Lecture notes on classical electrodynamics

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Contents

I. ELECTROMAGNETIC WAVES IN SIMPLE MATTER

A. Wave equation

The interaction of electromagnetic wave with matter is a huge subject, with lots of fascinating physics and important applications. EM wave with a wide range of frequency is an essential probe of material properties, widely used in laboratories around the globe. In this chapter we start with the basics.

Recall that the Maxwell equations in matter are,

$$\begin{aligned}
\nabla\cdot\mathbf{D} &= \rho_f, &(1.1)\\
\nabla\cdot\mathbf{B} &= 0, &(1.2)\\
\nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}, &(1.3)\\
\nabla\times\mathbf{H} &= \mathbf{J}_f+\frac{\partial\mathbf{D}}{\partial t}. &(1.4)
\end{aligned}$$

The so called **constitutive relations** between fields are,

$$\begin{aligned}
\mathbf{D} &= \varepsilon_0\mathbf{E}+\mathbf{P}=\varepsilon\mathbf{E}, &(1.5)\\
\mathbf{H} &= \frac{1}{\mu_0}\mathbf{B}-\mathbf{M}=\frac{1}{\mu}\mathbf{B}. &(1.6)
\end{aligned}$$

For monochromatic EM plane wave, we have

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \mathbf{E}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, &(1.7)\\
\mathbf{B}(\mathbf{r},t) &= \mathbf{B}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, &(1.8)
\end{aligned}$$

similarly for $\mathbf{D},\mathbf{H}$. The constitutive relations become

$$\begin{aligned}
\mathbf{D}(\mathbf{k},\omega) &= \varepsilon(\mathbf{k},\omega)\mathbf{E}(\mathbf{k},\omega), &(1.9)\\
\mathbf{H}(\mathbf{k},\omega) &= \frac{\mathbf{B}(\mathbf{k},\omega)}{\mu(\mathbf{k},\omega)}. &(1.10)
\end{aligned}$$

Suppose we are exploring a small range of frequency so that ε and μ are nearly constant, independent of $(\mathbf{k},\omega)$, then the Maxwell equations are very similar to those in vacuum. One only needs to replace ε_0,μ_0 with ε,μ, and most of the earlier results remain valid. We call this type of material *simple matter*.

Note: 1. Even if we treat all of the material parameters in the Mawxell equations, ε,μ, *and* σ, as constants. The conductivity may lead to an effective ε_{eff} that depends on ω. This variation can be neglected if we focus only on low frequency (see next Chap).
2. The constants, ε,μ can be complex numbers (in the complex notation) due to energy dissipation, and some of the physics in vacuum could change qualitatively (see I.A.3).
3. Out of the four fields $\mathbf{E},\mathbf{D},\mathbf{B},\mathbf{H}$, we can choose two of them as primary (one from $\mathbf{E},\mathbf{D}$, one from $\mathbf{B},\mathbf{H}$), and the other two as derived. There are four possible choices. In the following, we will choose $\mathbf{E},\mathbf{H}$ as primary fields.

1. Plane wave

Consider a monochromatic plane wave,

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \mathbf{E}_0e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, &(1.11)\\
\mathbf{H}(\mathbf{r},t) &= \mathbf{H}_0e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. &(1.12)
\end{aligned}$$

Suppose $\rho=0$ and $\mathbf{J}=0$, then we have

$$\begin{aligned}
\mathbf{k}\cdot\mathbf{E}_0 &= 0, &(1.13)\\
\mathbf{k}\cdot\mathbf{H}_0 &= 0, &(1.14)\\
\mathbf{k}\times\mathbf{E}_0 &= \omega\mu\mathbf{H}_0, &(1.15)\\
\mathbf{k}\times\mathbf{H}_0 &= -\omega\varepsilon\mathbf{E}_0. &(1.16)
\end{aligned}$$

From the third and the fourth equations, one has

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_0)=-\omega^2\varepsilon\mu\mathbf{E}_0. \quad (1.17)$$

With the BAC-CAB rule, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$, and the fact that $\mathbf{E}_0$ is perpendicular to the direction of propagation $\mathbf{k}$, we get

$$k^2=\omega^2\varepsilon\mu,\ \text{or}\ \omega(k)=\frac{k}{\sqrt{\varepsilon\mu}}. \quad (1.18)$$

The phase velocity of light in matter is,

$$\begin{aligned}
c_n &= \frac{\omega}{k},\quad (\omega=c_nk) &(1.19)\\
&= \frac{1}{\sqrt{\varepsilon\mu}}=\frac{c}{n}, &(1.20)
\end{aligned}$$

where c is the velocity of light in vacuum, and n is the **index of refraction**,

$$n = \sqrt{\varepsilon_r \mu_r} \equiv \sqrt{\frac{\varepsilon}{\varepsilon_0}\frac{\mu}{\mu_0}}. \tag{1.21}$$

2. Equation of continuity for energy

Let's analyze the Eq. of continuity for energy again, but now for EM fields in *matter*. The derivation is very similar to the one for the case in vacuum. The rate of work done by EM fields on charged particle is,

$$\frac{dW_{mech}}{dt} = \int dv \mathbf{J}_f \cdot \mathbf{E}. \tag{1.22}$$

From $\mathbf{J}_f = \nabla \times \mathbf{H} - \partial \mathbf{D}/\partial t$, one has

$$\frac{dW_{mech}}{dt} = -\int dv \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \int dv \nabla \times \mathbf{H} \cdot \mathbf{E}. \tag{1.23}$$

The second integral is equal to

$$\int dv \left[-\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{H} \cdot \nabla \times \mathbf{E} \right]. \tag{1.24}$$

With Faraday's law, $\nabla \times \mathbf{E}$ can be replaced by $-\partial \mathbf{B}/\partial t$. It follows that

$$\int dv \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) + \int dv \nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\int dv \mathbf{J}_f \cdot \mathbf{E}. \tag{1.25}$$

The is the Eq of continuity for energy in integral form.

If ε, μ are independent of field strength (called *linear medium*), then the integrand of the first term on the LHS can be written as the derivative, $\partial u_{EM}/\partial t$, of the energy density,

$$u_{EM} = \frac{1}{2}\mathbf{E} \cdot \mathbf{D} + \frac{1}{2}\mathbf{H} \cdot \mathbf{B}. \tag{1.26}$$

After the integration, it is the derivative, dU_{EM}/dt, of total field energy. Identify the Poynting vector as,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{1.27}$$

then Eq. (1.25) can be written as

$$\frac{d}{dt}U_{EM} + \int_S d\mathbf{a} \cdot \mathbf{S} = -\int_V dv \mathbf{J}_f \cdot \mathbf{E}. \tag{1.28}$$

3. Energy dissipation

Consider an EM wave passing through matter in the absence of $\mathbf{J}_f$. Then Eq. (1.25) becomes,

$$\int dv \left(\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) + \int d\mathbf{a} \cdot (\mathbf{E} \times \mathbf{H}) = 0. \tag{1.29}$$

Suppose the material is not magnetic, $\mu = \mu_0$, then $\mathbf{B} = \mu_0 \mathbf{H}$. Allow the dielectric constant to be complex,

$$\varepsilon = \varepsilon' + i\varepsilon''. \tag{1.30}$$

We will show that the imaginary part ε'' is related to energy dissipation.

Consider uniform fields,

$$\begin{aligned} \mathbf{E}(t) &= \mathbf{E}_0 \cos \omega t = Re\left(\mathbf{E}_0 e^{-i\omega t}\right), & (1.31)\\ \mathbf{D}(t) &= Re\left(\varepsilon \mathbf{E}_0 e^{-i\omega t}\right) & (1.32)\\ &= \varepsilon' \mathbf{E}_0 \cos \omega t + \varepsilon'' \mathbf{E}_0 \sin \omega t. & (1.33) \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{E}_0 \cos \omega t \cdot \left(-\varepsilon' \omega \mathbf{E}_0 \sin \omega t + \varepsilon'' \omega \mathbf{E}_0 \cos \omega t\right) \\ &= \frac{d}{dt}\left(\frac{\varepsilon'}{2}|\mathbf{E}(t)|^2\right) + \varepsilon'' \omega |\mathbf{E}(t)|^2. & (1.34) \end{aligned}$$

The first term is the usual electric field energy in u_{EM}. Move the second term to the RHS of the Eq. of continuity, we have

$$\frac{d}{dt}U_{EM} + \int_S d\mathbf{a} \cdot \mathbf{S} = -\int_V dv \varepsilon'' \omega |\mathbf{E}(t)|^2. \tag{1.35}$$

The term on the RHS is a source or sink of energy, depending on the sign of ε''. That is, a positive ε'' is an effective material parameter for energy dissipation. This integral gives the EM energy absorbed by the medium per unit time (i.e., power loss).

Like the damped oscillator in classical mechanics, we can define a Q-factor for the damping (of a monochromatic plane wave). For a homogeneous material,

$$\begin{aligned} Q &\equiv \omega \times \frac{\langle \text{energy stored} \rangle_T}{\langle \text{power loss} \rangle_T} & (1.36)\\ &= \omega \frac{\varepsilon' \langle |\mathbf{E}|^2 \rangle_T}{\varepsilon'' \omega \langle |\mathbf{E}|^2 \rangle_T} & (1.37)\\ &= \frac{\varepsilon'}{\varepsilon''}. & (1.38) \end{aligned}$$

The Q-factor has no dimension. It is large (i.e. little damping) if ε'' is small.

B. Reflection and refraction

We now consider the reflection and refraction of an EM wave at the interface of different media. But before doing that, we need to have the boundary condition of electromagnetic fields.

1. Boundary condition

In Chap 3 and Chap 10, we have studied the boundary condition (BC) of electrostatic field and magnetostatic field. Now we will investigate the BC for general *dynamic* EM fields. Let's start from the Maxwell equations,

$$\oint_S \mathbf{D}\cdot d\mathbf{a} = \int_V \rho\, dv, \tag{1.39}$$

$$\oint_S \mathbf{B}\cdot d\mathbf{a} = 0, \tag{1.40}$$

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a}, \tag{1.41}$$

$$\oint_C \mathbf{H}\cdot d\mathbf{r} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot d\mathbf{a}. \tag{1.42}$$

These are the integral forms of Eqs. (1.1), (1.2) (1.3), (1.4).

Let's apply the first equation to a closed surface S near the interface of two media, as shown in Fig. 1. S is the surface of a tiny box with area Δa and thickness Δw. The thickness would eventually shrink to zero. The left-hand side (LHS) of Eq. (1.39) gives

$$\oint_S \mathbf{D}\cdot d\mathbf{a} = (\mathbf{D}_2 - \mathbf{D}_1)\cdot \hat{\mathbf{n}}\Delta a. \tag{1.43}$$

The flux through the side surface can be ignored since the thinkness Δw of the box approaches zero. On the other hand, the right-hand side (RHS) gives

$$\int_V \rho\, dv = \sigma\Delta a, \tag{1.44}$$

where σ is the surface charge density. Equating LHS with RHS, one then has

$$(\mathbf{D}_2 - \mathbf{D}_1)\cdot \hat{\mathbf{n}} = \sigma. \tag{1.45}$$

The same method can be applied to Eq. (1.40) to get

$$(\mathbf{B}_2 - \mathbf{B}_1)\cdot \hat{\mathbf{n}} = 0. \tag{1.46}$$

If there is no surface charge, then the *normal* components of both $\mathbf{D}$ and $\mathbf{B}$ need to be continuous across the interface.

Next, let's apply the Eq. (1.41) to a closed loop C near the interface of two media, as shown in Fig. 1. C is a tiny rectangular loop with length $\Delta\ell$ and width Δw. The latter would eventually shrink to zero. The LHS of Eq. (1.41) gives

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = (\mathbf{E}_2 - \mathbf{E}_1)\cdot \hat{\boldsymbol{\ell}}\Delta\ell. \tag{1.47}$$

The contribution from two short sides can be ignored since the width Δw of the loop is nearly zero. On the other hand, since the area of the rectangle approaches zero, the RHS gives

$$-\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a} = 0. \tag{1.48}$$

Equating LHS with RHS, and noting that $\hat{\boldsymbol{\ell}} = \hat{\mathbf{t}}\times\hat{\mathbf{n}}$, one then has

$$\hat{\mathbf{t}}\times\hat{\mathbf{n}}\cdot(\mathbf{E}_2 - \mathbf{E}_1) = \hat{\mathbf{t}}\cdot\hat{\mathbf{n}}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0. \tag{1.49}$$

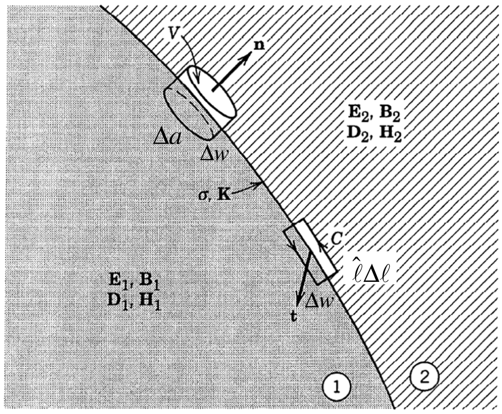

FIG. 1 An interface between two media. Fig. from Jackson, 1998

This is valid for any tangent vector $\hat{\mathbf{t}}$ (which is perpendicular to the surface of the rectangle). Thus

$$\hat{\mathbf{n}}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0. \tag{1.50}$$

The same method can be applied to Eq. (1.42). It is left as an exercise to show that

$$\hat{\mathbf{n}}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}, \tag{1.51}$$

where $\mathbf{K}$ is the surface current density. If there is no surface current, then the *tangential* components of both $\mathbf{E}$ and $\mathbf{H}$ have to be continuous across the interface.

2. Reflection and refraction

In Fig. 1, there is a monochromatic plane wave incident from below the interface. The incident wave vector $\mathbf{k}_i$ lies on the $x-z$ plane. The reflected wave and refracted wave propagate along $\mathbf{k}_r$ and $\mathbf{k}_t$. The associated electric fields are (in the complex notation),

$$\mathbf{E}_i(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_i e^{i(\mathbf{k}_i\cdot\mathbf{r}-\omega t)}, \tag{1.52}$$

$$\mathbf{E}_r(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_r e^{i(\mathbf{k}_r\cdot\mathbf{r}-\omega t)}, \tag{1.53}$$

$$\mathbf{E}_t(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_t e^{i(\mathbf{k}_t\cdot\mathbf{r}-\omega t)}. \tag{1.54}$$

Each of these fields is perpendicular to the direction of propagation.

From Eq. (1.3), one has

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}. \tag{1.55}$$

Recall that $\omega/k = c/n = 1/\sqrt{\varepsilon\mu}$, hence

$$\frac{c}{n}\mathbf{B} = \hat{\mathbf{k}}\times\mathbf{E}. \tag{1.56}$$

Thus,

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B} = \sqrt{\frac{\varepsilon}{\mu}}\hat{\mathbf{k}}\times\mathbf{E}. \tag{1.57}$$

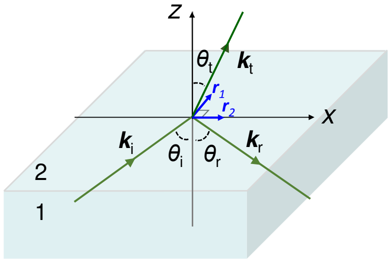

FIG. 2 An EM wave enters the interface from medium 1 below. Part of it is reflected and the rest is transmitted to medium 2.

Define the **wave impedance** as follows,

$$Z \equiv \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}, \tag{1.58}$$

then

$$Z\mathbf{H} = \hat{\mathbf{k}} \times \mathbf{E}. \tag{1.59}$$

Thus, we have

$$Z_1\mathbf{H}_i = \hat{\mathbf{k}}_i \times \mathbf{E}_i, \; Z_1\mathbf{H}_r = \hat{\mathbf{k}}_r \times \mathbf{E}_r, \tag{1.60}$$
$$Z_2\mathbf{H}_t = \hat{\mathbf{k}}_t \times \mathbf{E}_t. \tag{1.61}$$

According to the BC given in Eqs. (1.45), (1.46) (1.75), (1.76), the following components need to be continuous across the boundary (that has no surface charge and surface current):

Normal components D_n, B_n (or εE_n, μH_n), and

Tangential components $\mathbf{E}_\parallel$, $\mathbf{H}_\parallel$.

We emphasize that these apply to any location $\mathbf{r}$ on the boundary and at any instant of time t.

The boundary conditions dictate the relations between the fields on two sides of the interface. First, the frequencies ω of the three fields need to be the same in order for the temporal phase $e^{-i\omega t}$ from different fields to be congruent. For example, the frequency of a beam of green-light laser entering water would not change.

Furthermore, in what follows, we will show that

1). By matching the spatial *phases* of fields, one can get (a) the law of reflection, and (b) the law of refraction (Snell's law).

2). By matching the *amplitudes* of fields, one obtains the Fresnel equations.

1). *Match the spatial phases*

To match the spatial phases of electric field, one needs to have the following relations at any point $\mathbf{r}$ on the boundary,

$$\mathbf{k}_i \cdot \mathbf{r} = \mathbf{k}_r \cdot \mathbf{r} = \mathbf{k}_t \cdot \mathbf{r}. \tag{1.62}$$

First, choose $\mathbf{r} \perp \mathbf{k}_i$ and lies on the surface ($\mathbf{k}_i \cdot \mathbf{r} = 0$), as the $\mathbf{r}_1$ in Fig. 1. Then $\mathbf{k}_r, \mathbf{k}_t$ are also perpendicular to $\mathbf{r}$. Thus, $\mathbf{k}_i, \mathbf{k}_r, \mathbf{k}_t$ are lying on the same plane (this defines the **incident plane**).

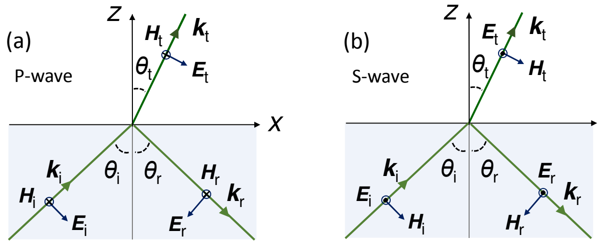

FIG. 3 (a) The p-wave has its $\mathbf{E}_i$ lying on the incident plane. (b) The s-wave has its $\mathbf{E}_i$ perpendicular to the incident plane.

Second, choose $\mathbf{r} \parallel \hat{\mathbf{x}}$ and lies on the surface, as the $\mathbf{r}_2$ in Fig. 1. Then one has

$$k_{ix} = k_{rx} = k_{tx}, \tag{1.63}$$
$$\text{or } k_i \sin\theta_i = k_r \sin\theta_r = k_t \sin\theta_t, \tag{1.64}$$

where $\theta_i, \theta_r, \theta_t$ are the incident angle, the reflection angle, and the refraction angle shown in Fig. 1. Since $k_i = k_r$, the first equality gives the **law of reflection**,

$$\theta_i = \theta_r. \tag{1.65}$$

Since $k = n\omega/c$, the second equality gives the **Snell's law**,

$$n_1 \sin\theta_i = n_2 \sin\theta_t. \tag{1.66}$$

Note that the Maxwell equations have not been explicitly used so far. So similar relations could also apply to other types of wave.

Note that $k_{iz} = -k_{rz}$, and

$$k_{tz} = \sqrt{k_t^2 - k_{tx}^2} \tag{1.67}$$
$$= \frac{\omega}{c}\sqrt{n_2^2 - n_1^2 \sin^2\theta_i}. \tag{1.68}$$

This is a real number if $n_2 > n_1$. However, it can be an imaginary number if $n_2 < n_1$ *and* $\sin\theta_i > n_2/n_1$, so that

$$k_{tz} = \frac{\omega}{c} i \sqrt{n_1^2 \sin^2\theta_i - n_2^2} \equiv i\kappa. \tag{1.69}$$

When this happens, for the refracted wave

$$e^{i(\mathbf{k}_t \cdot \mathbf{r} - \omega t)} = e^{i(k_{tx}x - \omega t)} e^{ik_{tz}z} \tag{1.70}$$
$$= e^{i(k_{tx}x - \omega t)} e^{-\kappa z}. \tag{1.71}$$

The first exponential is a plane wave moving along x direction. The second exponential decays along z, hence the wave cannot propagate along that direction, Thus, the field energy would be reflected back to medium 1. This is the **total internal reflection**.

2). *Match the amplitudes*

As we have mentioned, $\varepsilon E_n(\mathbf{r}, t)$, $\mu H_n(\mathbf{r}, t)$, $\mathbf{E}_\parallel(\mathbf{r}, t)$, and $\mathbf{H}_\parallel(\mathbf{r}, t)$ need to be continuous across the boundary. Since the phase factors that depend on $\mathbf{r}, t$ have been taken cared of, what's left to match is the amplitudes of field,

$$\varepsilon\mathcal{E}_n, \mu\mathcal{H}_n, \boldsymbol{\mathcal{E}}_\parallel, \boldsymbol{\mathcal{H}}_\parallel. \tag{1.72}$$

It follows that,

$$\begin{aligned} \varepsilon_1(\mathcal{E}_{iz}+\mathcal{E}_{rz}) &= \varepsilon_2\mathcal{E}_{tz}, & (1.73)\\ \mu_1(\mathcal{H}_{iz}+\mathcal{H}_{rz}) &= \mu_2\mathcal{H}_{tz}, & (1.74)\\ \boldsymbol{\mathcal{E}}_{i\parallel}+\boldsymbol{\mathcal{E}}_{r\parallel} &= \boldsymbol{\mathcal{E}}_{t\parallel}, & (1.75)\\ \boldsymbol{\mathcal{H}}_{i\parallel}+\boldsymbol{\mathcal{H}}_{r\parallel} &= \boldsymbol{\mathcal{H}}_{t\parallel}. & (1.76) \end{aligned}$$

Once $\boldsymbol{\mathcal{E}}$ is known, then $\boldsymbol{\mathcal{H}}$ can be determined. Thus only two equations are required. Here we choose to solve the last two equations.

Let's focus on two special cases: $\boldsymbol{\mathcal{E}}_i$ is parallel to or perpendicular to the incident plane. The field $\boldsymbol{\mathcal{E}}_i$ pointing to other direction can be considered as a superposition of this two special cases.

• $\boldsymbol{\mathcal{E}}_i \parallel x-z$ plane

Such an incident wave is called p-polarized wave (p for *parallel*), or TM-polarized wave, since the magnetic field is transverse to the incident plane. According to the geometry in Fig. 2(a), Eqs. (1.75) and (1.76) give

$$\begin{aligned} \cos\theta_i(\mathcal{E}_i-\mathcal{E}_r) &= \cos\theta_t\mathcal{E}_t, & (1.77)\\ \mathcal{H}_i+\mathcal{H}_r &= \mathcal{H}_t, & (1.78)\\ Z\mathcal{H}=\mathcal{E} \;\rightarrow\; \frac{\mathcal{E}_i+\mathcal{E}_r}{Z_1} &= \frac{\mathcal{E}_t}{Z_2}. & (1.79) \end{aligned}$$

This can be solved to give

$$\begin{aligned} r_p \equiv \frac{\mathcal{E}_r}{\mathcal{E}_i} &= \frac{Z_1\cos\theta_i - Z_2\cos\theta_t}{Z_1\cos\theta_i + Z_2\cos\theta_t}, & (1.80)\\ t_p \equiv \frac{\mathcal{E}_t}{\mathcal{E}_i} &= \frac{2Z_2\cos\theta_i}{Z_1\cos\theta_i + Z_2\cos\theta_t}. & (1.81) \end{aligned}$$

These are the **Fresnel equations** for the p-wave.

• $\boldsymbol{\mathcal{E}}_i \perp x-z$ plane

This incident wave is called s-polarized wave (s for *senkrecht* in German), or TE-polarized wave, since the electric field is transverse to the incident plane. According to the geometry in Fig. 2(b), Eqs. (1.75) and (1.76) give

$$\begin{aligned} \mathcal{E}_i+\mathcal{E}_r &= \mathcal{E}_t, & (1.82)\\ \cos\theta_i(\mathcal{H}_i-\mathcal{H}_r) &= \cos\theta_t\mathcal{H}_t, & (1.83)\\ Z\mathcal{H}=\mathcal{E} \;\rightarrow\; Z_2\cos\theta_i(\mathcal{E}_i-\mathcal{E}_r) &= Z_1\cos\theta_t\mathcal{E}_t. & (1.84) \end{aligned}$$

This can be solved to give

$$\begin{aligned} r_s = \frac{\mathcal{E}_r}{\mathcal{E}_i} &= \frac{Z_2\cos\theta_i - Z_1\cos\theta_t}{Z_2\cos\theta_i + Z_1\cos\theta_t}, & (1.85)\\ t_s = \frac{\mathcal{E}_t}{\mathcal{E}_i} &= \frac{2Z_2\cos\theta_i}{Z_2\cos\theta_i + Z_1\cos\theta_t}. & (1.86) \end{aligned}$$

These are the **Fresnel equations** for the s-wave.

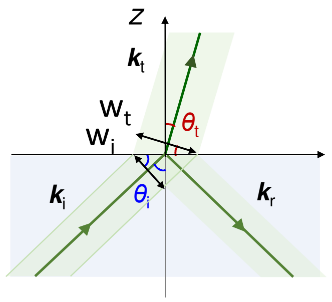

FIG. 4 The widths of the incident beam and the refracted beam are w_i and w_t respectively.

For non-magnetic materials (or for $\mu_1=\mu_2$), one can replace the wave impedance ($Z=\sqrt{\mu/\varepsilon}$) with the index of refraction ($n=\sqrt{\varepsilon\mu}c$),

$$\frac{Z_1}{Z_2} \rightarrow \frac{n_2}{n_1}. \tag{1.87}$$

The Fresnel equations then become

$$\begin{aligned} r_p \equiv \frac{\mathcal{E}_r}{\mathcal{E}_i} &= \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_2\cos\theta_i + n_1\cos\theta_t}, & (1.88)\\ t_p \equiv \frac{\mathcal{E}_t}{\mathcal{E}_i} &= \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t}; & (1.89)\\ r_s = \frac{\mathcal{E}_r}{\mathcal{E}_i} &= \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}, & (1.90)\\ t_s = \frac{\mathcal{E}_t}{\mathcal{E}_i} &= \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t}. & (1.91) \end{aligned}$$

For normal incidence ($\theta_i=0$), both sets of the Fresnel equations give similar results,

$$\begin{aligned} r_{s/p} &= \pm\frac{Z_2-Z_1}{Z_2+Z_1} = \pm\frac{n_1-n_2}{n_1+n_2}, & (1.92)\\ t_{s/p} &= \frac{2Z_2}{Z_2+Z_1} = \frac{2n_1}{n_1+n_2}. & (1.93) \end{aligned}$$

The second equality is valid only when the replacement in Eq. (1.87) applies. Even though r_s and r_p have opposite signs, these two actually give the same physical prediction for reflection: If $n_1>n_2$, then the reflected field does no suffer a flip of phase; whereas for $n_1<n_2$, there is a phase flip. This is similar to the phase flip (or not) for a pulse propagating through the joint connecting two ropes with different densities.

3. Transport of energy

Using the complex notation,

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= Re\left[\boldsymbol{\mathcal{E}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] & (1.94)\\ &= \frac{1}{2}\left[\boldsymbol{\mathcal{E}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+\boldsymbol{\mathcal{E}}^*e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right], & (1.95)\\ \mathbf{H}(\mathbf{r},t) &= Re\left[\boldsymbol{\mathcal{H}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] & (1.96)\\ &= \frac{1}{2}\left[\boldsymbol{\mathcal{H}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+\boldsymbol{\mathcal{H}}^*e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right], & (1.97) \end{aligned}$$

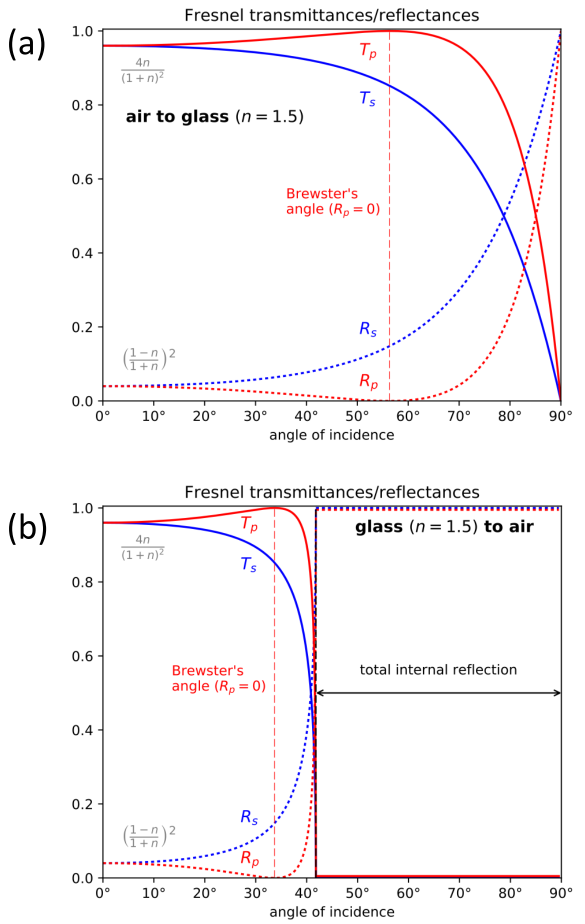

FIG. 5 (a) Reflection and refraction coefficients for p-wave and s-wave that propagate from air to glass. $R_p = 0$ at the Brewster angle. (b) Reflection and refraction coefficients for p-wave and s-wave that propagate from glass to air. $R_{s/p} = 1$ beyond the critical angle. Figs from Wikipedia.

the Poynting vector is,

$$\mathbf{S} = Re\mathbf{E} \times Re\mathbf{H} \tag{1.98}$$
$$= \frac{1}{4}\left(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}^* + \boldsymbol{\mathcal{E}}^* \times \boldsymbol{\mathcal{H}}\right) + \text{oscillating part.} \tag{1.99}$$

After taking a temporal average over periods, the oscillating part vanishes, hence

$$\langle \mathbf{S} \rangle_T = \frac{1}{2} Re\left(\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}^*\right), \quad Z\boldsymbol{\mathcal{H}} = \hat{\mathbf{k}} \times \boldsymbol{\mathcal{E}} \tag{1.100}$$
$$= \frac{1}{2Z}|\boldsymbol{\mathcal{E}}|^2 \hat{\mathbf{k}}. \tag{1.101}$$

Apply this to the reflection of wave. Define the **reflection coefficient** as follows (the normal vector $\hat{\mathbf{n}}$ points up),

$$R \equiv \left| \frac{\langle \mathbf{S}_r \rangle_T \cdot \hat{\mathbf{n}}}{\langle \mathbf{S}_i \rangle_T \cdot \hat{\mathbf{n}}} \right| \tag{1.102}$$
$$= \frac{|\mathcal{E}_r|^2}{|\mathcal{E}_i|^2} = r^2, \quad r^2 \equiv \frac{|\mathcal{E}_r|^2}{|\mathcal{E}_i|^2}. \tag{1.103}$$

This is the ratio of the energy flows perpendicular to the boundary surface. Similarly, the **transmission coefficient** is defined as,

$$T \equiv \left| \frac{\langle \mathbf{S}_t \rangle_T \cdot \hat{\mathbf{n}}}{\langle \mathbf{S}_i \rangle_T \cdot \hat{\mathbf{n}}} \right| \tag{1.104}$$
$$= \frac{Z_1 |\hat{\mathbf{k}}_t \cdot \hat{\mathbf{n}}|}{Z_2 |\hat{\mathbf{k}}_i \cdot \hat{\mathbf{n}}|} \frac{|\mathcal{E}_t|^2}{|\mathcal{E}_i|^2} \tag{1.105}$$
$$= \frac{Z_1 \cos\theta_t}{Z_2 \cos\theta_i} t^2, \quad t^2 \equiv \frac{|\mathcal{E}_t|^2}{|\mathcal{E}_i|^2}. \tag{1.106}$$

Since the energy of the incident wave either goes through the surface or reflects back, so one should have

$$R + T = 1. \tag{1.107}$$

Note that if $\mu_1 = \mu_2$, and the width of the wave is finite in extent, then T can also be written as,

$$T = \frac{n_2 w_t}{n_1 w_i} \frac{|\mathcal{E}_t|^2}{|\mathcal{E}_i|^2}, \tag{1.108}$$

where w_i, w_t are the widths of the beams (Fig. 3).

The angular dependence of R and T are shown in Fig. (for $\mu_1 = \mu_2$). Let's consider two special cases.
1. For normal incidence ($\theta_i = 0$),

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2. \tag{1.109}$$

The usual glass-air boundary has $n_2/n_1 = 1.5$, and $R \simeq 4$ %.

2. In Fig. 4(a), if $\theta_i \to 90$ degrees, then $R \to 1$, and most of the wave is reflected back. This is called **grazing reflection**. That is why a piece of acrylic plate works as a mirror when your eyesight is nearly parallel to the plate. Also, you are probably familiar with the fact that the surface of a serene lake could reflect the mountain far away like a mirror.

4. Brewster angle

In Fig. 4, for p-wave (but not for s-wave), there is a special angle θ_B with no reflection ($R = 0$). This is the **Brewster angle**. According to Eq. (1.88), it happens when

$$n_2 \cos\theta_i = n_1 \cos\theta_t. \tag{1.110}$$

Together with Snell's law,

$$n_1 \sin\theta_i = n_2 \sin\theta_t, \tag{1.111}$$

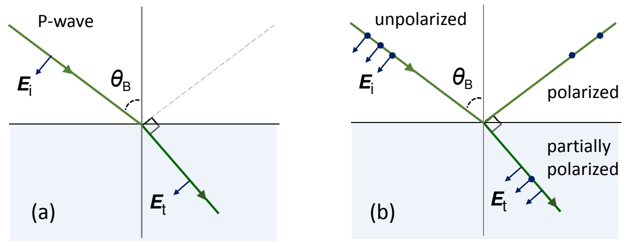

FIG. 6 (a) A p-wave has no reflection at the Brewster angle, when the reflected wave and the refracted wave are perpendicular to each other. (b) An unpolarized EM wave becomes polarized after reflection.

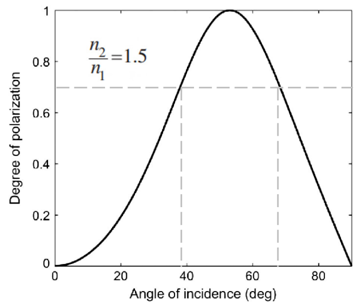

FIG. 7 The angular dependence of the degree of polarization after reflection. Note that the degree of polarization is over 70 % around an interval of 30 degrees.

we have

$$\begin{aligned}\cos^2\theta_i + \sin^2\theta_i &= \cos^2\theta_t + \sin^2\theta_t \\ &= \left(\frac{n_2}{n_1}\right)^2 \cos^2\theta_i + \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_i.\end{aligned} \tag{1.112}$$

Thus,

$$\left[\left(\frac{n_2}{n_1}\right)^2 - 1\right]\cos^2\theta_i + \left[\left(\frac{n_1}{n_2}\right)^2 - 1\right]\sin^2\theta_i = 0. \tag{1.113}$$

It follows that (write θ_i as θ_B)

$$\tan\theta_B = \frac{n_2}{n_1}. \tag{1.114}$$

If $n_2/n_1 = 1.5$, then $\theta_B \simeq 56.3$ degrees.

As an exercise, show that you can also write

$$r_s = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}, \tag{1.115}$$

$$r_p = +\frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}. \tag{1.116}$$

Thus, $r_p = 0$ when $\theta_i + \theta_t = \pi/2$ (see Fig. 5(a)).

• **Polarization by reflection**

Suppose an incident wave is an equal mixture of p wave and s wave. If it propagates along the Brewster angle, then the reflected wave would consist only of s-wave, since the p-wave does not reflect. Thus, a non-polarized

FIG. 8 A sunglass with polarizers. One of the lens can effectively block out reflected light from the ground.

incident wave becomes fully polarized upon reflection at the Brewster angle (Fig. 5(b)). If θ_i is close to θ_B, even though the effect is not as perfect, we still get partial polarization, and most of the reflection is s-wave. One can quantify the degree of polarization with

$$\Pi \equiv \left|\frac{R_s - R_p}{R_s + R_p}\right|. \tag{1.117}$$

See Fig. 6 for its angular dependence.

Since the reflection is mostly s-wave, especially near θ_B, one can use a polarizer to filter out some of the reflected light. For example, when you're at a seashore in a sunny day, a sunglass made of polarizers that block s-wave (with an electrical field parallel to the ground) can filter out the glaring reflection of sunshine on the surface of water. If you are to choose the orientation of the polarizer, then which one in Fig. 7 should you choose?

5. Total internal reflection

If an EM wave passes from a dense medium to a dilute medium ($n_1 > n_2$) with a large incident angle θ_i, then there could be **total internal reflection**. Snell's law tells us that

$$n_1 \sin\theta_i = n_2 \sin\theta_t. \tag{1.118}$$

If $n_1 > n_2$, then when θ_i approaches a θ_c given by

$$n_1 \sin\theta_c = n_2, \tag{1.119}$$

the refraction angle θ_t approaches $\pi/2$. Beyond angle θ_c, there is no real-valued solution for θ_t. What happens is that the wave is totally reflected when $\theta_i > \theta_c$. If $n_2/n_1 = 1/1.5$, then $\theta_c \simeq 41.8$ degrees.

In the following, we will investigate this phenomenon in more details.

• **Penetration depth**

Consider the refracted wave,

$$\mathbf{E}_t(\mathbf{r}, t) = \boldsymbol{\mathcal{E}}_t e^{i(k_{tx}x + k_{tz}z - \omega t)}. \tag{1.120}$$

Recall that

$$k_{tx} = k_{ix}, \tag{1.121}$$

$$k_{tz} = \sqrt{k_t^2 - k_{tx}^2}, \quad k_t = \frac{n_2}{c}\omega \tag{1.122}$$

$$= \frac{\omega}{c}\sqrt{n_2^2 - n_1^2\sin^2\theta_i} \tag{1.123}$$

$$\theta_i > \theta_c \to \quad = \frac{\omega}{c} i\sqrt{n_1^2\sin^2\theta_i - n_2^2} \equiv i\kappa. \tag{1.124}$$

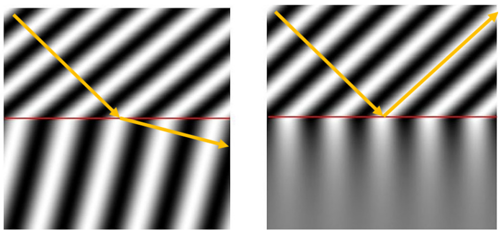

FIG. 9 An incident wave with (a) the usual reflection/refraction and (b) total internal reflection. The latter has the evanescent wave. Figs from Wikipedia.

Thus, when $\theta_i > \theta_c$,

$$\mathbf{E}_t(\mathbf{r},t) = \boldsymbol{\mathcal{E}}_t e^{-\kappa z} e^{i(k_{tx}x-\omega t)}. \qquad (1.125)$$

The wave propagates along x-direction, with an amplitude decaying along z- direction (Fig. 8). Such a type of refracted wave is called **evanescent wave**. It has a penetration depth,

$$\delta \equiv \frac{1}{\kappa} = \frac{n_2\lambda_t/2\pi}{\sqrt{n_1^2\sin^2\theta_i - n_2^2}}, \qquad (1.126)$$

in which c/ω is replaced by $n_2\lambda_t/2\pi$, and λ_t is the wave length in medium 2. The penetration depth blows up if $\theta_i \to \theta_c$ (when there is little refracted wave).

• **Frustrated total internal reflection**

For most of the θ_i in Eq. (1.126), the penetration depth of the evanescent wave is of the order of wavelength λ_t. Suppose before the wave decays completely, another dense medium is placed near the boundary, then it can continue with its journey as in Fig. 9(a). This is called frustrated total internal reflection (FTIR). A similar phenomenon appears in the quantum tunnelling of a particle through a barrier. A smart application of the FTIR can be found in some fingerprint scanner (Fig. 9(b)).

• **Phase velocity**

Because of the factor $e^{i(k_{tx}x-\omega t)}$ in Eq. (1.125), the evanescent wave would move near the boundary along x direction. It has the phase velocity,

$$v_p = \frac{\omega}{k_{tx}} = \frac{c}{n_1\sin\theta_i} > \frac{c}{n_1} \equiv v_{p1}. \qquad (1.127)$$

On the other hand,

$$v_p = \frac{c}{n_1\sin\theta_i} = \frac{c/n_2}{(n_1/n_2)\sin\theta_i} \qquad (1.128)$$

$$= \frac{\sin\theta_c}{\sin\theta_i}\frac{c}{n_2} < \frac{c}{n_2} \equiv v_{p2}. \qquad (1.129)$$

The velocities v_{p1}, v_{p2} are the phase velocities in medium 1 and medium 2, and

$$v_{p1} < v_p < v_{p2}. \qquad (1.130)$$

• **Transport of energy**

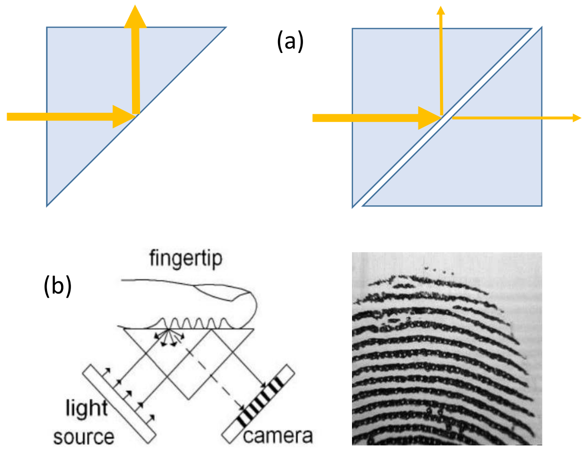

FIG. 10 (a) A light is reflected by a prism because of the total internal reflection. The total internal reflection is frustrated when two prisms are close to each other. (b) FTIR can be used to take fingerprint. The figure is from R. Trebino's ppt at frog.gatech.edu/talks.html.

Repeat the calculation in Sec. I.B.3 for the evanescent wave (Eq. (1.100)), then its not difficult to see that,

$$\langle \mathbf{S}\rangle_T = \frac{1}{2}Re\left(\boldsymbol{\mathcal{E}}_t \times \boldsymbol{\mathcal{H}}_t^*\right)e^{-2\kappa z}, \quad \mu\omega\boldsymbol{\mathcal{H}} = \mathbf{k}\times\boldsymbol{\mathcal{E}}$$

$$= \frac{1}{2\mu\omega}Re\left[\boldsymbol{\mathcal{E}}_t \times (\mathbf{k}_t^* \times \boldsymbol{\mathcal{E}}_t^*)\right]. \qquad (1.131)$$

Using the BAC-CAB rule for cross product, one has,

$$\boldsymbol{\mathcal{E}}_t \times (\mathbf{k}_t^* \times \boldsymbol{\mathcal{E}}_t^*) = \mathbf{k}_t^*|\boldsymbol{\mathcal{E}}_t|^2 - \boldsymbol{\mathcal{E}}_t^*(\boldsymbol{\mathcal{E}}_t\cdot\mathbf{k}_t^*). \qquad (1.132)$$

Since the electric field is transverse, $\boldsymbol{\mathcal{E}}_t\cdot\mathbf{k}_t^* = 0$ (see Eq. (1.137)), the second term is zero. Therefore,

$$\langle \mathbf{S}\rangle_T\cdot\hat{\mathbf{z}} = \frac{1}{2\mu\omega}Re\left[(\mathbf{k}_t^*\cdot\hat{\mathbf{z}})|\boldsymbol{\mathcal{E}}_t|^2\right]e^{-2\kappa z}. \qquad (1.133)$$

Recall that

$$\mathbf{k}_t = k_{tx}\hat{\mathbf{x}} + i\kappa\hat{\mathbf{z}}, \qquad (1.134)$$

thus $Re(\mathbf{k}_t^*\cdot\hat{\mathbf{z}}) = Re(-i\kappa) = 0$, and

$$\langle \mathbf{S}\rangle_T\cdot\hat{\mathbf{z}} = 0. \qquad (1.135)$$

There is no energy flowing to the other side along the z-direction, as expected.

On the other hand,

$$\langle \mathbf{S}\rangle_T\cdot\hat{\mathbf{x}} = \frac{k_{tx}}{2\mu\omega}|\boldsymbol{\mathcal{E}}_t|^2 e^{-2\kappa z} \neq 0. \qquad (1.136)$$

Thus, energy flows near the boundary along the x-direction and diminishes along the z-direction.

You might be wondering: If all of the field energy is reflected back, then why can there be extra energy moving along the boundary? Be aware that our analysis applies to plane wave with *infinite* width, but not to a beam of

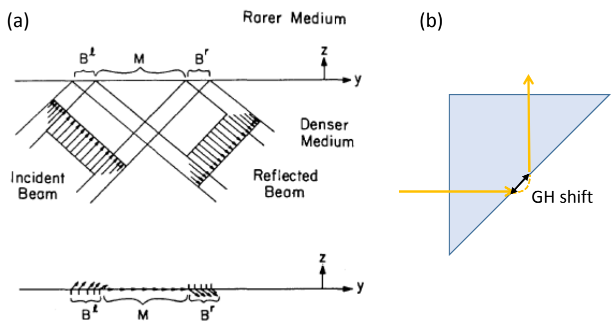

FIG. 11 (a) A beam with *finite* width is reflected due to total internal reflection. Energy flow of the evanescent wave is shown below (Puri and Birman, 1986). (b) Goos-Hänchen effect.

wave with finite width. If you study the reflection of a wave with finite extent (Fig. 10(a)), which is a more difficult problem, you'll find out that the energy can move along the boundary with a distance roughly the width of the EM wave. The energy of the evanescent wave would eventually return back to medium 1.

Because a beam of light venture a little bit to the other side before coming back, the entry point on the boundary and the exit point would differ slightly (Fig. 10(b)) Such a *longitudinal shift* of the light beam upon total internal reflection is called **Goos-Hänchen effect**. This shift is of the order of penetration depth, and can be enhanced when $\theta_i \simeq \theta_c$.

- **Polarization**

We can also investigate the polarization of the evanescent wave. From $\nabla \cdot \mathbf{E} = 0$, one has

$$\mathbf{k}_t \cdot \boldsymbol{\mathcal{E}}_t = k_{tx}\mathcal{E}_{tx} + k_{ty}\mathcal{E}_{ty} = 0. \tag{1.137}$$

Since $k_{tz} = i\kappa$, we have

$$\frac{\mathcal{E}_{tx}}{\mathcal{E}_{tz}} = -\frac{i\kappa}{k_i \sin\theta_i}. \tag{1.138}$$

Recall that $\boldsymbol{\mathcal{E}} = E_{01}\hat{\mathbf{e}}_1 \pm iE_{02}\hat{\mathbf{e}}_2$ represents elliptically polarized wave. Thus, the evanescent wave is elliptically polarized, with its $\mathbf{E}$ vector rotating within the incident plane.

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