Lecture notes on classical electrodynamics

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(Dated: May 2, 2022)

Contents

I. ELECTROMAGNETIC WAVES IN VACUUM

A. Wave equation

In this chapter, we consider EM field in vacuum with $\rho = 0, \mathbf{J} = 0$, so that

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, && (1.1) \\ \nabla \cdot \mathbf{B} &= 0, && (1.2) \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, && (1.3) \\ \nabla \times \mathbf{B} &= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. && (1.4) \end{aligned}$$

This leads to

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E} &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}, && (1.5) \\ \nabla \times \nabla \times \mathbf{B} &= -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}. && (1.6) \end{aligned}$$

Using $\nabla \times \nabla \times \mathbf{v} = \nabla\nabla \cdot \mathbf{v} - \nabla^2 \mathbf{v}$, we have

$$\begin{aligned} \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0, && (1.7) \\ \nabla^2 \mathbf{B} - \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} &= 0. && (1.8) \end{aligned}$$

Differential equation of this form is called **wave equation**. To solve it, we need to have the initial condition. Also, the Maxwell equations are still required to link $\mathbf{E}$ with $\mathbf{B}$.

B. Plane waves

Consider the following wave equation,

$$\nabla^2 \mathbf{w} - \frac{1}{c^2}\frac{\partial^2 \mathbf{w}}{\partial t^2} = 0. \quad (1.9)$$

Let's narrow down the scope of the problem and assume that the field is uniform along x, y directions. That is, $\mathbf{w}(\mathbf{r}, t) = \mathbf{w}(z, t)$ depends only on z and t, so that $\nabla^2 \mathbf{w} = \partial^2 \mathbf{w}/\partial z^2$. The wave equation can be written as,

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right)\mathbf{w} = 0. \quad (1.10)$$

Its solution must be of the following form,

$$\mathbf{w}(z, t) = \mathbf{f}(z + ct) + \mathbf{g}(z - ct), \ \forall\ \mathbf{f}, \mathbf{g}. \quad (1.11)$$

These two wave forms $\mathbf{f}, \mathbf{g}$ move down and up along z-axis with velocity c.

To prove it, define new variables,

$$\begin{aligned} \xi &\equiv z + ct, && (1.12) \\ \eta &\equiv z - ct. && (1.13) \end{aligned}$$

Or,

$$\begin{aligned} z &= \frac{1}{2}(\xi + \eta), && (1.14) \\ ct &= \frac{1}{2}(\xi - \eta). && (1.15) \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \frac{1}{2}\left(\frac{\partial}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}\right), && (1.16) \\ \frac{\partial}{\partial \eta} &= \frac{1}{2}\left(\frac{\partial}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\right). && (1.17) \end{aligned}$$

Thus, Eq. (1.10) becomes

$$\frac{\partial^2 \mathbf{w}}{\partial \xi \partial \eta} = 0. \quad (1.18)$$

It has two independent solutions, $\mathbf{w}(\xi)$ and $\mathbf{w}(\eta)$, or $\mathbf{f}(z + ct)$ and $\mathbf{g}(z - ct)$. A general solution is just a superposition of the two.

Now, $\mathbf{w}$ can be $\mathbf{E}$ or $\mathbf{B}$ in Eqs. (1.7),(1.8). but they still need to be restricted by the Maxwell equations. These restrictions will tell us that

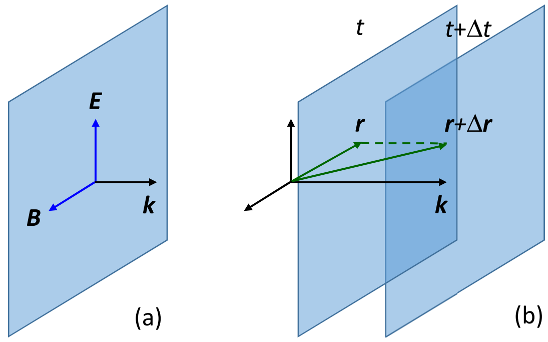

FIG. 1 (a) The vectors $\mathbf{E}, \mathbf{B}, \mathbf{k}$ are perpendicular to each other. (b) The wavefront at time t moves along $\mathbf{k}$ to a new location at $t + \Delta t$.

1. $E_z = B_z = 0$. That is, there is no longitudinal component of the EM field.
2. $c\mathbf{B} = \hat{\mathbf{z}} \times \mathbf{E}$. That is, $\mathbf{E}, \mathbf{B}, \hat{\mathbf{z}}$ are perpendicular to each other.

For a proof, see Sec. 16.3 of Zangwill, 2013.

Since the amplitude of the wave depends only on z (for a fixed t), thus its **wavefront** consist of *planes* moving along z-axis. This is called **plane wave**, with **wavevector** $\mathbf{k} = k\hat{\mathbf{z}}$, $k = 2\pi/\lambda$ (see Fig. 1(a)).

If a plane wave is propagating along a general direction $\hat{\mathbf{k}}$, then just replace

$$z \pm ct \;\to\; \hat{\mathbf{k}} \cdot \mathbf{r} \pm ct, \tag{1.19}$$

$$k(z \pm ct) \;\to\; \mathbf{k} \cdot \mathbf{r} \pm \omega t, \;\; \omega = ck. \tag{1.20}$$

Thus,

$$\mathbf{E}_\perp(\mathbf{r}, t) = \mathbf{E}_\perp(\mathbf{k} \cdot \mathbf{r} \pm \omega t), \tag{1.21}$$

$$\mathbf{B}_\perp(\mathbf{r}, t) = \mathbf{B}_\perp(\mathbf{k} \cdot \mathbf{r} \pm \omega t), \;\; c\mathbf{B}_\perp = \hat{\mathbf{k}} \times \mathbf{E}_\perp. \tag{1.22}$$

We have added a subscript to emphasize that the fields are transverse to $\hat{\mathbf{k}}$.

At an instant t, the wavefront with phase ϕ_0 consists of points $\mathbf{r}$ satisfying

$$\mathbf{k} \cdot \mathbf{r} - \omega t = \phi_0. \tag{1.23}$$

At the next instant $t + \Delta t$, this wavefront moves to $\mathbf{r} + \Delta\mathbf{r}$ (see Fig. 1(b)), and

$$\mathbf{k} \cdot (\mathbf{r} + \Delta\mathbf{r}) - \omega(t + \Delta t) = \phi_0. \tag{1.24}$$

Hence, the wavefront moves with velocity

$$v_p = \frac{\Delta\mathbf{r} \cdot \hat{\mathbf{k}}}{\Delta t} = \frac{\omega}{k}. \tag{1.25}$$

This is the **phase velocity**.

1. Energy and momentum

The energy density of the EM field at $\mathbf{r}, t$ is

$$u_{EM}(\mathbf{r}, t) = \frac{\varepsilon_0}{2}\left[E^2(\mathbf{r}, t) + c^2 B^2(\mathbf{r}, t)\right]. \tag{1.26}$$

For a transverse wave, $\mathbf{E}, \mathbf{B}, \mathbf{k}$ are perpendicular to each other, thus

$$c^2 B^2 = (\hat{\mathbf{k}} \times \mathbf{E}) \cdot (\hat{\mathbf{k}} \times \mathbf{E}) = E^2. \tag{1.27}$$

That is, the energy densities from $\mathbf{E}$ field and $\mathbf{B}$ field are the same. Hence,

$$u_{EM}(\mathbf{r}, t) = \varepsilon_0 E^2(\mathbf{r}, t). \tag{1.28}$$

The Poynting vector is given by,

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \tag{1.29}$$

$$= \frac{1}{\mu_0 c}|\mathbf{E}|^2\hat{\mathbf{k}} = u_{EM} c\hat{\mathbf{k}}, \tag{1.30}$$

which is equal to energy density $\times$ velocity. The momentum density of field is,

$$\mathbf{g}(\mathbf{r}, t) = \frac{\mathbf{S}(\mathbf{r}, t)}{c^2} = \frac{u_{EM}(\mathbf{r}, t)}{c}\hat{\mathbf{k}}. \tag{1.31}$$

2. Monochromatic plane wave

We now focus on the plane wave that is **monochromatic**. That is, it only has a single frequency, such as

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t), \tag{1.32}$$

where $\mathbf{E}_0$ is a constant vector.

It is convenient to introduce the **complex notation** and write

$$\mathbf{E}(\mathbf{r}, t) = Re\left[\boldsymbol{\mathcal{E}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right], \tag{1.33}$$

$$\mathbf{B}(\mathbf{r}, t) = Re\left[\boldsymbol{\mathcal{B}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\right], \;\; c\boldsymbol{\mathcal{B}} = \hat{\mathbf{k}} \times \boldsymbol{\mathcal{E}} \tag{1.34}$$

in which $\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}$ are constant *complex* vectors. If $\boldsymbol{\mathcal{E}}$ is real, then we are back to Eq. (1.32). If not, then the imaginary parts would contribute to phase shifts, since

$$\boldsymbol{\mathcal{E}} = |\mathcal{E}_1|e^{i\phi_1}\hat{\mathbf{e}}_1 + |\mathcal{E}_2|e^{i\phi_2}\hat{\mathbf{e}}_2., \tag{1.35}$$

in which $\hat{\mathbf{e}}_{1,2}$ are unit vectors perpendicular to $\mathbf{k}$.

If a calculation involves only linear equations of fields from start to finish, then the prefix *Re* can be dropped during the calculation. One only needs to take the real part of the end result to get the solution. However, if quadratic fields (e.g., energy density) are involved, then it is safer to keep the prefix *Re* explicit during the calculation to avoid mistakes. For example,

$$(Re\; e^{-i\omega t}) \times (Re\; e^{-i\omega t})) = \cos^2 \omega t, \tag{1.36}$$

$$\text{but} \;\; Re\; (e^{-i\omega t} \times e^{-i\omega t}) = \cos 2\omega t, \tag{1.37}$$

which are obviously different.

That is, the energy density should be written as

$$u_{EM} = \frac{\varepsilon_0}{2}\left(|Re\mathbf{E}|^2 + c^2|Re\mathbf{B}|^2\right). \tag{1.38}$$

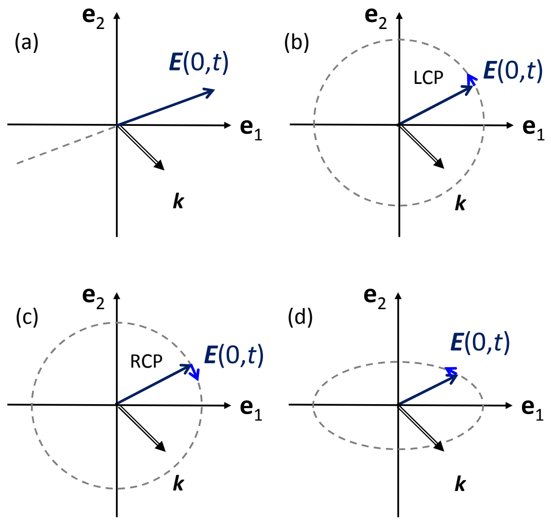

FIG. 2 The trajectory of the electric field for (a) Linear polarization, (b) Left circular polarization, (c) Right circular polarization, and (d) Elliptic polarization. The vector $\mathbf{k}$ is perpendicular to the $\hat{\mathbf{e}}_1 - \hat{\mathbf{e}}_2$ plane.

Since

$$
\begin{aligned}
Re\mathbf{E} &= \frac{1}{2}\left(\mathbf{E}+\mathbf{E}^*\right), &(1.39)\\
Re\mathbf{B} &= \frac{1}{2}\left(\mathbf{B}+\mathbf{B}^*\right), &(1.40)
\end{aligned}
$$

we have for monochromatic plane wave,

$$
\begin{aligned}
u_{EM} &= \frac{\varepsilon_0}{4}\left(\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}}^*+c^2\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\mathcal{B}}^*\right) &(1.41)\\
&+ \frac{\varepsilon_0}{4}Re\left[\left(\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}}+c^2\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\mathcal{B}}\right)e^{2i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right].
\end{aligned}
$$

The first part is static, while the second part oscillates in space and time. When one takes the time average of u_{EM} over one period (or many periods), then the second part vanishes. Recall that the electric field and the magnetic field contribute equally to the field energy, therefore,

$$
\begin{aligned}
\langle u_{EM}\rangle_T(\mathbf{r}) &= \frac{\varepsilon_0}{2}|\boldsymbol{\mathcal{E}}|^2, &(1.42)\\
\langle\mathbf{S}\rangle_T(\mathbf{r}) &= \langle u_{EM}\rangle_T\ c\hat{\mathbf{k}}, &(1.43)\\
\langle\mathbf{g}\rangle_T(\mathbf{r}) &= \frac{\langle u_{EM}\rangle_T}{c}\hat{\mathbf{k}}. &(1.44)
\end{aligned}
$$

Note that the first expression differs from $u_{EM} = \varepsilon_0 E^2(\mathbf{r},t)$ obtained earlier in Eq. (1.28) by a factor of two.

Example

A radio station transmits 10-kW EM wave with a frequency of 100 MHz. One kilometer away, the Poynting vector is

$$\langle S\rangle_T = \frac{10000}{4\pi 1000^2} = \frac{10^{-2}}{4\pi}\ \frac{\mathrm{W}}{\mathrm{m}^2}. \tag{1.45}$$

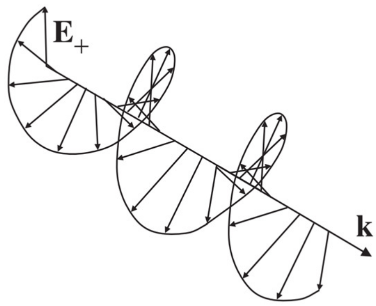

FIG. 3 A snapshot of the EM wave $\mathbf{E}_+(\mathbf{r},0)$ at $t=0$ with left circular polarization. Fig. from Zangwill, 2013.

From

$$\langle S\rangle_T = \frac{\varepsilon_0}{2}E_0^2 c, \tag{1.46}$$

one gets $E_0 = 0.775$ V/m. You may check that the magnetic field $B_0 = 2.58\times10^{-9}$ T. Remember that $\mathbf{S}$ is energy current density (energy/area·time). Therefore, the energy incident normally on a square plate with area $A = 1$ m^2 in one minute is

$$\langle S\rangle_T A\Delta t = 0.48\ J. \tag{1.47}$$

C. Polarization

1. Linear polarization

Let's continue with the monochromatic plane wave. By definition, its direction of polarization is the direction of its $\mathbf{E}$. The wave in Eq. (1.32) has linear polarization. Using complex notation,

$$
\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) &(1.48)\\
&= Re\left[\boldsymbol{\mathcal{E}}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right], &(1.49)\\
\boldsymbol{\mathcal{E}} &= \mathbf{E}_0 = E_{01}\hat{\mathbf{e}}_1+E_{02}\hat{\mathbf{e}}_2, &(1.50)
\end{aligned}
$$

in which $\mathbf{E}_0$ is real, and $(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2)$ is an orthogonal basis perpendicular to $\mathbf{k}$ (Fig. 2(a)).

2. Circular polarization

For an EM wave with circular polarization, the tip of the $\mathbf{E}$ vector draws out a circle. For example,

$$\mathbf{E}(\mathbf{r},t) = E_0\hat{\mathbf{e}}_1\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)\mp E_0\hat{\mathbf{e}}_2\sin(\mathbf{k}\cdot\mathbf{r}-\omega t). \tag{1.51}$$

At a fixed location $\mathbf{r}=0$,

$$\mathbf{E}_\pm(0,t) = E_0\hat{\mathbf{e}}_1\cos(\omega t)\pm E_0\hat{\mathbf{e}}_2\sin(\omega t). \tag{1.52}$$

The vector $\mathbf{E}_+(0,t)$ (or $\mathbf{E}_-(0,t)$) rotates counter-clockwise (or clockwise). We say that it has left (or right) circular polarization (Fig. 2(b), (c)). Note that here the handedness is defined *from the perspective of the source*, rather than the receiver (Fig. 3). That is, your thumb points to $-\mathbf{k}$, rather than $+\mathbf{k}$. However, not everyone likes this convention, and you may find an opposite definition in some literatures.

To write this in complex notation, choose

$$\boldsymbol{\mathcal{E}}_\pm = E_0(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2),\ \ L/R \tag{1.53}$$

which is a complex vector. Then

$$\begin{aligned}\mathbf{E}_\pm(\mathbf{r},t) &= Re\left[\boldsymbol{\mathcal{E}}_\pm e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] &(1.54)\\ &= Re\left[E_0\hat{\mathbf{e}}_1 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \pm E_0 i\hat{\mathbf{e}}_2 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] &(1.55)\\ &= E_0\hat{\mathbf{e}}_1\cos(\mathbf{k}\cdot\mathbf{r}-\omega t) \mp E_0\hat{\mathbf{e}}_2\sin(\mathbf{k}\cdot\mathbf{r}-\omega t).\end{aligned}$$

This is indeed the circular polarization in Eq. (1.51).

It is obvious that if the oscillations along $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ differ in magnitude, then we have elliptic polarization (Fig. 2(d)). That is,

$$\begin{aligned}\text{Linear pol.}\ \ \boldsymbol{\mathcal{E}} &= E_{01}\hat{\mathbf{e}}_1 + E_{02}\hat{\mathbf{e}}_2, &(1.56)\\ \text{Circular pol.}\ \ \boldsymbol{\mathcal{E}}_\pm &= E_0\hat{\mathbf{e}}_1 \pm iE_0\hat{\mathbf{e}}_2, &(1.57)\\ \text{Elliptic pol.}\ \ \boldsymbol{\mathcal{E}}_\pm &= E_{01}\hat{\mathbf{e}}_1 \pm iE_{02}\hat{\mathbf{e}}_2. &(1.58)\end{aligned}$$

A linearly polarized wave can be considered as an superposition of circular-polarized waves,

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= E_{01}\hat{\mathbf{e}}_1 + E_{02}\hat{\mathbf{e}}_2 &(1.59)\\ &= \alpha E_0(\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2) + \beta(\hat{\mathbf{e}}_1 - i\hat{\mathbf{e}}_2), &(1.60)\end{aligned}$$

where

$$\alpha E_0 = \frac{1}{2}(E_{01} - iE_{02}), \tag{1.61}$$

$$\beta E_0 = \frac{1}{2}(E_{01} + iE_{02}). \tag{1.62}$$

We can define an alternative basis for circular polarization,

$$\begin{aligned}\hat{\mathbf{e}}_+ &\equiv \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2), &(1.63)\\ \hat{\mathbf{e}}_- &\equiv \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_1 - i\hat{\mathbf{e}}_2). &(1.64)\end{aligned}$$

Then,

$$\begin{aligned}\boldsymbol{\mathcal{E}} &= E_{01}\hat{\mathbf{e}}_1 + E_{02}\hat{\mathbf{e}}_2 &(1.65)\\ &= E_-\hat{\mathbf{e}}_+ + E_+\hat{\mathbf{e}}_-, &(1.66)\end{aligned}$$

where

$$E_\pm \equiv \frac{1}{\sqrt{2}}(E_{01} \pm iE_{02}). \tag{1.67}$$

Since $|E_+| = |E_-|$, a linear polarization wave is an equal superposition of a LCP wave and a RCP wave.

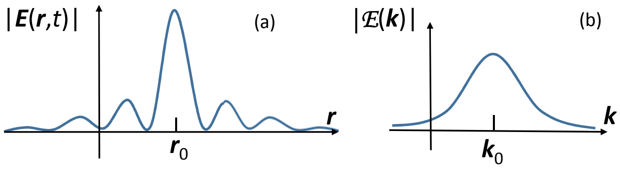

FIG. 4 A wavepacket (a) and its Fourier components (b) that are centered at $\mathbf{k}_0$.

D. General plane wave

For a plane wave that is *not* monochromatic, it can be decomposed as a superposition of monochromatic waves (Fig. 4),

$$\mathbf{E}(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3}\boldsymbol{\mathcal{E}}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \tag{1.68}$$

We have used the complex notation but dropped the prefix *Re*. Also,

$$\begin{aligned}c\mathbf{B}(\mathbf{r},t) &= \int \frac{d^3k}{(2\pi)^3}\hat{\mathbf{k}}\times\boldsymbol{\mathcal{E}}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} &(1.69)\\ &= \hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t). &(1.70)\end{aligned}$$

1. Energy density

We now analyze the energy density of a general plane wave. Since

$$c^2|Re\mathbf{B}(\mathbf{r},t)|^2 = |\hat{\mathbf{k}}\times Re\mathbf{E}(\mathbf{r},t)|^2 = |Re\mathbf{E}|^2, \tag{1.71}$$

one has

$$\begin{aligned}u_{EM}(\mathbf{r},t) &= \frac{\varepsilon_0}{2}\left(|Re\mathbf{E}(\mathbf{r},t)|^2 + c^2|Re\mathbf{B}(\mathbf{r},t)|^2\right)\\ &= \varepsilon_0|Re\mathbf{E}(\mathbf{r},t)|^2. &(1.72)\end{aligned}$$

Now,

$$\begin{aligned}|Re\mathbf{E}|^2 &= \left|\frac{1}{2}(\mathbf{E}+\mathbf{E}^*)\right|^2 &(1.73)\\ &= \frac{1}{2}|\mathbf{E}|^2 + \frac{1}{4}\left(\mathbf{E}\cdot\mathbf{E} + \mathbf{E}^*\cdot\mathbf{E}^*\right). &(1.74)\end{aligned}$$

In the first term, we have

$$\mathbf{E}^*\cdot\mathbf{E} = \int\frac{d^3k'}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}\boldsymbol{\mathcal{E}}^*(\mathbf{k}')\cdot\boldsymbol{\mathcal{E}}(\mathbf{k})e^{i[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}-(\omega-\omega')t]}, \tag{1.75}$$

which shows complicated interference between monochromatic waves. This can be simplified after integration over $\mathbf{r}$. Using

$$\int dv e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}'), \tag{1.76}$$

and $\omega = ck$, one has

$$\int dv|\mathbf{E}|^2 = \int\frac{d^3k}{(2\pi)^3}|\boldsymbol{\mathcal{E}}(\mathbf{k})|^2. \tag{1.77}$$

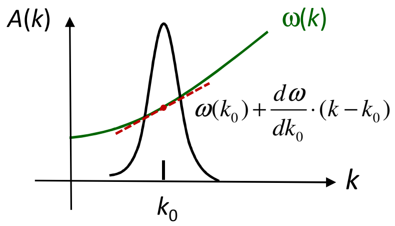

FIG. 5 When $\omega(k)$ is smoother than $A(k)$ near k_0, it can be approximated by the dotted straight line, which is the Taylor expansion of $\omega(k)$ to the first order.

Similarly,

$$\int dv \mathbf{E}\cdot\mathbf{E} = \int \frac{d^3k}{(2\pi)^3}\boldsymbol{\mathcal{E}}(\mathbf{k})\cdot\boldsymbol{\mathcal{E}}(-\mathbf{k})e^{-2i\omega t}. \quad (1.78)$$

For practical cases, this term (and its complex conjugate) is oscillating rapidly and can be neglected after time average.

Finally, the total field energy is

$$U_{EM}(t) = \int dv u_{EM}(\mathbf{r},t). \quad (1.79)$$

After time average, we have

$$\langle U_{EM}\rangle_T = \frac{\varepsilon_0}{2}\int \frac{d^3k}{(2\pi)^3}|\boldsymbol{\mathcal{E}}(\mathbf{k})|^2. \quad (1.80)$$

Similarly,

$$\langle \mathbf{P}_{EM}\rangle_T = \frac{\varepsilon_0}{2c}\int \frac{d^3k}{(2\pi)^3}\hat{\mathbf{k}}|\boldsymbol{\mathcal{E}}(\mathbf{k})|^2. \quad (1.81)$$

2. Group velocity of wavepacket

A wavepacket has a localized waveform, which is a superposition of plane waves. For simplicity, we study the velocity of a scalar wavepacket $u(\mathbf{r},t)$. The same analysis applies not only to EM wave, but also to acoustic wave, electron wave ... etc.

Suppose at $t=0$, we have a smooth bump $u(\mathbf{r},0)$, then how does it evolve to $u(\mathbf{r},t)$? To solve this, first decompose the initial wavepacket into a superposition of plane waves,

$$u(\mathbf{r},0) = \int \frac{d^3k}{(2\pi)^3}A(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (1.82)$$

The coefficient $A(\mathbf{k})$ is localized around some location $\mathbf{k}_0$ in $\mathbf{k}$-space. Each plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$ moves with phase velocity $\mathbf{v}_p = (\omega/k)\hat{\mathbf{k}}$. That is, after time t,

$$e^{i\mathbf{k}\cdot\mathbf{r}} \to e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}_p t)} = e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \quad (1.83)$$

Therefore, after superposition,

$$u(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3}A(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \quad (1.84)$$

In an isotropic medium with linear dispersion, $\omega(\mathbf{k}) = vk$. However, in general, $\omega(\mathbf{k})$ can be nonlinear. Suppose $\omega(\mathbf{k})$ is a smooth function of $\mathbf{k}$ compared to $A(\mathbf{k})$, then we can expand (see Fig. 5)

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_0) + \frac{\partial\omega}{\partial\mathbf{k}_0}\cdot(\mathbf{k}-\mathbf{k}_0) + \cdots. \quad (1.85)$$

It follows that,

$$\begin{aligned} u(\mathbf{r},t) &\simeq e^{-i\omega_0 t + i\frac{\partial\omega}{\partial\mathbf{k}_0}\cdot\mathbf{k}_0}\int \frac{d^3k}{(2\pi)^3}A(\mathbf{k})e^{i\mathbf{k}\cdot\left(\mathbf{r}-\frac{\partial\omega}{\partial\mathbf{k}_0}t\right)} \\ &= e^{i\phi}u\left(\mathbf{r}-\frac{\partial\omega}{\partial\mathbf{k}_0}t, 0\right). \end{aligned} \quad (1.86)$$

That is, to this order of approximation, the wavepacket is moving rigidly (without the change of shape) with velocity,

$$\mathbf{v}_g = \frac{\partial\omega}{\partial\mathbf{k}_0}. \quad (1.87)$$

This is called **group velocity**. The energy and momentum of an EM pulse propagates with the velocity of $\mathbf{v}_g$, not $\mathbf{v}_p$.

If we go beyond the first order of the expansion in Eq. (1.85), then the shape of the wavepacket would no longer remain the same. Its width would grow with time. This is the known as **wavepacket expansion**.

References

Zangwill, A., 2013, *Modern electrodynamics* (Cambridge Univ. Press, Cambridge).