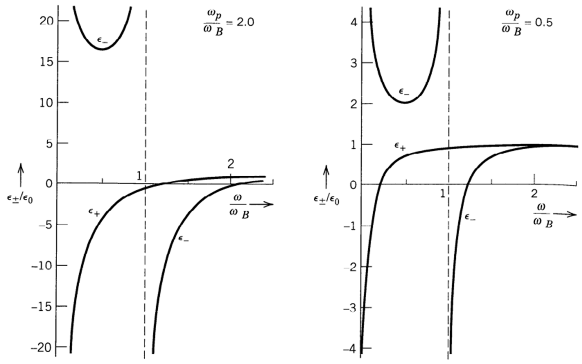

FIG. 6 Dielectric functions for circularly polarized waves. The figures are from Jackson, 1998.

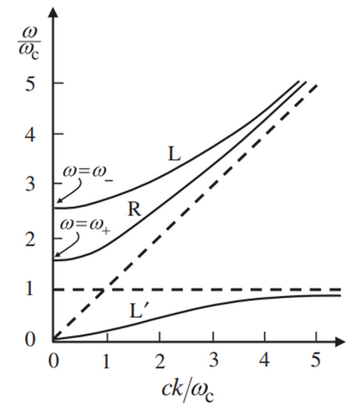

FIG. 7 Dispersion curves for EM waves in a magnetized plasma ($\omega_p = 2\omega_c$). The branch of helicon wave is indicated by L'.The figure is from Zangwill, 2013

propagate through it, and the copper appears transparent (Note: the frequency of violet light is up to 790 THz).

The ionosphere in the atmosphere has ionized electrons with density $n \simeq 10^{10} \sim 10^{12}/\mathrm{m}^3$. Thus, $f_p \simeq 10^6 \sim 10^7$ Hz. So the radio wave with lower frequency would be reflected back to earth, while that with higher frequency could propagate through. For comparison, the frequency of FM radio ranges from 88 MHz to 108 MHz.

3. Appleton model of magnetized plasma

We now consider a plasma in a magnetic field. For example, the ionosphere mentioned above is in a geomagnetic field. At high altitude, $B_0 \simeq 0.3$ G. An electron in this field executes cyclotron motion with frequency,

$$\omega_c = \frac{eB_0}{m} \simeq 5.3 \times 10^6 \text{ rad/s}, \tag{1.80}$$

which is close to the plasma frequency ω_p and thus is not negligible.

Adding the Lorentz force, and neglecting the damping force, the equation of motion (Eq. (1.56)) for an electron becomes

$$m\frac{d^2\mathbf{r}}{dt^2} = -e\mathbf{E}_0 e^{-i\omega t} - e\frac{d\mathbf{r}}{dt} \times \mathbf{B}_0. \tag{1.81}$$

Assume $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, and consider steady-state solution $\mathbf{r}(t) = \mathbf{r}_0 e^{-i\omega t}$, then one can get

$$\begin{pmatrix} -m\omega^2 & -ie\omega B_0 & 0 \\ ie\omega B_0 & -m\omega^2 & 0 \\ 0 & 0 & -m\omega^2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = -e \begin{pmatrix} E_{0x} \\ E_{0y} \\ E_{0z} \end{pmatrix}. \tag{1.82}$$

The solution is,

$$\mathbf{r}_0 = \frac{e}{m(\omega^2 - \omega_c^2)} \begin{pmatrix} 1 & -i\frac{\omega_c}{\omega} & 0 \\ +i\frac{\omega_c}{\omega} & 1 & 0 \\ 0 & 0 & 1 - \left(\frac{\omega_c}{\omega}\right)^2 \end{pmatrix} \mathbf{E}_0. \tag{1.83}$$

The polarization

$$\mathbf{P}(t) = (-e)n\mathbf{r}(t) = \mathbf{P}_0 e^{-i\omega t}, \tag{1.84}$$

and

$$\mathbf{P}_0 = \varepsilon_0 \chi_e \mathbf{E}_0. \tag{1.85}$$

Thus,

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \begin{pmatrix} 1 & -i\frac{\omega_c}{\omega} & 0 \\ +i\frac{\omega_c}{\omega} & 1 & 0 \\ 0 & 0 & 1 - \left(\frac{\omega_c}{\omega}\right)^2 \end{pmatrix}. \tag{1.86}$$

The dielectric matrix is of the form

$$\varepsilon = \begin{pmatrix} \varepsilon_1 & i\varepsilon_2 & 0 \\ -i\varepsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}. \tag{1.87}$$

It has the following eigenvalues

$$\varepsilon_1 \pm \varepsilon_2, \varepsilon_3. \tag{1.88}$$

The corresponding eigenvectors are,

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.89}$$

These are also the eigenvectors of the matrix in Eq. (1.83).

These first two eigenvectors tell us that, if

$$\boldsymbol{\mathcal{E}}_0 = E_0(\hat{\mathbf{x}} \mp i\hat{\mathbf{y}}), \;\; R/L \tag{1.90}$$

which represent right circular polarization (RCP) wave and left circular polarization (LCP) wave propagating *along* $\mathbf{B}_0$ direction, then (see Eq. (1.83))

$$\mathbf{r}_0 = r_0(\hat{\mathbf{x}} \mp i\hat{\mathbf{y}}), \tag{1.91}$$

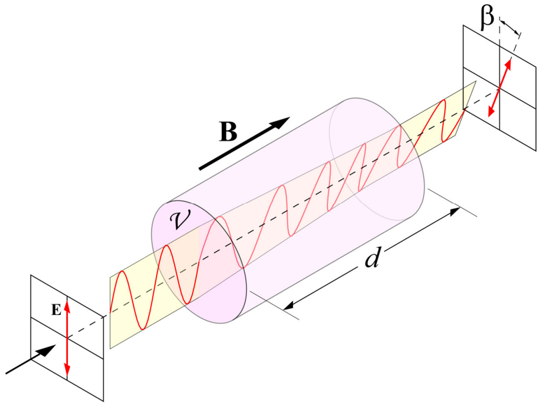

FIG. 8 Faraday rotation of a linearly polarized wavepassing through a magnetized plasma. The Fig. is from Wikipedia.

and $\mathbf{r} \parallel \boldsymbol{\mathcal{E}}$. The dielectric functions of this two eigenmodes are (Fig. 6)

$$\varepsilon_\pm(\omega) = \varepsilon_0 \left[1 - \frac{\omega_p^2}{\omega^2 \left(1 \pm \frac{\omega_c}{\omega}\right)} \right], \ R/L. \tag{1.92}$$

That is, the velocities of RCP wave and LCP wave are different. The plasma is not only anisotropic, but is also **birefringent**.

For $\varepsilon_+(\omega)$, there is only one curve; but for ε_- there are two. The dielectric function $\varepsilon_\pm(\omega) = 0$ when

$$\omega_\pm = \mp \left(\frac{\omega_c}{2}\right) + \sqrt{\left(\frac{\omega_c}{2}\right)^2 + \omega_p^2}. \tag{1.93}$$

The wave cannot propagate when $\varepsilon_\pm < 0$.

Furthermore, from $\omega = kc/n$ and $n = \sqrt{\varepsilon/\varepsilon_0}$, one can plot the dispersion curves $\omega_\pm(k)$ for propagating waves in Fig. 7.

• **Faraday rotation**

Following the discussion above, suppose the magnetic field is weak, and that

$$\omega > \omega_p \gg \omega_c, \tag{1.94}$$

then

$$\frac{\varepsilon_\pm}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \pm \frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2} + O\left(\frac{\omega_c^2}{\omega^2}\right). \tag{1.95}$$

From this we have the index of refraction,

$$n_\pm = \sqrt{\varepsilon_\pm/\varepsilon_0} \simeq n_0 \pm \frac{f}{2n_0} B_0, \tag{1.96}$$

where

$$n_0 = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}, \quad f(\omega) = \frac{e}{m\omega}\frac{\omega_p^2}{\omega^2}. \tag{1.97}$$

That is, the velocities of two opposite circular-polarized wave differ by an amount proportional to B_0.

A *linear-polarized* wave can be considered as an equal superposition of two opposite circular-polarized waves, we'll show below that such a difference in velocities would rotate the plane of linear polarization. First, recall that a vector can be expanded with two types of bases,

$$\begin{aligned} \mathbf{v} &= v_1 \hat{\mathbf{e}}_1 + v_2 \hat{\mathbf{e}}_2 & (1.98)\\ &= v_- \hat{\mathbf{e}}_+ + v_+ \hat{\mathbf{e}}_-, & (1.99) \end{aligned}$$

where

$$\begin{aligned} v_\pm &= \frac{1}{\sqrt{2}}(v_1 \pm i v_2), & (1.100)\\ \hat{\mathbf{e}}_\pm &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i \hat{\mathbf{e}}_2). & (1.101) \end{aligned}$$

In a birefringent medium,

$$\begin{aligned} \boldsymbol{\mathcal{D}} &= \varepsilon_0 \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{P}} & (1.102)\\ &= (\varepsilon_0 \mathcal{E}_- + \mathcal{P}_-)\hat{\mathbf{e}}_+ + (\varepsilon_0 \mathcal{E}_+ + \mathcal{P}_+)\hat{\mathbf{e}}_- & (1.103)\\ &= \varepsilon_- \mathcal{E}_- \hat{\mathbf{e}}_+ + \varepsilon_+ \mathcal{E}_+ \hat{\mathbf{e}}_- \quad (LCP + RCP). & (1.104) \end{aligned}$$

Suppose a linearly polarized wave has the following electric field at the origin,

$$\begin{aligned} \mathbf{E}(0,t) &= E_0 e^{-i\omega t} \hat{\mathbf{e}}_1 & (1.105)\\ &= \frac{1}{\sqrt{2}}\left(E_0 \hat{\mathbf{e}}_+ + E_0 \hat{\mathbf{e}}_-\right) e^{-i\omega t}. & (1.106) \end{aligned}$$

After propagating a distance L in the birefringent medium, the electric field becomes ($k_\pm \equiv \omega \frac{n_\pm}{c}$),

$$\begin{aligned} &\mathbf{E}(L,t)\\ &= \frac{E_0}{\sqrt{2}}\left(e^{ik_- L}\hat{\mathbf{e}}_+ + e^{ik_+ L}\hat{\mathbf{e}}_-\right) e^{-i\omega t} \qquad (1.107)\\ &= \frac{E_0}{2}\left[\left(e^{ik_- L} + e^{ik_+ L}\right)\hat{\mathbf{e}}_1 + i\left(e^{ik_- L} - e^{ik_+ L}\right)\hat{\mathbf{e}}_2\right] e^{-i\omega t}\\ &= E_0 \left(\cos\frac{\Delta k L}{2}\hat{\mathbf{e}}_1 + \sin\frac{\Delta k L}{2}\hat{\mathbf{e}}_2\right) e^{i\left(\frac{k_+ + k_-}{2}L - \omega t\right)}, \end{aligned}$$

in which

$$\Delta k = k_+ - k_- = \Delta n \frac{\omega}{c}, \text{ and } \Delta n = n_+ - n_-. \tag{1.108}$$

Obviously, the plane of polarization has rotated by an angle (Fig. 8)

$$\begin{aligned} \phi = \frac{\Delta k L}{2} &= \Delta n \frac{\omega L}{2c} & (1.109)\\ &= V B_0 L, & (1.110) \end{aligned}$$

where $V = \frac{ne^3}{2\varepsilon_0 m^2 c}\frac{1}{\omega^2}$ is the **Verdet constant**.

This Faraday rotation can be observed when an EM wave propagate through, e.g., a piece of metal, the ionosphere, or an interstellar medium (all are in a magnetic field).

• **Helicon wave**

Put a piece of metal in a magnetic field, e.g., sodium in a magnetic field $B = 1$ T. The plasma frequency and the cyclotron frequency are roughly $\omega_p \simeq 9.2 \times 10^{15}$ rad/s, $\omega_c \simeq 1.8 \times 10^{11}$ rad/s. At low frequency with

$$\omega \ll \omega_c, \tag{1.111}$$

one has, from Eq. (1.92),

$$\frac{\varepsilon_\pm}{\varepsilon_0} \simeq 1 \mp \frac{\omega_p^2}{\omega_c \omega}. \tag{1.112}$$

If $\omega < \omega_p^2/\omega_c$, then only the part ε_- allows a wave to propagate. Such a wave is left circularly polarized. Recall that when $B = 0$, no wave can propagate when $\omega < \omega_p$.

If $\omega \ll \omega_p^2/\omega_c$, then the refraction index

$$n_-(\omega) \simeq \frac{\omega_p}{\sqrt{\omega_c \omega}} \tag{1.113}$$

can be very large at low frequency. Thus the velocity of wave propagation would be very small. This type of wave in magnetized plasma is called **helicon wave** (see Sec 14.5 of Quinn and Yi, 2018). It has the dispersion relation,

$$\omega = \frac{\omega_c c^2}{\omega_p^2} k^2. \tag{1.114}$$

For the example of sodium above, $\omega_p \simeq 10^{16}$ rad/s, $\omega_c \simeq 10^{11}$ rad/s, and $\omega \simeq 10^3$ rad/s, we have $n \simeq 10^9$, and

$$v_p = \frac{c}{n} \simeq 0.3 \text{ m/s}. \tag{1.115}$$

Its group velocity is twice of this value.

In the ionosphere, $\omega_p \simeq 5 \times 10^6$ rad/s, $\omega_c \simeq 10^7$ rad/s. For $\omega \simeq 10 \sim 10^5$ rad/s, we have $n \simeq 100 \sim 1$. A thunderstorm in the atmosphere could trigger helicon waves with a broad range of frequency. These waves would follow the field lines of the geomagnetic field and reach the north pole or the south pole. The waves with higher (lower) frequency would propagate faster (slower). A receiver near the north pole, for example, would pick up these radio signals. It sounds like a whistler when the frequency falls within the audible range (20 Hz $\sim$ 20 kHz). Thus, it is called **whistler** in radio signal, which falls from high pitch to low pitch in seconds. You can find out recordings of them from the internet.

E. Left-handed material

We conclude this chapter with a peculiar type of medium with negative ε *and* μ. Recall that

$$\frac{\mathcal{N}}{c} = \sqrt{\varepsilon_r \mu_r}. \tag{1.116}$$

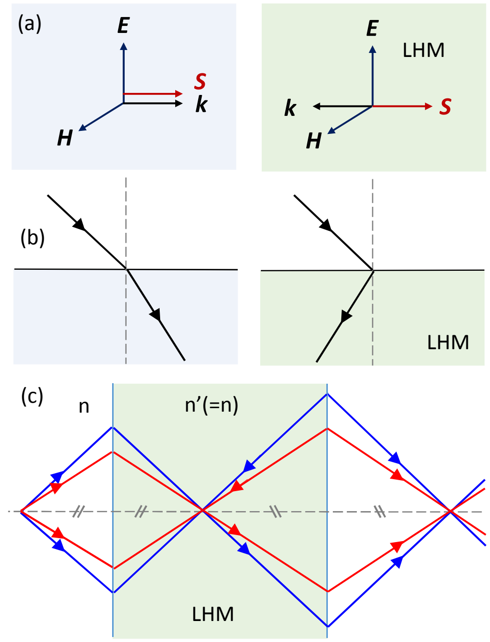

FIG. 9 (a) The $(\mathbf{E}, \mathbf{H}, \mathbf{k})$ triad in right-handed material (left) and left-handed material (right). (b) The refraction of wave in right-handed material (left) and left-handed material (right). (c) The perfect lens made from the left-handed material.

If $\varepsilon < 0$, *or* $\mu < 0$, then $\mathcal{N}$ is an imaginary number and a wave cannot propagate. However, what if both ε *and* μ are negative, so that $\mathcal{N}$ is still real? Such a possibility is first raised in Veselago, 1968. Even though there is no material with such a strange property, this is still something that could be explored theoretically. In fact, in the past two decades, people have begun to fabricate devices with arrays of tiny electronic components that could have this strange behavior. See p. 640 of Zangwill, 2013 for more details. This works only when the wavelength is much larger than the size of each component (which plays the role of an atom), and only for a limited range of frequency. But this restriction might be relaxed with more researches in the future.

Veselago found that, when $\varepsilon, \mu < 0$ and n is real, the wave can propagate, but with several surprising properties. For example,

1. The $(\mathbf{E}, \mathbf{H}, \mathbf{k})$ triad becomes left-handed.

Recall that

$$\mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H}, \tag{1.117}$$
$$\mathbf{k} \times \mathbf{H} = -\omega \varepsilon \mathbf{E}. \tag{1.118}$$

They can be written as,

$$\frac{n}{c} \hat{\mathbf{k}} \times \mathbf{E} = \mu \mathbf{H}, \tag{1.119}$$
$$\frac{n}{c} \hat{\mathbf{k}} \times \mathbf{H} = -\varepsilon \mathbf{E}, \tag{1.120}$$

where the $(\mathbf{E}, \mathbf{H}, \hat{\mathbf{k}})$ triad is right-handed.

If $\varepsilon, \mu > 0$, then the $(\mathbf{E}, \mathbf{H}, \mathbf{k})$ triad is right handed.

If $\varepsilon, \mu < 0$, then for these equations to hold simultaneously, we must choose $n < 0$ and $\mathbf{k} = -|n|\omega/c\hat{\mathbf{k}}$. Thus, the $(\mathbf{E}, \mathbf{H}, \mathbf{k})$ triad becomes left handed (Fig. 9(a)). This type of medium is called **left-handed materials**.

2. The energy of EM field flows backward.

The Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the energy current density of EM field. In the usual material, $\mathbf{S} \parallel \hat{\mathbf{k}}$. But in a left-hand material, $\mathbf{S} \parallel -\hat{\mathbf{k}}$. That is, the energy (and momentum) flows backward. The phase velocity given by $\mathbf{v}_p = (c/|n|)\hat{\mathbf{k}}$ is still along the $\hat{\mathbf{k}}$ direction, while the group velocity, which is the velocity of a wavepacket carrying field energy momentum, is along $\mathbf{S}$. Thus, the directions of $\mathbf{v}_p$ and $\mathbf{v}_g$ are opposite.

3. Refraction angle becomes negative.

Without giving a proof, we state that if medium 1 is the usual material, and medium 2 is LH material, then the Snell's law is

$$n_1 \sin\theta_i = -|n_2| \sin\theta_t. \tag{1.121}$$

Thus, the refraction angle is negative, and a refraction wave would bend backward (Fig. 9(b)).

If $n_2 = -n_1$, then something special happens. A divergent beam of rays would be re-focused after passing through the interface. This can serve as a *perfect lens* that is free of chromatic aberration (Fig. 9(c)). See Pendry and Smith, July, 2006 for more details.

4. Doppler effect is reversed.

In a usual medium, the light of a luminous object moving away from you would be red-shifted. But if it is moving away from you in a LH material, then its light would be blue-shifted.

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