Lecture notes on classical electrodynamics

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(Dated: April 17, 2023)

CONTENTS

I. GENERAL ELECTROMAGNETIC FIELDS

A. Electromagnetic potentials and gauge transformations

It is known in vector calculus that the divergence of the curl of any vector field $\mathbf{W}$ is zero, and the curl of the gradient of any scalar field W is zero. The reverse is also true: if the divergence of a vector field $\mathbf{V}$ is zero, then $\mathbf{V}$ can be written as a curl; if the curl of $\mathbf{V}$ is zero, then $\mathbf{V}$ can be written as a gradient:

$$\nabla\cdot\mathbf{V}=0 \quad\leftrightarrow\quad \mathbf{V}=\nabla\times\mathbf{W}, \tag{1.1}$$
$$\nabla\times\mathbf{V}=0 \quad\leftrightarrow\quad \mathbf{V}=\nabla W. \tag{1.2}$$

Recall that two of the Maxwell equations without a source term are,

$$\nabla\cdot\mathbf{B} = 0, \tag{1.3}$$
$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} = 0. \tag{1.4}$$

The first equation allows us to write

$$\mathbf{B}=\nabla\times\mathbf{A}, \tag{1.5}$$

where $\mathbf{A}$ is the **vector potential**. The second equation gives

$$\nabla\times(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t})=0. \tag{1.6}$$

It follows that,

$$\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t}. \tag{1.7}$$

Substitute Eqs. (1.5) and (1.7) to the other two Maxwell equations,

$$\nabla\cdot\mathbf{E} = \frac{\rho_0}{\varepsilon_0}, \tag{1.8}$$
$$\nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{J}, \tag{1.9}$$

we have

$$\nabla^2\phi+\frac{\partial}{\partial t}\nabla\cdot\mathbf{A} = -\frac{\rho_0}{\varepsilon_0}, \tag{1.10}$$
$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = -\mu_0\mathbf{J}. \tag{1.11}$$

Originally, we have four *first-order* partial differential eqs for $\mathbf{E}$ and $\mathbf{B}$, now we have two *second-order* partial differential eqs for ϕ and $\mathbf{A}$.

1. Gauge transformation

The fields $\mathbf{E}$ and $\mathbf{B}$ can be measured in experiments through the Lorentz force, while the potentials ϕ and $\mathbf{A}$ cannot. The latter are not uniquely defined in Eqs. (1.5) and (1.7). Indeed, suppose $\phi,\mathbf{A}$ and $\phi',\mathbf{A}'$ are related by the following **gauge transformation**,

$$\mathbf{A}' = \mathbf{A}+\nabla\Lambda, \tag{1.12}$$
$$\phi' = \phi-\frac{\partial\Lambda}{\partial t}, \tag{1.13}$$

where $\Lambda(\mathbf{r},t)$ can be any smooth function, then they give the same electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$.

We can take advantage of this gauge degree of freedom to simplify Eqs. (1.10) and (1.11): Given $\phi,\mathbf{A}$, we can choose a Λ such that

$$\nabla\cdot\mathbf{A}' = 0 \text{ (Coulomb gauge)}, \tag{1.14}$$
$$\text{or } \nabla\cdot\mathbf{A}'+\frac{1}{c^2}\frac{\partial\phi'}{\partial t} = 0 \text{ (Lorenz gauge)}. \tag{1.15}$$

For example, suppose $\nabla\cdot\mathbf{A}\neq 0$, then we demand

$$\nabla\cdot\mathbf{A}'=\nabla\cdot\mathbf{A}+\nabla^2\Lambda=0. \tag{1.16}$$

The gauge function Λ needs to satisfy

$$\nabla^2\Lambda=-\nabla\cdot\mathbf{A}. \tag{1.17}$$

The right hand side is supposedly known. Similar to the Poisson equation for charge, we know that under proper boundary condition ($\Lambda \to 0$ as $r \to \infty$), there is a formal solution,

$$\Lambda(\mathbf{r},t) = \frac{1}{4\pi}\int dv' \frac{\nabla'\cdot\mathbf{A}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.18}$$

So it is always possible to choose a new vector potential $\mathbf{A}'$ that satisfies the Coulomb gauge. This works similarly for the Lorenz gauge, which is not elaborated here.

2. Coulomb gauge

Under the Coulomb gauge, Eqs. (1.10) and (1.11) become

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}, \tag{1.19}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t}. \tag{1.20}$$

The first equation is the same as the Poisson equation for static charge, except that the $\rho(\mathbf{r},t)$ here can depend on time. It follows that,

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int dv' \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.21}$$

Even thought this looks the same as the equation from the QES approximation (see Ch 14), the formula here is actually exact, without any approximation. Under the Coulomb gauge, one can easily obtain the scalar potential, but *not* the vector potential (Jackson, 2002).

A remark: Since the time in the $\rho(\mathbf{r},t)$ of Eq. (1.21) is not retarded, it *appears* that the effect of the disturbance from the source is instantaneous. This is not true. For example, in Prob. 6.20 of Jackson, 1998, you are given a dipole source that flashes on and off at $t=0$,

$$\rho(\mathbf{r},t) = \delta(x)\delta(y)\delta'(z)\delta(t) \tag{1.22}$$

$$J_z(\mathbf{r},t) = -\delta(x)\delta(y)\delta(z)\delta'(t). \tag{1.23}$$

The problem is to find out the electric field it generates. The result shows that the field *is* actually retarded.

The right hand side of Eq. (1.20) can be simplified. But before doing that, we need to learn the **Helmholtz theorem**. It says that any vector field $\mathbf{V}(\mathbf{r})$ can be decomposed as two parts,

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2, \tag{1.24}$$

where $\nabla\times\mathbf{V}_1 = 0$, and $\nabla\cdot\mathbf{V}_2 = 0$. This is similar to the fact that, given a vector $\mathbf{v}$, it is always possible to decompose it into two orthogonal directions (Fig. 1). One is parallel to $\hat{\mathbf{k}}$, the other is perpendicular to $\hat{\mathbf{k}}$,

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp \tag{1.25}$$

$$= \hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{v}) - \hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{v}). \tag{1.26}$$

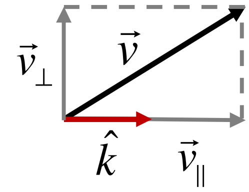

FIG. 1 Decompose a vector $\mathbf{v}$ into two components.

In momentum space, $\nabla\sim\mathbf{k}$, and $\nabla\times\mathbf{V}_1\sim\mathbf{k}\times\mathbf{V}_1=0$, and $\nabla\cdot\mathbf{V}_2\sim\mathbf{k}\cdot\mathbf{V}_2=0$. Thus we call $\mathbf{V}_1$ the *longitudinal part*, and $\mathbf{V}_2$ the *transverse part*. Since

$$\nabla\times(\nabla\times\mathbf{V}) = \nabla(\nabla\cdot\mathbf{V}) - \nabla^2\mathbf{V}, \tag{1.27}$$

the longitudinal part (written as $\mathbf{V}_\parallel$) satisfies,

$$\nabla^2\mathbf{V}_\parallel = \nabla(\nabla\cdot\mathbf{V}_\parallel). \tag{1.28}$$

From the Eq. of continuity and the Poisson equation, we have

$$\nabla\cdot\mathbf{J} = \nabla\cdot\mathbf{J}_\parallel = -\frac{\partial\rho}{\partial t} = \varepsilon_0\frac{\partial}{\partial t}\nabla^2\phi. \tag{1.29}$$

Take the gradient of both sides and use Eq. (1.28), it follows that,

$$\nabla^2\left(\mu_0\mathbf{J}_\parallel - \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi\right) = 0. \tag{1.30}$$

It is known that if $\nabla^2 F = 0$ everywhere, and F vanishes at infinity, then $F(\mathbf{r}) = 0$. A physical interpretation of this is that, if there is no charge everywhere, then the potential must be a constant, and equal to its value (zero) at infinity. Now, suppose that both terms in the equation above vanish at infinity, then

$$\mu_0\mathbf{J}_\parallel = \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi. \tag{1.31}$$

Therefore, on the right hand side of Eq. (1.20), only the transverse part is left,

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}_\perp. \tag{1.32}$$

That is, only the transverse part of the current, $\mathbf{J}_\perp$, can generate the vector potential, and hence the magnetic field.

3. Lorenz gauge

Using the Lorenz gauge in Eq. (1.15), Eqs. (1.10) and (1.11) become

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho_0}{\varepsilon_0}, \tag{1.33}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}. \tag{1.34}$$

This gauge condition is explicitly invariant under relativistic transformation. The potentials can be solved with the **method of Green function** (see Chap 20 of Zangwill, 2013).

First, we illustrate this method with a simple example: To solve the Poisson equation,

$$\nabla^2\phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon}, \tag{1.35}$$

we can first solve the potential g for a point charge at $\mathbf{r}'$,

$$\nabla^2 g_{\mathbf{r}'}(\mathbf{r}) = -\frac{\delta(\mathbf{r}-\mathbf{r}')}{\varepsilon}. \tag{1.36}$$

The solution is, of course,

$$g_{\mathbf{r}'}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.37}$$

The general charge distribution ρ can be considered as a superposition of point charges,

$$\rho(\mathbf{r}) = \int dv'\rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}'), \tag{1.38}$$

in which $\rho(\mathbf{r}')$ can be considered as the "weight" of point charges. By the *principle of superposition*, the potential ϕ would be a superposition of the potentials from these point charges,

$$\phi(\mathbf{r}) = \int dv'\rho(\mathbf{r}')g_{\mathbf{r}'}(\mathbf{r}) \tag{1.39}$$

$$= \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.40}$$

The function g for a point source is called the **Green function** in this approach.

Now, to solve the wave equation in Eq. (1.33), first solve the potential $g_{\mathbf{r}',t'}(\mathbf{r},t)$ for *a point source* in space and time,

$$\nabla^2 g - \frac{1}{c^2}\frac{\partial^2 g}{\partial t^2} = -\frac{\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')}{\varepsilon_0}. \tag{1.41}$$

For a general distribution of charges, the potential ϕ is a superposition of g's for point sources,

$$\phi(\mathbf{r},t) = \int dt'dv'\rho(\mathbf{r}',t')g_{\mathbf{r}',t'}(\mathbf{r},t). \tag{1.42}$$

We know that a point source generates a spherical shell of disturbance propagating at the speed of light with the form (see Sec. 20.3 of Zangwill, 2013),

$$g_{\mathbf{r}',t'}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.43}$$

Therefore,

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}, \tag{1.44}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int dv'\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}, \tag{1.45}$$

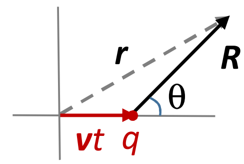

FIG. 2 The vector **R** (or **r**) points from the charge at present (or earlier) location to the observation point.

in which $t_r = t - |\mathbf{r}-\mathbf{r}'|/c$ is the **retarded time**. The electromagnetic fields can be calculated accordingly using $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$.

• Electromagnetic field of a moving charge

In Chap 14, we have calculated the electric and magnetic fields for a charge that moves *slowly*. Here we'll redo this without limiting the velocity. For a point charge moving with constant velocity $\mathbf{v} = v\hat{\mathbf{z}}$, its charge density and current density are,

$$\rho(\mathbf{r},t) = q\delta(x)\delta(y)\delta(z-vt), \tag{1.46}$$

$$\mathbf{J}(\mathbf{r},t) = q\mathbf{v}\delta(x)\delta(y)\delta(z-vt). \tag{1.47}$$

Usually we would prefer doing integration to get ϕ and $\mathbf{A}$ since the integral of Dirac delta function is trivial. This is not the case here, because the retarded time t_r inside the delta function in Eqs. (1.44) and (1.45) is a function of $\mathbf{r}'$ and would render the integration difficult.

Therefore, instead we solve this problem with the differential equations in Eqs. (1.33), (1.34). First, since the charge and current sources depend on the variable $z-vt$, one expects the potentials to depend on it as well,

$$\phi(x,y,z-vt) \quad\text{and}\quad \mathbf{A}(x,y,z-vt) = A(x,y,z-vt)\hat{\mathbf{z}}. \tag{1.48}$$

It can be seen from Eq. (1.45) that $\mathbf{A}$ has the z-component only. Define $\xi = z - vt$, then the differential equation for ϕ becomes,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \left(1-\frac{v^2}{c^2}\right)\frac{\partial^2\phi}{\partial\xi^2} = -\frac{q}{\varepsilon_0}\delta(x)\delta(y)\delta(\xi). \tag{1.49}$$

Define $1-(v/c)^2 = 1/\gamma^2$, and rescale the variable, $z' \equiv \gamma\xi = \gamma(z-vt)$, We then have

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z'^2} = -\frac{\gamma q}{\varepsilon_0}\delta(x)\delta(y)\delta(z'), \tag{1.50}$$

where $\delta(\xi) = \gamma\delta(\gamma\xi)$ has been used. The solution is similar to that of a point charge at rest,

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{\gamma q}{\sqrt{x^2+y^2+z'^2}} \tag{1.51}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\gamma q}{\sqrt{x^2+y^2+\gamma^2(z-vt)^2}}. \tag{1.52}$$

Similarly, for the vector potential, we have

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{\gamma q}{\sqrt{x^2+y^2+\gamma^2(z-vt)^2}}v\hat{\mathbf{z}}. \tag{1.53}$$

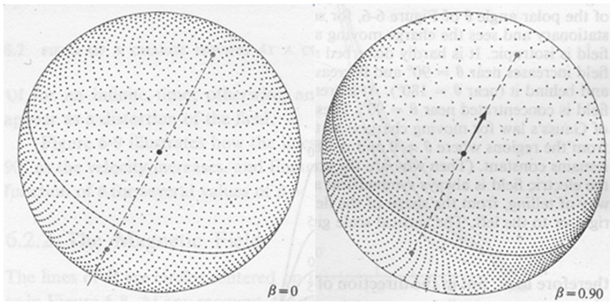

FIG. 3 Densities of electric field lines for a point charge at rest (a), moving with $v = 0.9c$ (b). Fig. from Lorrain and Corson, 1970

Use Eq. (1.7) to calculate the electric field and get,

$$\mathbf{E}(\mathbf{r},t) = \frac{\gamma q}{4\pi\varepsilon_0}\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (z-vt)\hat{\mathbf{z}}}{[x^2+y^2+\gamma^2(z-vt)^2]^{3/2}}. \tag{1.54}$$

The electric field is along the direction of $\mathbf{R}$ (Fig. 2), where

$$\mathbf{R} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (z-vt)\hat{\mathbf{z}}. \tag{1.55}$$

The angle θ between $\mathbf{R}$ and $\mathbf{v}$ satisfies

$$\sin^2\theta = \frac{x^2+y^2}{x^2+y^2+(z-vt)^2}. \tag{1.56}$$

Then the electric field can be written as,

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{R}}}{R^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}},\ \beta \equiv \frac{v}{c}. \tag{1.57}$$

When $\beta \neq 0$, the electric field is stronger in the transverse direction, and weaker at front and end (Fig. 3). It is as if the sphere of electric field has been squashed along the direction of motion. This result is first obtained by O. Heaviside. Such a **Heaviside ellipsoid** inspired the notion of **FitzGerald-Lorentz contraction**.

The magnetic field can also be calculated straightforwardly. It is left as an exercise to show that.

$$\mathbf{B} = \frac{\mathbf{v}}{c^2} \times \mathbf{E}, \tag{1.58}$$

similar to the relation for a slow charge (Fig. 4(a)).

Note that the denominator of Eq. (1.57) would blow up at $\beta = 1$, and become imaginary when $\beta > 1$. This indicates that it's impossible for the charge to move faster than light. After obtaining this result, Heaviside conjectured that when $v > c$, there would be electromagnetic shock wave, like the acoustic shock wave produced by a supersonic object in air (Fig. 4(b)). Even though this is not possible in vacuum, such a shock wave does exist in matter. When a charged particle moves faster than the speed of light in matter, it will generate a shock wave called the **Cerenkov radiation**, named after the person who first observed it (see Sec. 23.7 of Zangwill, 2013).

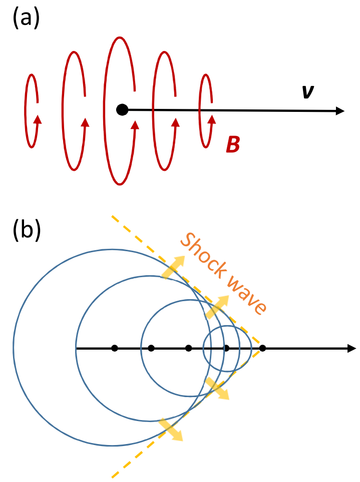

FIG. 4 (a) The magnetic field of a moving charge. (b) Cerenkov radiation as shock wave.

B. Equation of continuity for energy

We now calculate the mechanical work W_m done *by electromagnetic field on charged particles.* Suppose charge $\rho\Delta V$ is displaced by $\Delta\mathbf{r}$ due to fields, then the mechanical work is,

$$\Delta w_m = \rho\Delta V(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\Delta\mathbf{r}. \tag{1.59}$$

The rate of total work done is

$$\frac{dW_m}{dt} = \int dv\rho(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\mathbf{v} \tag{1.60}$$

$$= \int dv\mathbf{J}\cdot\mathbf{E},\quad \mathbf{J} = \rho\mathbf{v}. \tag{1.61}$$

Note that *the magnetic force does no work.*

We can use Ampere-Maxwell equation to relate current density with fields,

$$\nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{J} \tag{1.62}$$

It follows that,

$$\mathbf{J}\cdot\mathbf{E} = \frac{1}{\mu_0}\left(\mathbf{E}\cdot\nabla\times\mathbf{B} - \frac{1}{c^2}\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}\right). \tag{1.63}$$

Using a vector identity and Faraday's law, we have

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B}) \tag{1.64}$$

$$= -\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\nabla\times\mathbf{B}. \tag{1.65}$$

Therefore,

$$\mathbf{J}\cdot\mathbf{E} = -\frac{1}{\mu_0}\nabla\cdot(\mathbf{E}\times\mathbf{B}) - \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \varepsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}. \tag{1.66}$$

Define **electromagnetic energy density** as,

$$u_{EM} = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2. \tag{1.67}$$

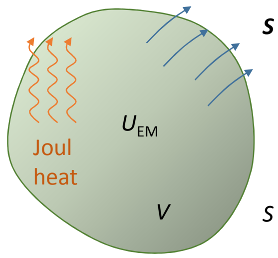

FIG. 5 The sum of the following three is conserved: 1. The EM field energy inside V, 2. The field energy flowing out of surface S, and 3. The mechanical work done by field on charged particles (Joul heat).

Also, define the **Poynting vector**,

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}. \tag{1.68}$$

Then Eq. (1.66) can be written as,

$$\frac{\partial u_{EM}}{\partial t} + \nabla\cdot\mathbf{S} = -\mathbf{J}\cdot\mathbf{E}. \tag{1.69}$$

This is the **Equation of continuity for energy**. The energy density u_{EM} of fields is similar to the charge density in the Eq. of continuity for charge. Therefore, $\mathbf{S}$ can be identified as **energy current density** for fields. You may check whether the dimension of $\mathbf{S}$ makes sense. First, it is not difficult to see that

$$[E] = [vB], \text{ and } [u_{EM}] = \left[\frac{B^2}{\mu_0}\right]. \tag{1.70}$$

Therefore,

$$[S] = \left[\frac{EB}{\mu_0}\right] = \left[\frac{vB^2}{\mu_0}\right] = [vu_{EM}]. \tag{1.71}$$

This is indeed the dimension of energy current density.

The term on the right hand side of Eq. (1.69) plays the role of a *source* or *sink*. If $\mathbf{J}\cdot\mathbf{E} > 0$, then $-\mathbf{J}\cdot\mathbf{E}$ is a sink for electromagnetic field energy. In this case, the field energy transfers to charged particles because the former does positive work W_m on the latter. In a conductor, mobile charges collide with ions and lose their kinetic energy, so that the mechanical energy turns into heat. This is the **Joul heat**.

The total electromagnetic field energy inside a volume V is,

$$U_{EM} = \int_V dv u_{EM}. \tag{1.72}$$

The flux of energy current flowing out of the surface S that bounds V is,

$$\int_V \nabla\cdot\mathbf{S} = \int_S d\mathbf{a}\cdot\mathbf{S}. \tag{1.73}$$

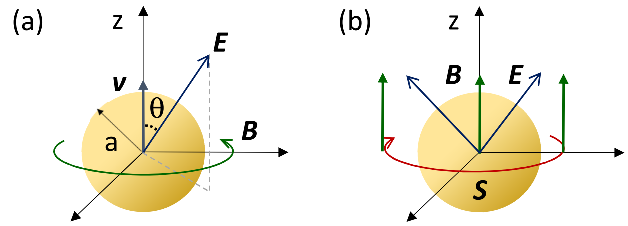

FIG. 6 (a) A charged sphere moving with velocity $\mathbf{v}$. (b) A charged sphere at rest in a uniform magnetic field.

Therefore, the integral form of Eq. (1.69) is (see Fig. 5),

$$\frac{d}{dt}U_{EM} + \int_S d\mathbf{a}\cdot\mathbf{S} + \int_V dv\mathbf{J}\cdot\mathbf{E} = 0. \tag{1.74}$$

The first term is the change of field energy (per unit time) inside V. The second is the flow of field energy (per unit time) through the boundary S of the region V. The third term is the loss (if $\mathbf{J}\cdot\mathbf{E} > 0$) of field energy to charged particles (per unit time). The net change of these three terms is zero, which is a manifestation of *the conservation of energy*.
Some remarks:

1. Note that $\mathbf{S}' = \mathbf{S} + \nabla\times\mathbf{C}$ gives the same energy flux as $\mathbf{S}$, since the flux integral is not changed by the curl. However, this addition could change the direction of $\mathbf{S}$. Chap 27, Vol II of Feynman *et al.*, 2010 has some discussion on such an ambiguity.
2. Based on the relation between mass and energy $E = mc^2$, u_{EM}/c^2 can be identified as the effective mass density associated with the field. Thus,

$$\mathbf{g} \equiv \frac{\mathbf{S}}{c^2} = \varepsilon_0\mathbf{E}\times\mathbf{B} \tag{1.75}$$

can be called the **electromagnetic momentum density**. The total EM momentum inside V is

$$\mathbf{P}_{EM} = \int_V dv\mathbf{g} = \varepsilon_0\int_V dv\mathbf{E}\times\mathbf{B}. \tag{1.76}$$

Note: A related quantity is the **radiation pressure**, which is $\mathbf{S}/c$.
3. Furthermore, $\boldsymbol{\ell}_{EM} \equiv \mathbf{r}\times\mathbf{g}$ can be identified as the **angular momentum density** of EM field (with respect to some reference point). The total angular momentum of the EM field inside V is

$$\mathbf{L}_{EM} = \varepsilon_0\int_V dv\mathbf{r}\times(\mathbf{E}\times\mathbf{B}). \tag{1.77}$$

This includes both orbital and (photon) spin angular momenta (Sec. 16.7.6. of Zangwill, 2013).

Example 1:

Find out the field energy and Poynting vector of a charged spherical shell moving with velocity $v \ll c$ (Fig. 6(a)). This shell with radius a is charged uniformly

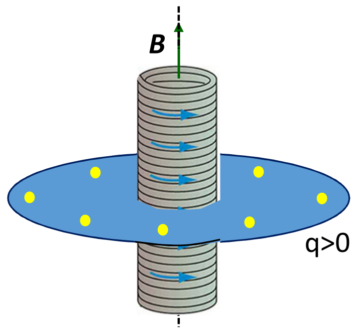

FIG. 7 A solenoid at the center of a disk with positive charges on its surface.

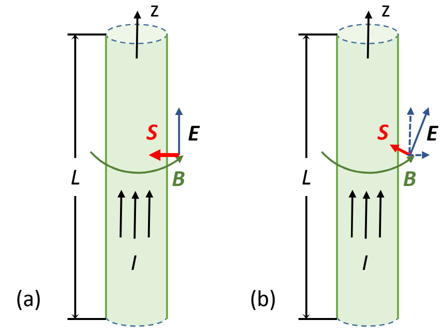

FIG. 8 (a) The EM field and Poynting vector surrounding a wire. (b) The electric field is tilted due to surface charge.

with charge q.
Solution:
A sphere at rest at the origin has the following field outside (no field inside),

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \tag{1.78}$$

Since $v \ll c$, we can rely on the quasi-electrostatic approximation (Chap 14). At the instant when the moving sphere passes through the origin, it has the same electric field as the one above. The electric field energy is

$$U_E = \frac{\varepsilon_0}{2}\int dv|\mathbf{E}|^2 = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2a}. \tag{1.79}$$

The magnetic field is

$$\mathbf{B} = \frac{\mathbf{v}}{c^2}\times\mathbf{E}, \quad \mathbf{v} = v\hat{\mathbf{z}}. \tag{1.80}$$

Sine $|\mathbf{v}\times\mathbf{E}| = vE\sin\theta$, we have

$$U_B = \frac{1}{2\mu_0}\int dv|\mathbf{B}|^2 = \frac{2}{3}\frac{U_E}{c^2}v^2. \tag{1.81}$$

Total EM field energy would be $U_E + U_B$.
The Poynting vector is

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} && (1.82)\\ &= \varepsilon_0\mathbf{E}\times(\mathbf{v}\times\mathbf{E}) && (1.83)\\ &= -\varepsilon_0 vE^2\sin\theta\hat{\boldsymbol{\theta}}. && (1.84) \end{aligned}$$

It flows from the south pole to the north pole. The EM momentum flowing upward is

$$\mathbf{P}_{EM} = \int dv\mathbf{g} = \frac{4}{3}\left(\frac{1}{c^2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2a}\right)\mathbf{v}. \tag{1.85}$$

Note that if you identify $U_E = m_0c^2$, $U_B = mv^2/2$, and $\mathbf{P}_{EM} = m\mathbf{v}$, then the effective masses, m_0, m, *do not* agree with each other. So it's advised not to interpret these EM quantities with traditional mechanical concept. See Chap 28, Vol II of Feynman *et al.*, 2010 for more discussions.

Note that as long as $\mathbf{E}$ and $\mathbf{B}$ coexist in a region, then there is EM momentum flowing around. For example, in Fig. 6(b), there is a charged sphere (not moving) in a uniform magnetic field $\mathbf{B}$. The electric field $\mathbf{E}$ is radial, and $\mathbf{B} = B\hat{\mathbf{z}}$. As a result, $\mathbf{g}$ is circulating the sphere endlessly. It's not difficult to see that the EM angular momentum $\mathbf{L}_{EM}$ with respect to the origin is also nonzero.

In Sec. 17-4, Vol II of Feynman *et al.*, 2010, Feynman proposed the following paradox: Consider a long solenoid at the center of a disk that can rotate (Fig. 7). There are positive charges fixed on the surface of the disk. At the beginning, nothing is moving, and a steady current generates a uniform magnetic field inside the solenoid. Then, we suddenly turn off the current. The changing magnetic flux through the solenoid induces a circular electric field outside the solenoid. As a result, the disk would start to rotate (counter-clockwise) since the charges are pushed by the electric force. This appears to violate the conservation of angular momentum, since no external torque has been applied to this device.

The resolution of this paradox relies on, of course, the fact that an EM field can carry angular momentum. Before the current is turned off, the Poynting vector $\mathbf{S}$ is circulating (counter-clockwise) inside the solenoid. Therefore, the angular momentum $\mathbf{L}_{EM}$ points up. When the current is turned off, such an angular momentum of EM field transfers to the charges (and the disk). Throughout this process, the total angular momentum of field and disk remains conserved.

1. Energy flow in a circuit

First, let's investigate the flow of electromagnetic energy around a straight wire. In Fig. 8(a), there is a straight wire with radius a and uniform current I. On the surface of the wire, the magnetic field is,

$$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{I}{a}\hat{\boldsymbol{\phi}}. \tag{1.86}$$

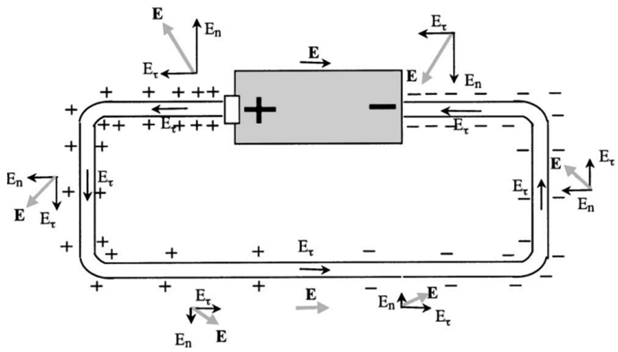

FIG. 9 The EM fields and surface charges surrounding a circuit. Fig. is from Galili and Goihbarg, 2005

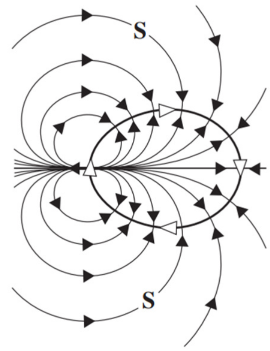

FIG. 10 An imaginary circuit that follows a dipole field line. together with the flow lines of $\mathbf{S}$. Fig. is from Zangwill, 2013.

Since $\mathbf{J} = \sigma \mathbf{E}$ and I is uniform, the electric field inside the wire is,

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{1}{\sigma} \frac{I}{\pi a^2} \hat{\mathbf{z}}. \tag{1.87}$$

This is also the field near the surface *outside* the wire, since the *tangential* component of $\mathbf{E}$ needs to be continuous across the boundary, $E_{1t} = E_{2t}$.

It follows that

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \frac{I}{\sigma \pi a^2} \frac{I}{2\pi a} (-\hat{\boldsymbol{\rho}}). \tag{1.88}$$

The Poynting vector points toward the wire. The total EM energy flowing into the wire within a length L is,

$$\left| \int_S d\mathbf{a} \cdot \mathbf{S} \right| = 2\pi a L S \tag{1.89}$$

$$= I^2 \rho \frac{L}{\pi a^2} = I^2 R, \tag{1.90}$$

where ρ and R are the resistivity and resistance of the wire, and $R = \rho \frac{L}{\pi a^2}$. As we have explained earlier, this energy would turn into Joul heat.

In this example, the energy flux flows inward. In reality, it also has to move *forward* along the wire in order to transport electric energy. That is, there is a part of $\mathbf{S} \parallel \hat{\mathbf{z}}$. So we need an electric field that deviates from the tangential direction (see Fig. 8(b)). Recall that the boundary condition for electric field is (see Chap 3)

$$\mathbf{E}_2 - \mathbf{E}_1 = \frac{\sigma_s}{\varepsilon_0} \hat{\mathbf{n}}_{1\to 2}, \tag{1.91}$$

where σ_s is surface charge density. Since there is no normal component inside the wire, $E_{1n} = 0$, thus on the outside,

$$E_{2n} = \frac{\sigma_s}{\varepsilon_0}. \tag{1.92}$$

The E_{2n} outside the wire can exist *only if there are surface charges.*

Fig. 9 is a schematic plot of the energy flow surrounding an electric circuit. For a report of an experimental observation of the surface charge, see Jefimenko, 1962.

An artificial setup in Fig. 10 offers a concrete example of the energy flow from a battery to its wire. The battery is simulated with an electric dipole. The wire lays along one of the field lines, so that the charges inside can be driven by the electric dipole field. Since the strength of E varies along the wire, the current cannot remain uniform if the conductivity is constant along the wire. To keep the current I fixed, σ needs to vary with location, such that

$$\mathbf{J} = \sigma(\mathbf{r}) \mathbf{E}(\mathbf{r}) = \text{const}. \tag{1.93}$$

Since the magnetic field $\mathbf{B}$ is transverse to the wire, and $\mathbf{E}$ is along the wire, the Poynting vector $\mathbf{S} \parallel \mathbf{E} \times \mathbf{B}$ is everywhere perpendicular to the wire. Their flow would coincide with the equipotential curves of the dipole field.

C. Electromagnetic momentum

Newton's third law, which states that action equals reaction, is no longer valid between two moving charges. Consider two point charges q_1, q_2, moving with velocities $\mathbf{v}_1, \mathbf{v}_2$. Charge-$i$ produces electromagnetic field $\mathbf{E}_i, \mathbf{B}_i$ ($i = 1, 2$). The forces $\mathbf{F}_{1,2}$ on charges $q_{1,2}$ are,

$$\mathbf{F}_1 = q_1(\mathbf{E}_2 + \mathbf{v}_1 \times \mathbf{B}_2), \tag{1.94}$$

$$\mathbf{F}_2 = q_2(\mathbf{E}_1 + \mathbf{v}_2 \times \mathbf{B}_1). \tag{1.95}$$

We know that

$$\mathbf{B}_1(\mathbf{r}_2) = \frac{\mathbf{v}_1}{c^2} \times \mathbf{E}_1(\mathbf{r}_2), \tag{1.96}$$

$$\mathbf{B}_2(\mathbf{r}_1) = \frac{\mathbf{v}_2}{c^2} \times \mathbf{E}_2(\mathbf{r}_1). \tag{1.97}$$

Therefore, in general the two magnetic forces, $q_1 \mathbf{v}_1 \times \mathbf{B}_2$ and $q_2 \mathbf{v}_2 \times \mathbf{B}_1$, are not opposite to each other, and $\mathbf{F}_1 \neq -\mathbf{F}_2$.

The problem, of course, is that the EM field also carries momentum, and we need to take this into account. Let's jump to a frame that stays with q_1, so that q_1 sits at rest at $\mathbf{r}_1$, and q_2 is located at $\mathbf{r}_2(t)$ moving with velocity $\mathbf{v}$. The electric fields are

$$\begin{aligned} \mathbf{E}_1 &= -\nabla\phi_1, && (1.98)\\ \mathbf{E}_2 &= -\nabla\phi_2 - \frac{\partial \mathbf{A}_2}{\partial t}. && (1.99)\end{aligned}$$

Even though charge q_2 produces a magnetic field, it does not have an effect on the charge q_1 at rest. The forces on q_1 and q_2 are

$$\begin{aligned} \mathbf{F}_1 &= q_1\mathbf{E}_2 = -q_1\nabla\phi_2(\mathbf{r}_1,t) - q_1\frac{\partial \mathbf{A}_2}{\partial t}(\mathbf{r}_1,t), && (1.100)\\ \mathbf{F}_2 &= q_2\mathbf{E}_1 = -q_2\nabla\phi_1(\mathbf{r}_2). && (1.101)\end{aligned}$$

Using the Coulomb gauge, we have

$$\begin{aligned} \phi_1(\mathbf{r}_2) &= \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|\mathbf{r}_2(t)-\mathbf{r}_1|}, && (1.102)\\ \phi_2(\mathbf{r}_1) &= \frac{1}{4\pi\varepsilon_0}\frac{q_2}{|\mathbf{r}_1-\mathbf{r}_2(t)|}. && (1.103)\end{aligned}$$

Therefore, $q_1\nabla_1\phi_2 = -q_2\nabla_2\phi_1$. The total force on these two particles is,

$$\mathbf{F}_1+\mathbf{F}_2\left(=\frac{d\mathbf{P}_{mech}}{dt}\right) = -q_1\frac{\partial \mathbf{A}_2}{\partial t} \neq 0. \qquad (1.104)$$

This is in the *absence* of any external force. To remedy this problem, define

$$\mathbf{P}_{EM} = q_1\mathbf{A}_2, \qquad (1.105)$$

so that

$$\frac{d}{dt}\mathbf{P}_{mech} + \frac{\partial}{\partial t}\mathbf{P}_{EM} = 0. \qquad (1.106)$$

$\mathbf{P}_{EM}$ is the momentum carried by the field.

Earlier we mentioned that $\mathbf{g} = \mathbf{S}/c^2$ is the momentum density of EM field. How would this relate to the definition of $\mathbf{P}_{EM}$ above? For this system with two charges, $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$, $\mathbf{B} = 0 + \mathbf{B}_2$. Therefore,

$$\begin{aligned} \mathbf{g} &= \varepsilon_0\mathbf{E}\times\mathbf{B} && (1.107)\\ &= \varepsilon_0\mathbf{E}_1\times\mathbf{B}_2 + \varepsilon_0\mathbf{E}_2\times\mathbf{B}_2. && (1.108)\end{aligned}$$

The second term is solely from the fields of q_2. It would diverge for a point charge. To avoid the divergence, one can start from a charged sphere with a finite radius a, calculate its field momentum (see Eq. (1.85)), then let $a \to 0$. In any case, this just leads to a constant (that diverges) times $\mathbf{v}$, so we will ignore this term and keep only the first term. It follows that

$$\mathbf{P}_{EM} = \varepsilon_0\int dv\mathbf{E}_1\times\mathbf{B}_2. \qquad (1.109)$$

Before demonstrating the equivalence of Eq. (1.105) and (1.109), let's prove the following equation:

$$\begin{aligned} &\int dv\,[\mathbf{E}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{E})]\\ &= \int dv\,(\mathbf{A}\nabla\cdot\mathbf{E} + \mathbf{E}\nabla\cdot\mathbf{A}). \qquad (1.110)\end{aligned}$$

pf: First, use

$$\begin{aligned} \nabla(\mathbf{a}\cdot\mathbf{b}) &= (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a}\\ &+ \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}). \qquad (1.111)\end{aligned}$$

Thus,

$$\begin{aligned} &\mathbf{E}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{E})\\ &= \nabla(\mathbf{E}\cdot\mathbf{A}) - (\mathbf{E}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{E}. \qquad (1.112)\end{aligned}$$

The first term on the RHS would vanish after integration over the whole space (assuming $\mathbf{E}\cdot\mathbf{A}\to 0$ at infinity), so let's focus on the second and the third terms:

$$\begin{aligned} E_j\partial_j A_i + A_j\partial_j E_i &= \partial_j(E_jA_i + E_iA_j)\\ &- A_i\partial_j E_j - E_i\partial_j A_j. \qquad (1.113)\end{aligned}$$

The first term on the RHS again would vanish after integration over space. Thus,

$$\begin{aligned} &\int dv\,[\mathbf{E}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{E})]\\ &= \int dv\,(\mathbf{A}\nabla\cdot\mathbf{E} + \mathbf{E}\nabla\cdot\mathbf{A}). \qquad (1.114)\end{aligned}$$

In the two-charge problem, $\mathbf{E}_1 = -\nabla\phi_1$, thus $\nabla\times\mathbf{E}_1 = 0$. The divergence $\nabla\cdot\mathbf{E}_1 = \rho_1/\varepsilon_0$. Also, under the Coulomb gauge, $\nabla\cdot\mathbf{A} = 0$. Putting all of these together, we have

$$\begin{aligned} \mathbf{P}_{EM} &= \varepsilon_0\int dv\mathbf{E}_1\times\mathbf{B}_2 && (1.115)\\ &= \varepsilon_0\int dv\mathbf{A}_2\nabla\cdot\mathbf{E}_1 && (1.116)\\ &= \int dv\rho_1\mathbf{A}_2. && (1.117)\end{aligned}$$

For a point charge, $\rho_1(\mathbf{r}) = q_1\delta(\mathbf{r}-\mathbf{r}_1)$, and

$$\mathbf{P}_{EM} = q_1\mathbf{A}_2(\mathbf{r}_1,t). \qquad (1.118)$$

Therefore, the two forms of EM momentum are indeed equivalent.

A remark: In classical mechanics, the Hamiltonian of a particle with charge q in an electromagnetic field is,

$$H = \frac{(\mathbf{p}-q\mathbf{A})^2}{2m} + \phi(\mathbf{r}). \qquad (1.119)$$

Momentum $\mathbf{p}$ is the **canonical momentum**, which is also the total momentum of this system. It is the sum of particle momentum and field momentum,

$$\mathbf{p} = m\dot{\mathbf{r}} + q\mathbf{A}. \qquad (1.120)$$

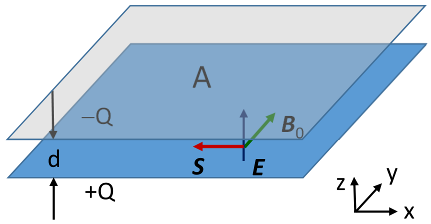

FIG. 11 (a) A parallel-plate capacitor in a uniform magnetic field $\mathbf{B}_0$.

The combination $\mathbf{p} - q\mathbf{A}$ in the Hamiltonian accounts only for the particle momentum. To *quantize* a system, we need to replace the *canonical momentum* with a differential operator,

$$\mathbf{p} \to \frac{\hbar}{i}\nabla. \tag{1.121}$$

An explanation about the replacement of $\mathbf{p}$, instead of $m\dot{\mathbf{r}}$, by the differential operator can be found in Vol III of Feynman *et al.*, 2010.

Example 2:

In Fig. 11, there is a parallel-plate capacitor with area A in a uniform magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{y}}$. Find out the electromagnetic momentum of this system.

Solution:

Let's ignore the fringe effect, so that the electric field is constant inside the capacitor, and drops to zero abruptly at the edge. The electric field inside is

$$\mathbf{E} = \frac{Q}{\varepsilon_0 A}\hat{\mathbf{z}}. \tag{1.122}$$

Therefore, the EM momentum is

$$\begin{aligned} \mathbf{P}_{EM} &= \varepsilon_0 \int_V dv \mathbf{E} \times \mathbf{B}_0 & (1.123)\\ &= \varepsilon_0 A d \mathbf{E} \times \mathbf{B}_0. & (1.124) \end{aligned}$$

An alternative, and more complicated approach is as follows: Treat the charged plates as a collection of electric dipoles. Each of the dipole moment is

$$\mathbf{p}_0 = -qd\hat{\mathbf{z}}. \tag{1.125}$$

The total dipole moment of the capacitor is

$$\mathbf{p}_T = -Qd\hat{\mathbf{z}} = -\varepsilon_0 A d\mathbf{E}. \tag{1.126}$$

The vector potential of a uniform magnetic field can be written as,

$$\mathbf{A} = \frac{1}{2}\mathbf{B}_0 \times \mathbf{r}. \tag{1.127}$$

Thus, the EM momentum for *one* dipole in $\mathbf{B}_0$ is

$$\begin{aligned} \mathbf{p}_{EM} &= \int dv \rho \mathbf{A} & (1.128)\\ &= \frac{1}{2}\mathbf{B}_0 \times \int dv \rho \mathbf{r} & (1.129)\\ &= \frac{1}{2}\mathbf{B}_0 \times \mathbf{p}_0. & (1.130) \end{aligned}$$

After summing over all of the dipoles, the total EM momentum is

$$\begin{aligned} \mathbf{P}_{EM} &= -\frac{1}{2}\mathbf{p}_T \times \mathbf{B}_0 & (1.131)\\ &= \frac{1}{2}\varepsilon_0 A d\mathbf{E} \times \mathbf{B}_0. & (1.132) \end{aligned}$$

This result differs from the previous one by a factor of two. The first result is actually wrong because of the negligence of fringe effect. The latter has the fringe effect included and is correct. Thus, the fringe field has a significant effect on field momentum. However, when one calculates the electrostatic field *energy* of the capacitor, the fringe field can be neglected, with negligible error. The reason is that U_E depends on E^2, instead of linearly on E, as $\mathbf{P}_{EM}$ does.

If you connect the two plates with a wire, the capacitor would discharge, and the EM momentum disappears eventually. This poses another paradox: conservation of momentum seems to be violated since there is no external force acting on the capacitor. For a resolution of this paradox, see Prob. 3.

1. Equation of continuity for momentum

So far we have derived the equations of continuity for charge and energy. They are the result of *local* conservation of charge and energy respectively. Since momentum is also conserved, there is the equation of continuity for momentum, which is derived below. First, charge and current in electromagnetic field subject to the mechanical force,

$$\mathbf{F}_{mech} = \int dv(\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}). \tag{1.133}$$

The integrand, $\mathbf{f}_{mech} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$, is called **force density**. We relate the sources to fields via the Maxwell equations,

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0}, & (1.134)\\ \nabla \times \mathbf{B} &= \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. & (1.135) \end{aligned}$$

With the help of the following identity and Faraday's law,

$$\begin{aligned} \nabla(\mathbf{a}\cdot\mathbf{b}) &= (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a}\\ &+ (\mathbf{a}\times\nabla)\times\mathbf{b} + (\mathbf{b}\times\nabla)\times\mathbf{a}, & (1.136) \end{aligned}$$

it's not difficult to show that

$$\begin{aligned} \mathbf{f}_{mech} &= \rho\mathbf{E} + \mathbf{J}\times\mathbf{B} & (1.137)\\ &= \varepsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} - \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}\\ &= -\frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t} + \varepsilon_0\left[(\nabla\cdot\mathbf{E})\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{E} - \frac{1}{2}\nabla(\mathbf{E}\cdot\mathbf{E})\right]\\ &+ \frac{1}{\mu_0}\left[(\nabla\cdot\mathbf{B})\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{1}{2}\nabla(\mathbf{B}\cdot\mathbf{B})\right]. & (1.138) \end{aligned}$$

The sum of the two square brackets can be written as the divergence of a tensor T, called **Maxwell stress tensor**,

$$\mathsf{T} \equiv \varepsilon_0 \left[\mathbf{EE} + c^2 \mathbf{BB} - \frac{1}{2}(E^2 + c^2 B^2) \right]. \quad (1.139)$$

We have used the *dyadic notation*: putting two vectors side by side turns it into a matrix (or a 2nd-rank tensor), with the matrix elements,

$$(\mathbf{EE})_{ij} \equiv E_i E_j. \quad (1.140)$$

That is,

$$T_{ij} = \varepsilon_0 \left[E_i E_j + c^2 B_i B_j - \frac{\delta_{ij}}{2}(E^2 + c^2 B^2) \right], \quad (1.141)$$

which is symmetric, $T_{ji} = T_{ij}$. Also,

$$\begin{aligned} (\nabla \cdot \mathsf{T})_j &= \partial_i T_{ij} && (1.142)\\ &= \varepsilon_0 \partial_i (E_i E_j) + \cdots && (1.143)\\ &= \varepsilon_0 (\partial_i E_i) E_j + \varepsilon_0 E_i \partial_i E_j + \cdots, && (1.144) \end{aligned}$$

which is equal to the last two terms of Eq. (1.138).

Therefore, Eq. (1.138) can be written as

$$\mathbf{f}_{mech} \left(= \frac{d\mathbf{p}_{mech}}{dt} \right) = -\frac{1}{c^2}\frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot \mathsf{T}, \quad (1.145)$$

or, since $\mathbf{g} = \mathbf{S}/c^2$, one has

$$\frac{d\mathbf{p}_{mech}}{dt} + \frac{\partial \mathbf{g}}{\partial t} = \nabla \cdot \mathsf{T}. \quad (1.146)$$

This is the **Eq. of continuity for momentum**. We can check the dimension of T to have some idea about this quantity. From the equation above, one has

$$\frac{[T]}{[x]} = \frac{1}{[t]}\frac{[P]}{[V]}, \quad (1.147)$$

therefore

$$[T] = [v]\frac{[P]}{[V]}. \quad (1.148)$$

This has the dimension of (velocity)×(momentum density). Since (velocity)×(energy density) is called energy current density, T can be called *momentum current density*.

Integration of momentum density over a volume V gives the momentum inside that volume,

$$\begin{aligned} \mathbf{P}_{mech} &= \int_V dv\ \mathbf{p}_{mech}, && (1.149)\\ \mathbf{P}_{EM} &= \int_V dv\ \mathbf{g}. && (1.150) \end{aligned}$$

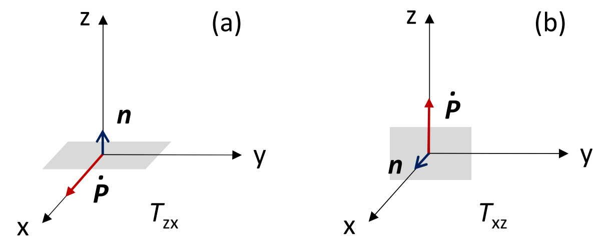

FIG. 12 (a) A force along $\hat{\mathbf{x}}$ acting on an area element $da\hat{\mathbf{z}}$. (b) A force along $\hat{\mathbf{z}}$ acting on an area element $da\hat{\mathbf{x}}$.

Thus, the integral form of Eq. (1.146) is,

$$\begin{aligned} \frac{d\mathbf{P}_{mech}}{dt} + \frac{d\mathbf{P}_{EM}}{dt} &= \int_V dv\ \nabla \cdot \mathsf{T} && (1.151)\\ &= \int_S d\mathbf{a} \cdot \mathsf{T}. && (1.152) \end{aligned}$$

For an area element, we have

$$\Delta \dot{P}_j \simeq \Delta a_i T_{ij}. \quad (1.153)$$

Hence,

$$T_{ij} \simeq \frac{\Delta \dot{P}_j}{\Delta a_i} = \frac{\text{force along } j}{\text{area with normal along } i}, \quad (1.154)$$

which is stress (or pressure) indeed. It is a shear stress when $i \neq j$. The shear stresses T_{zx} and T_{xz} are shown in Fig. 12. Note that $T_{xz} = T_{zx}$ since T_{ij} is symmetric.

Consider two special circumstances: 1. If there is only electric field, then the mechanical force density

$$\mathbf{f}_{mech} = \rho \mathbf{E} = \nabla \cdot \mathsf{T}, \quad (1.155)$$

where

$$\mathsf{T} = \varepsilon_0 \left(\mathbf{EE} - \frac{1}{2}E^2 \right). \quad (1.156)$$

The EM momentum density $\mathbf{g} = 0$. If there is only $\mathbf{B}$ field, then

$$\mathbf{f}_{mech} = \mathbf{J} \times \mathbf{B} = \nabla \cdot \mathsf{T}, \quad (1.157)$$

where

$$\mathsf{T} = \frac{1}{\mu_0} \left(\mathbf{BB} - \frac{1}{2}B^2 \right). \quad (1.158)$$

Note that in this two cases, the LHS's are quite different, but the RHS's look similar.

Given a device surrounded by an electric or a magnetic field, it could be difficult to calculate the force on the LHS, if the distribution of ρ or $\mathbf{J}$ couldn't be found easily. In this case, it might be easier to calculate the RHS if we can find out the distribution of $\mathbf{E}(\mathbf{r})$ or $\mathbf{B}(\mathbf{r})$ (Kirtly Jr., 2005).

Example 3:

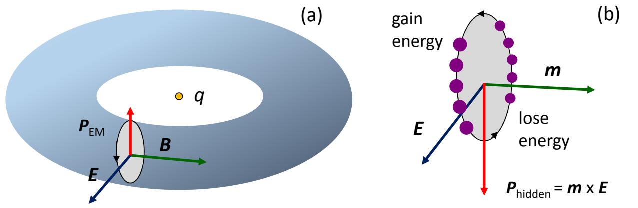

FIG. 13 (a) A point charge q at the center of a toroidal solenoid. (b) The charges moving in one coil of the solenoid.

Consider a charged plate on the $x-y$ plane. Suppose there is a uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$ above but no field below. Find out the electric force (per unit area) on the surface.
Solution:
The stress tensor above the plate is

$$T_{ij} = \varepsilon_0 \left(E_i E_j - \frac{\delta_{ij}}{2} E^2 \right) \quad (1.159)$$
$$= \frac{\varepsilon_0}{2} \begin{pmatrix} -E^2 & 0 & 0 \\ 0 & -E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix} \quad (1.160)$$

The force on a surface with area A is ($d\mathbf{a} = da\hat{\mathbf{z}}$)

$$\int_S d\mathbf{a} \cdot \mathsf{T} = \int_S da T_{zz} = \frac{\varepsilon_0}{2} E^2 A. \quad (1.161)$$

Thus the force per unit area is $\varepsilon_0 E^2/2$ (points up). We can reach the same result using $\mathbf{f}_{mech} = \sigma_s \mathbf{E}_s$, where σ_s is surface charge density. The tricky part is that $\mathbf{E}_s$ is the field from other part of the charged conductor that excludes the point of interest, rather than $\mathbf{E}$ itself. In this example, $\mathbf{E}_s = \mathbf{E}/2$ (see Sec. 3.4.3 of Zangwill, 2013).

2. Hidden momentum

Let's conclude this chapter with yet another paradox, first raised by Shockley and James (Shockley and James, 1967). Here is a simpler version proposed by Vaidman, 1990. In Fig. 13(a), there is a point charge $q (> 0)$ at the center of a toroidal solenoid. The charge has a radial electric field penetrating through the solenoid. The magnetic field inside the solenoid is circular, and there is no magnetic field outside. Now, if you calculate $\mathbf{S}$ or $\mathbf{g}$, the vector points up no matter where you are inside the solenoid. Therefore, total field momentum $\mathbf{P}_{EM}$ points up. The total momentum of this system seems to be nonzero, again in the absence of any external force. This does not make sense, since we do not expect the solenoid to lift up by itself without any external force.

The resolution of this paradox is rather subtle. One has to consider the electron momentum inside the wire. In Fig. 13(b), we see that when an electron moves near the top of the coil, it is accelerated by the electric field; when it moves down to the bottom, it is de-accelerated. When it gains (loses) velocity, it also gains (loses) some mass due to relativistic effect, $m(v) = \gamma(v) m_0$. Therefore, it carries a larger momentum when it is flowing down, and a smaller momentum up. The net effect is that there is a mechanical momentum (from electrons) flowing *downward*,

$$\mathbf{p}_{hidden} = \mathbf{m} \times \mathbf{E}, \quad (1.162)$$

where $\mathbf{m}$ is the magnetic moment of the coil. This is called the **hidden momentum**. It turns out that this cancels with $\mathbf{P}_{EM}$, and conservation of total momentum is again rescued. For more details, see Sec. 15.7 of Zangwill, 2013.

Problems:
1. (a) Starting from Eq. (1.54), verify that the electric field can be written as,

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\hat{\mathbf{R}}}{R^2} \frac{1-\beta^2}{(1-\beta^2 \sin^2\theta)^{3/2}}, \ \beta \equiv \frac{v}{c}. \quad (1.163)$$

(b) Take the curl of the vector potential in Eq. (1.53), show that

$$\mathbf{B} = \frac{\mathbf{v}}{c^2} \times \mathbf{E}, \quad (1.164)$$

where $\mathbf{E}$ is given in Eq. (1.54).
2. In Example 1, given the fields in Eqs. (1.78) and (1.80), show that

$$\mathbf{S} = -\varepsilon_0 v E^2 \sin\theta \hat{\boldsymbol{\theta}}. \quad (1.165)$$

and

$$\mathbf{P}_{EM} = \frac{\mu_0}{4\pi} \frac{2q^2}{3a} \mathbf{v}. \quad (1.166)$$

3. In Example 2, the EM momentum of the parallel-plate capacitor vanishes after the capacitor is discharged. During the discharge, current is flowing along the wire. Therefore, the wire in a magnetic field experiences a Lorentz force. Show that the total impulse delivered to the wire by this Lorentz force is equal to the EM momentum in Eq. (1.124) before discharge (see Chap 8 of Griffiths, 2017). Thus, the momentum of this system remains conserved.
Note: As we have explained in the text, the result in Eq. (1.124) is wrong by a factor of 2. Therefore, the analysis here is not entirely satisfactory. One can see Babson *et al.*, 2009 for more discussions.

REFERENCES

Babson, David, Stephen P. Reynolds, Robin Bjorkquist, and David J. Griffiths (2009), "Hidden momentum, field momentum, and electromagnetic impulse," American Journal of Physics **77** (9), 826–833.

Feynman,, Leighton, and Sands (2010), *The Feynman Lectures on Physics*, new millennium ed. (California Institute of Technology) www.feynmanlectures.caltech.edu/info/.

Galili, Igal, and Elisabetta Goihbarg (2005), "Energy transfer in electrical circuits: A qualitative account," American Journal of Physics **73**, 141.

Griffiths, David J (2017), *Introduction to Electrodynamics*, 4th ed. (Cambridge University Press).

Jackson, J D (1998), *Classical Electrodynamics*, 3rd ed. (Wiley).

Jackson, J D (2002), "From lorenz to coulomb and other explicit gauge transformations," American Journal of Physics **70** (9), 917–928.

Jefimenko, Oleg (1962), "Demonstration of the electric fields of current-carrying conductors," American Journal of Physics **30**, 19.

Kirtly Jr., James L (2005), "Chap 1 of the course notes on electric machines," Web.mit.edu/course/6/6.685/www/notes.htm.

Lorrain, Paul, and Dale Corson (1970), *Electromagnetic fields and waves* (W. H. Freeman).

Shockley, W, and R. P. James (1967), ""try simplest cases" discovery of "hidden momentum" forces on "magnetic currents"," Phys. Rev. Lett. **18**, 876–879.

Vaidman, Lev (1990), "Torque and force on a magnetic dipole," American Journal of Physics **58**, 978.

Zangwill, Andrew (2013), *Modern electrodynamics* (Cambridge Univ. Press, Cambridge).