Lecture notes on classical electrodynamics

Ming-Che Chang

Department of Physics,
National Taiwan Normal University, Taipei,
Taiwan

(Dated: March 13, 2023)

CONTENTS

I. DYNAMICS AND QUASISTATIC FIELDS

A. Maxwell equations in vacuum

From what we have learned so far, together with Faraday's law, we have

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{1.1}$$
$$\nabla\cdot\mathbf{B} = 0, \tag{1.2}$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{1.3}$$
$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}, \tag{1.4}$$

Note that two of the equations above have source terms (charge, current) on the right-hand side, while the other two do not. Because of charge conservation, the sources have to satisfy the equation of continuity,

$$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0. \tag{1.5}$$

Eqs. (1.1) and (1.4) give

$$\rho = \varepsilon_0\nabla\cdot\mathbf{E}, \tag{1.6}$$
$$\mathbf{J} = \frac{1}{\mu_0}\nabla\times\mathbf{B}. \tag{1.7}$$

Substitute them to Eq.(1.5), one gets

$$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0+\varepsilon_0\frac{\partial}{\partial t}\nabla\cdot\mathbf{E}\neq 0. \tag{1.8}$$

That is, the set of equations above is *not* consistent with charge conservation.

To fix this problem, one can add a term to Ampere's law,

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\Box. \tag{1.9}$$

Repeat the calculation above to get

$$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=-\frac{1}{\mu_0}\nabla\cdot\Box+\varepsilon_0\frac{\partial}{\partial t}\nabla\cdot\mathbf{E}. \tag{1.10}$$

For this to be zero, we need

$$\Box=\varepsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}+\nabla\times\mathbf{K}, \tag{1.11}$$

where $\mathbf{K}$ is an arbitrary function. We will drop the second term, and be aware of this possible addition.

After the revision, Eq. (1.4) becomes **Ampere-Maxwell's law**,

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t},\quad \frac{1}{c^2}=\varepsilon_0\mu_0 \tag{1.12}$$
$$= \mu_0(\mathbf{J}+\mathbf{J}_D),\quad \mathbf{J}_D\equiv\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{1.13}$$

The effective current density $\mathbf{J}_D$ produced by a changing electric field is called the **displacement current**, which is first introduced by Maxwell. Without this term, the Maxwell equations would fail to encompass electromagnetic waves.

The *integral form* of the Maxwell equations are

$$\int_S d\mathbf{a}\cdot\mathbf{E} = \frac{Q}{\varepsilon_0}, \tag{1.14}$$
$$\int_S d\mathbf{a}\cdot\mathbf{B} = 0, \tag{1.15}$$
$$\oint_C d\mathbf{r}\cdot\mathbf{E} = -\int_S d\mathbf{a}\cdot\frac{\partial\mathbf{B}}{\partial t}, \tag{1.16}$$
$$\oint_C d\mathbf{r}\cdot\mathbf{B} = \mu_0 I+\frac{1}{c^2}\int_S d\mathbf{a}\cdot\frac{\partial\mathbf{E}}{\partial t}, \tag{1.17}$$

in which C is the boundary of surface S. These are called in turn,

1. **Gauss's law**: The electric flux through a closed surface S is proportional to the electric charges Q inside the surface.

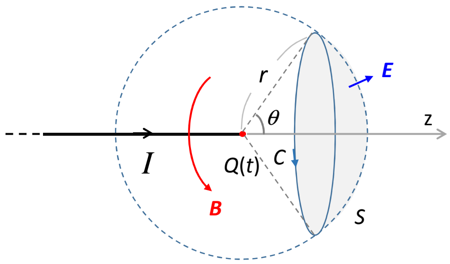

FIG. 1 A steady current flowing along a wire with an open end.

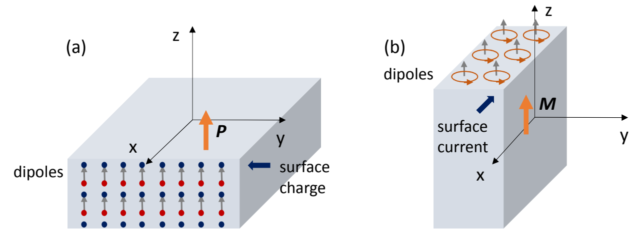

FIG. 2 (a) A semi-infinite dielectric below the $x-y$ plane. (b) A semi-infinite magnet to the left of the $x-z$ plane.

2. **"Magnetic" Gauss's law**: The magnetic flux over any closed surface S is zero. That is, there can be no magnetic monopole. *If* there are magnetic monopoles, then one needs to put magnetic charge density on the right hand side, just like the electric case.
3. **Faraday's law**: The electric circulation around a closed loop C (which is the boundary of a surface S) is related to the rate of change of the magnetic flux through surface S.
4. **Ampere-Maxwell's law**: The magnetic circulation around a closed loop C (which is the boundary of a surface S) is related to the current I passing through C and the rate of change of the electric flux through surface S.

Example: The semi-infinite wire in Fig. 1 has an open end at the origin. Suppose one can maintain a steady current I in the wire, so that charges $Q(t) = It$ would build up steadily with time at the end point. Find out the electric field and magnetic field generated by this wire.
Solution
The electric field is simply

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{Q(t)}{r^2}\hat{\mathbf{r}}. \tag{1.18}$$

Adopt spherical coordinate, and choose its z-axis to be along the wire, then the magnetic field is static and has the form (see p.311 of Zangwill, 2013),

$$\mathbf{B}(\mathbf{r}) = B_\phi(r,\theta)\hat{\boldsymbol{\phi}}. \tag{1.19}$$

Instead of Ampere's law, we need to use Ampere-Maxwell's law to find the magnetic field,

$$\oint_C d\mathbf{r}\cdot\mathbf{B} = \mu_0\int_S d\mathbf{a}\cdot\mathbf{J} + \frac{1}{c^2}\int_S d\mathbf{a}\cdot\frac{\partial\mathbf{E}}{\partial t}. \tag{1.20}$$

Suppose one wants to find the magnetic field at $\mathbf{r}$, then draw a circle C around the z-axis passing through $\mathbf{r}$, as shown in Fig. 1. Choose the surface S with boundary C to be a spherical cap from a sphere centered at the origin. There is no electric current passing through S. However, the displacement current is nonzero,

$$\frac{1}{c^2}\int_S d\mathbf{a}\cdot\frac{\partial\mathbf{E}}{\partial t} = \frac{I}{4\pi\varepsilon_0 c^2}\int_{\text{cap}}\frac{\hat{\mathbf{r}}\cdot d\mathbf{a}}{r^2}. \tag{1.21}$$

The integral is just the solid angle of the spherical cap with respect to the origin, which is $2\pi(1-\cos\theta)$.

Thus, Eq. (1.20) leads to

$$B_\phi 2\pi r\sin\theta = 0 + \frac{\mu_0}{2}I(1-\cos\theta). \tag{1.22}$$

It follows that

$$\mathbf{B}(r,\theta) = \frac{\mu_0 I}{4\pi}\frac{1-\cos\theta}{r\sin\theta}\hat{\boldsymbol{\phi}}. \tag{1.23}$$

It diverges at $\theta=\pi$, when $\mathbf{r}$ approaches the wire.

You may check that, if one chooses S to be the complementary surface to the cap on the sphere, then Eq. (1.20) gives

$$B_\phi 2\pi r\sin\theta = \mu_0 I - \frac{\mu_0}{2}I(1+\cos\theta). \tag{1.24}$$

This would lead to the same result in Eq. (1.23). Note that if there is no displacement current, then this two approaches would give contradictory results.

B. Polarization and magnetization

The electromagnetic properties of *matter* are characterized by **polarization P** and **magnetization M**,

$$\begin{aligned}\mathbf{P}(\mathbf{r}) &= \text{electric dipole moments per unit volume,}\\ \mathbf{M}(\mathbf{r}) &= \text{magnetic dipole moments per unit volume.}\end{aligned}$$

The "volume" here is a small one surrounding the point $\mathbf{r}$. Therefore, rigorously speaking, the coordinate $\mathbf{r}$ in $\mathbf{P}$ and $\mathbf{M}$ should be understood as **coarse-grained** points, with the size of a molecule or above.

Non-uniform polarization or magnetization would generate effective charge and current,

$$\rho_P = -\nabla\cdot\mathbf{P}, \tag{1.25}$$
$$\mathbf{J}_M = \nabla\times\mathbf{M}. \tag{1.26}$$

These are called **polarization charge** (Ch 6) and **magnetization current** (Ch 13). See the latex notes for Chaps 4 and 11 for detailed explanations. Here we just use two simple examples to illustrate these:

First, in Fig. 2(a) there is a semi-infinite dielectric with uniform polarization,

$$\mathbf{P} = P_0\theta(-z)\hat{z}, \tag{1.27}$$

in which θ is the step function. Its polarization charge density is,

$$\rho_P = -\nabla\cdot\mathbf{P} = P_0\delta(z)\hat{z}. \tag{1.28}$$

We can see from the figure that the bulk is charge-neutral, and only the outer-most electrons can be exposed. So its reasonable for the polarization charges to reside on the surface of the dielectric.

Second, in Fig. 2(b) there is a semi-infinite magnet with uniform magnetization,

$$\mathbf{M} = M_0\theta(-y)\hat{z}. \tag{1.29}$$

Its magnetization current density is,

$$\mathbf{J}_M = \nabla\times\mathbf{M} = -M_0\delta(y)\hat{x}. \tag{1.30}$$

That is, magnetization current flows only on the surface of the magnet. In Fig. 2(b), we see that molecular currents generate magnetic dipoles. Near the interface between neighboring current loops, the currents flow along opposite directions and cancel with each other. Thus, there is no current inside the bulk, and only the outer-most current exposed.

Both the polarization charge and the magnetization current in the examples are bounded to molecules. They cannot move away like free electrons in metals.

1. Polarization current

Changing polarization would produce **polarization current $\mathbf{J}_P$**. These charges need to be conserved locally, and we have the following Eq. of continuity,

$$\nabla\cdot\mathbf{J}_P + \frac{\partial\rho_P}{\partial t} = 0. \tag{1.31}$$

Plug in $\rho_P = -\nabla\cdot\mathbf{P}$, we get

$$\nabla\cdot\left(\mathbf{J}_P - \frac{\partial\mathbf{P}}{\partial t}\right) = 0. \tag{1.32}$$

Therefore,

$$\mathbf{J}_P = \frac{\partial\mathbf{P}}{\partial t} + \nabla\times\mathbf{\Lambda}. \tag{1.33}$$

In the example earlier, if $P_0(t)$ in Eq. (1.27) increases linearly with time, then the polarization charges would keep building up near the boundary, as if there is a current $\partial\mathbf{P}/\partial t$ flowing up toward the boundary.

In addition, the first part in Eq. (1.33) produces a magnetic moment,

$$\mathbf{m}_P = \frac{1}{2}\int dv\mathbf{r}\times\frac{\partial\mathbf{P}}{\partial t}, \tag{1.34}$$

which gives a magnetization $\mathbf{M}_P$, whose curl contributes to the polarization current,

$$\mathbf{J}_P = \frac{\partial\mathbf{P}}{\partial t} + \nabla\times\mathbf{M}_P. \tag{1.35}$$

We will show that $\mathbf{M}_P = \mathbf{P}\times\mathbf{v}$ for a rigid body of *moving* dielectric.

Before moving on, a short note on **total derivative**: Suppose a function $F(\mathbf{r}(t), t)$ depends on certain flow $\mathbf{r}(t)$ in space. Its total derivative is,

$$\frac{dF}{dt} \equiv \lim_{dt\to 0}\frac{F(\mathbf{r}+d\mathbf{r}, t+dt) - F(\mathbf{r}, t)}{dt} \tag{1.36}$$

$$= \frac{\partial F}{\partial t} + \frac{d\mathbf{r}}{dt}\cdot\frac{\partial F}{\partial\mathbf{r}}. \tag{1.37}$$

For a rigid distribution $F(\mathbf{r}, t)$ that moves with the flow, $F(\mathbf{r}+d\mathbf{r}, t+dt) = F(\mathbf{r}, t)$, and $dF/dt = 0$. The same is true if F is replaced with a vector function $\mathbf{F}$.

Back to physics: Since the distribution of polarization is rigid within the dielectric, we have

$$\frac{d\mathbf{P}}{dt} = \frac{\partial\mathbf{P}}{\partial t} + \mathbf{v}\cdot\frac{\partial\mathbf{P}}{\partial\mathbf{r}} = 0. \tag{1.38}$$

Furthermore,

$$\mathbf{J}_P = \rho_P\mathbf{v} = -\mathbf{v}\nabla\cdot\mathbf{P}. \tag{1.39}$$

Therefore, from, Eqs. (1.35) and (1.38), we have

$$\nabla\times\mathbf{M}_P = -\mathbf{v}\nabla\cdot\mathbf{P} + (\mathbf{v}\cdot\nabla)\mathbf{P} \tag{1.40}$$

$$= \nabla\times(\mathbf{P}\times\mathbf{v}). \tag{1.41}$$

That is, $\mathbf{M}_P = \mathbf{P}\times\mathbf{v}$, and

$$\mathbf{J}_P = \frac{\partial\mathbf{P}}{\partial t} + \nabla\times(\mathbf{P}\times\mathbf{v}). \tag{1.42}$$

According to the special theory of relativity, a moving magnetic dipole generates an electric dipole moment; while a moving electric dipole generates a magnetic dipole moment. As a result, one has

$$\mathbf{P}_{eff} = \mathbf{v}\times\mathbf{M}, \tag{1.43}$$

$$\mathbf{M}_{eff} = \mathbf{P}\times\mathbf{v}. \tag{1.44}$$

Therefore, the magnetization $\mathbf{M}_P$ obtained in the analysis above is consistent with the $\mathbf{M}_{eff}$ in the theory of special relativity.

C. Maxwell equations in matter

For the Maxwell equations in matter, we need to include the effective charge and current considered above.

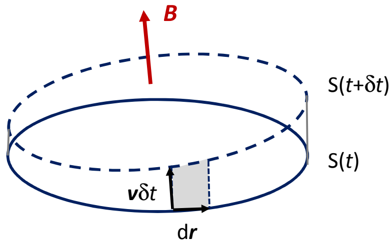

FIG. 3 A surface moves from $S(t)$ to $S(t+\delta t)$ in a magnetic field.

The first and the last Maxwell equations become,

$$\begin{aligned}\nabla\cdot\mathbf{E} &= \frac{\rho}{\varepsilon_0}+\frac{\rho_P}{\varepsilon_0} &(1.45)\\ &= \frac{\rho}{\varepsilon_0}-\frac{\nabla\cdot\mathbf{P}}{\varepsilon_0}, &(1.46)\\ \nabla\times\mathbf{B} &= \mu_0\left(\mathbf{J}+\mathbf{J}_D+\mathbf{J}_M+\mathbf{J}_P\right) &(1.47)\\ &= \mu_0\left(\mathbf{J}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}+\nabla\times\mathbf{M}+\frac{\partial\mathbf{P}}{\partial t}\right). &(1.48)\end{aligned}$$

Introducing

$$\begin{aligned}\mathbf{D} &= \varepsilon_0\mathbf{E}+\mathbf{P}, &(1.49)\\ \mathbf{H} &= \frac{\mathbf{B}}{\mu_0}-\mathbf{M}, &(1.50)\end{aligned}$$

then we have

$$\begin{aligned}\nabla\cdot\mathbf{D} &= \rho_f, &(1.51)\\ \nabla\cdot\mathbf{B} &= 0, &(1.52)\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}, &(1.53)\\ \nabla\times\mathbf{H} &= \mathbf{J}_f+\frac{\partial\mathbf{D}}{\partial t}. &(1.54)\end{aligned}$$

We have added the subscript f to ρ and $\mathbf{J}$ to emphasize that these are *free* (instead of *bounded*) charge and current.

D. Electromagnetic induction

First, let's study an identity crucial to the formulation of Faraday's law. In Eq. (1.16), we see a time derivative within the integrand. If the surface S of integration is static, then

$$\int_S d\mathbf{a}\cdot\frac{\partial\mathbf{B}}{\partial t}=\frac{d}{dt}\int_S d\mathbf{a}\cdot\mathbf{B}=\frac{d\Phi_B}{dt}, \quad (1.55)$$

which is just the change of the magnetic flux Φ_B through S. However, if the surface S is changing with time, then we need the **flux theorem**,

$$\frac{d}{dt}\int_{S(t)} d\mathbf{a}\cdot\mathbf{B}(t)=\int_{S(t)} d\mathbf{a}\cdot\frac{\partial\mathbf{B}}{\partial t}-\oint_{C(t)} d\mathbf{r}\cdot\mathbf{v}\times\mathbf{B}, \quad (1.56)$$

where $C(t)$ is the boundary of $S(t)$.
Pf: When both $\mathbf{B}$ and S are changing with time, the change of magnetic flux

$$\begin{aligned}\delta\left(\int d\mathbf{a}\cdot\mathbf{B}\right) &= \int_{S(t+\delta t)} d\mathbf{a}\cdot\mathbf{B}(t+\delta t)-\int_{S(t)} d\mathbf{a}\cdot\mathbf{B}(t)\\ &= \int_{S(t+\delta t)} d\mathbf{a}\cdot\mathbf{B}(t+\delta t)-\int_{S(t+\delta t)} d\mathbf{a}\cdot\mathbf{B}(t)\\ &+ \int_{S(t+\delta t)} d\mathbf{a}\cdot\mathbf{B}(t)-\int_{S(t)} d\mathbf{a}\cdot\mathbf{B}(t). \quad (1.57)\end{aligned}$$

With $\delta\mathbf{B}(t)=\mathbf{B}(t+\delta t)-\mathbf{B}(t)$, the first half is abbreviated as,

$$\int_{S(t+\delta)} d\mathbf{a}\cdot\delta\mathbf{B}(t)\simeq\int_{S(t)} d\mathbf{a}\cdot\delta\mathbf{B}(t), \quad (1.58)$$

with an error that is of the second order. The second half is denoted as,

$$\int(\delta d\mathbf{a})\cdot\mathbf{B}(t)=\int_{S(t+\delta t)} d\mathbf{a}\cdot\mathbf{B}(t)-\int_{S(t)} d\mathbf{a}\cdot\mathbf{B}(t). \quad (1.59)$$

In short,

$$\delta\left(\int d\mathbf{a}\cdot\mathbf{B}\right)=\int_{S(t)} d\mathbf{a}\cdot\delta\mathbf{B}(t)+\int(\delta d\mathbf{a})\cdot\mathbf{B}(t). \quad (1.60)$$

The first change is due to the variation of $\mathbf{B}(t)$, while the second is due to the variation of $S(t)$.

In Fig. 3, the magnetic flux out of the pill-like surface is zero (since there is no magnetic charge inside). That is,

$$\begin{aligned}0=\int_{pill} da\hat{\mathbf{n}}\cdot\mathbf{B} &= \int_{S(t+\delta t)} d\mathbf{a}\cdot\mathbf{B}-\int_{S(t)} d\mathbf{a}\cdot\mathbf{B}\\ &+ \oint_{C(t)}(d\mathbf{r}\times\mathbf{v}\delta t)\cdot\mathbf{B}, \quad (1.61)\end{aligned}$$

in which $\hat{\mathbf{n}}$ points out of the pill-like surface. The area element of the side surface (the grey area in Fig. 2) is $d\mathbf{r}\times\mathbf{v}\delta t$. Therefore, the second change

$$\int(\delta d\mathbf{a})\cdot\mathbf{B}=-\oint_{C(t)} d\mathbf{r}\cdot\mathbf{v}\times\mathbf{B}\delta t. \quad (1.62)$$

Divide Eq. (1.60) with δt, then Eq. (1.56) follows. QED

As a result, the integral form of the Faraday's law reads,

$$\begin{aligned}\oint_C d\mathbf{r}\cdot\mathbf{E} &= -\int_S d\mathbf{a}\cdot\frac{\partial\mathbf{B}}{\partial t} &(1.63)\\ &= -\frac{d}{dt}\int_S d\mathbf{a}\cdot\mathbf{B}-\oint_C d\mathbf{r}\cdot\mathbf{v}\times\mathbf{B}. &(1.64)\end{aligned}$$

Or,

$$\int_S d\mathbf{a}\cdot\frac{\partial\mathbf{B}}{\partial t}+\oint_C d\mathbf{r}\cdot\mathbf{v}\times\mathbf{B}=-\frac{d\Phi_B}{dt}. \quad (1.65)$$

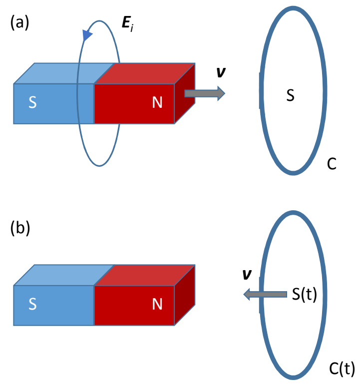

FIG. 4 (a) The magnet is moving. (b) The wire is moving.

The first term on the right hand side is called the **transformer EMF**, the second the **motional EMF**. The sum of the two, the total **EMF** (**electromotive force**) is proportional to the change of the magnetic flux Φ_B through a circuit.

1. A thought experiment

Einstein's 1905 paper, *On the electrodynamics of moving bodies*, begins with the following thought experiment: A magnet and a coil are moving with respect to each other at a constant velocity. In Fig. 4(a), we stay with the coil and see the magnet coming. Because of the changing magnetic field, an electric field $\mathbf{E}_i$ is induced surrounding the moving magnet. Such a field provides an EMF that drives the electrons inside the coil.

On the other hand, in Fig. 4(b), we stay with the magnet and see the coil coming. There is no induced field because the magnetic field is static. Nevertheless, the electrons in the coil are carried by the moving conductor, thus experience the Lorentz force.

Even though this two different views give the same electromotive force around the circular wire, there is a subtle difference regarding the existence (or not) of $\mathbf{E}_i$. Aiming to solve this puzzle, Einstein discovers the theory of special relativity along the way.

In mathematical terms, the EMF in case (a), aka the *transformer* EMF, is given as,

$$\mathcal{E}_i = \oint_C d\mathbf{r}\cdot\mathbf{E}_i = -\int_S d\mathbf{a}\cdot\frac{\partial\mathbf{B}}{\partial t}. \tag{1.66}$$

The EMF in case (b), aka the *motional* EMF, is given as,

$$\mathcal{E}_m = \oint_C d\mathbf{r}\cdot(\mathbf{v}\times\mathbf{B}). \tag{1.67}$$

In general, the total EMF,

$$\mathcal{E}_F = \oint_C d\mathbf{r}\cdot(\mathbf{E}_i + \mathbf{v}\times\mathbf{B}) \tag{1.68}$$

$$= -\frac{d\Phi_B}{dt}. \tag{1.69}$$

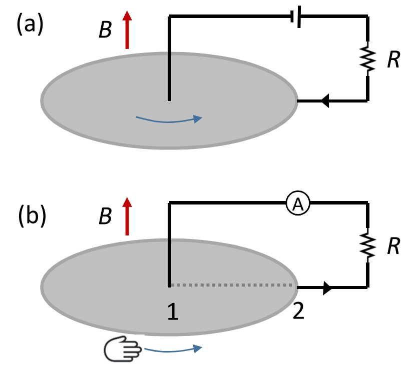

FIG. 5 (a) Motor mode (b) Generator mode

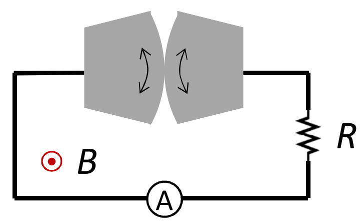

FIG. 6 Feynman's device in a uniform magnetic field that points out of the paper.

No matter who is moving, the EMF can be calculated with the changing magnetic flux $d\Phi_B/dt$.

Even though the flux theorem helps us connecting the two cases, Einstein's puzzle regarding the existence of $\mathbf{E}_i$ is still there. Its resolution requires the **Lorentz transformation** (see Sec. 22.6 of Zangwill, 2013). Suppose a frame K' is moving with velocity $\mathbf{v}$ with respect to a rest frame K. The electromagnetic fields in frames K and K' are $\mathbf{E}, \mathbf{B}$, and $\mathbf{E}', \mathbf{B}'$ respectively. Then, the **special theory of relativity** tells us that at low velocity ($v \ll c$),

$$\mathbf{E}' \simeq \mathbf{E} + \mathbf{v}\times\mathbf{B}, \tag{1.70}$$

$$\mathbf{B}' \simeq \mathbf{B} - \frac{\mathbf{v}}{c^2}\times\mathbf{E}. \tag{1.71}$$

Thus, whether there is $\mathbf{E}_i$ or not would depend on an observer's standpoint. There is no conflict as long as both observers agree on such a transformation. Let K be the coordinate frame of the magnet, and K' that of the coil, then in the frame K, an observer sees $\mathbf{E}_i = 0$. On the other hand, in the frame K', an observer sees $\mathbf{E}' = \mathbf{v}\times\mathbf{B}$, which is the induced electric field.

2. Faraday's disk generator (1831)

In Fig. 5, a conducting disk in a uniform magnetic field is connected with wires to form a circuit. It can operate in two different modes. The first is the *motor mode*: The battery generates a current that flows to the disk and is picked up by the wire at the center of the disk.

TABLE I EMF in Faraday disk experiments

Disk	Magnet	EMF
rotate	fix	nonzero
fix	rotate	zero
rotate	rotate	nonzero

The charge current under the magnetic field experiences a Lorentz force, which pushes the disk to rotate counter-clockwise. As a result, electric energy is transformed to mechanical energy.

The second is the *generator mode*: There is no battery to drive the current. Instead, an external force sets the disk spinning. The charges in the spinning disk experience a motional EMF, and a current due to free charges starts flowing in the circuit. As a result, mechanical energy transforms to electric energy.

Suppose the disk with radius a is spinning with angular frequency ω. Only the segment between points 1 and 2 in Fig. 5(b) contributes to the motional EMF. Thus,

$$\mathcal{E}_F = \int_1^2 d\mathbf{r}\cdot(\mathbf{v}\times\mathbf{B}) \qquad (1.72)$$

$$= \int_0^a drr\omega B = \frac{1}{2}\omega a^2 B. \qquad (1.73)$$

Note that even though Eq. (1.68) is valid, Eq. (1.69) fails, since the flux through the disk remains the same and $d\Phi_B/dt = 0$. In order to be safe, it is advised to apply the *flux rule*, $\mathcal{E}_F = -d\Phi_B/dt$, only to a circuit of thin wire.

A circuit devised by Feynman is shown in Fig. 6 (see Sec. 17-2, Vol II of Feynman *et al.*, 2010). Two conducting plates touch each other by a point that serve as a pathway of current. The shape of the plate is designed in such a way that, with a slow rotation, the contact point moves swiftly up and down. As a result, the magnetic flux through the circuit changes rapidly. This would generate a large current if the flux rule is correct. However, since the plate is barely moving, we do not expect to see a large current. Indeed, Eq. (1.68) would predict little current, which turns out to be case. The lesson is, again, don't apply the flux rule if the circuit is not made of thin wires only.

Instead of spinning the disk, if we spin the magnet (Fig. 7), would there be current? A related question is, if one rotates the magnet and the disk together (without relative motion), would there be current? According to observation, we have the results shown in Table 1 (Kelly, 1998). These results indicate that the magnetic field lines do not rotate with the magnet. (In these experiments, the connecting wires do not rotate with the setup.)

Let's consider another variation of the Faraday disk. Instead of Fig. 5(b), the wire is deformed to form a loop above the disk (see Fig. 8). Under the generator mode, there is a magnetic field $\mathbf{B}$ at the beginning. When we start spinning the disk, current begins to flow. Thus, the current loop above generates a magnetic field $\mathbf{B}'$ (also points up). We now can remove the external field $\mathbf{B}$ slowly. The disk would keep spinning and the current keep flowing because $\mathbf{B}'$ is still there. That is, this is a *self-sustained* Faraday generator. Eventually Joul heat and other dissipations would wear the energy out and stop the motion. It is believed that the geomagnetic field of the earth is maintained in a similar way.

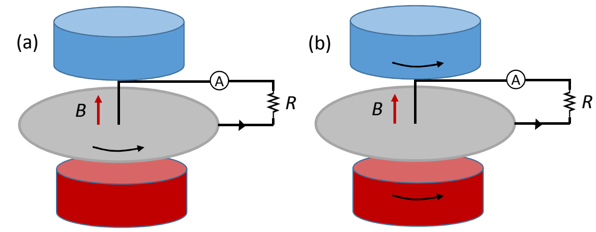

FIG. 7 (a) The disk is spinning. (b) The magnets are spinning.

E. Quasi-static fields

Again the Maxwell equations in vacuum are,

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad (1.74)$$

$$\nabla\cdot\mathbf{B} = 0, \qquad (1.75)$$

$$\nabla\times\mathbf{E} = \underline{-\frac{\partial\mathbf{B}}{\partial t}}, \qquad (1.76)$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \underline{\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}}, \qquad (1.77)$$

From the third and the fourth equations, a changing magnetic field induces an electric field, and a changing electric field induces a magnetic field. Therefore, the electric and magnetic fields generate each other, and the self-sustaining electromagnetic field can propagate in vacuum far away from its source.

We'll show that if *electric charge* varies slowly, then the $\partial\mathbf{B}/\partial t$ term in the third equation can be ignored. Therefore, one can determine the electric field from the first and the third equations, without knowing $\mathbf{B}$. This is called **quasi-electrostatic field**.

On the other hand, if *electric current* varies slowly, then the displacement current in the last equation can be ignored. Thus, one can determine the magnetic field from the second and the last equations, without knowing $\mathbf{E}$. This is called **quasi-magnetostatic field**.

The decoupling of electric and magnetic fields helps simplifying a calculation significantly.

When can one say that a variation is "slow"? Consider a system with oscillating charge or current. Suppose the

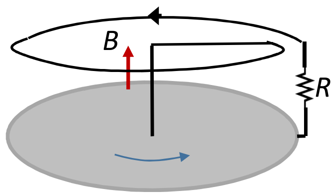

FIG. 8 A self-sustained dynamo

length scale of a system of our interest is ℓ, and the period of oscillation is $T = 1/f$. A process is slow if

$$\ell/c \ll T \tag{1.78}$$

That is, there is little lag between cause (from a source) and effect (on an observation point). This condition can also be written as,

$$f \ll c/\ell, \text{ or } \ell \ll \lambda, \tag{1.79}$$

where $\lambda = c/f$ is the wavelength of the electromagnetic disturbance.

For example, for a capacitor connected to a current oscillating at 6000 Hz, $\lambda \simeq 50$ km. Thus, this system is quasi-electrostatic as long as its detector is not placed far away. For a power line with a 60 Hz current, the corresponding $\lambda \simeq 5000$ km. Therefore, a power line stretching, for example, 300 km satisfies quasi-magnetostatic condition. A microprocessor chip that operates at 3 GHz has $\lambda \simeq 0.1$ m. This oscillation can still be roughly considered as slow.

1. Slowly time-varying charge in vacuum

We now show that when charge varies slowly, the $\partial\mathbf{B}/\partial t$ term can be ignored. First, we get the zeroth order fields, $\mathbf{E}_0$ and $\mathbf{B}_0$, from the Maxwell equations if all of the terms with time-derivative are dropped. That is,

$$\begin{aligned} \nabla\cdot\mathbf{E}_0 &= \frac{\rho}{\varepsilon_0}, &(1.80)\\ \nabla\cdot\mathbf{B}_0 &= 0, &(1.81)\\ \nabla\times\mathbf{E}_0 &= 0, &(1.82)\\ \nabla\times\mathbf{B}_0 &= \mu_0\mathbf{J}. &(1.83)\end{aligned}$$

To estimate the order of magnitude of each term, replacing ∇ with $1/\ell$, and $\partial/\partial t$ with ω, so that

$$\begin{aligned} E_0 &\sim \frac{\rho}{\varepsilon_0}\ell, &(1.84)\\ B_0 &\sim \mu_0 J\ell. &(1.85)\end{aligned}$$

Next, consider the first-order correction. We have

$$\begin{aligned} \nabla\cdot(\mathbf{E}_0+\mathbf{E}_1) &= \frac{\rho}{\varepsilon_0}, &(1.86)\\ \nabla\cdot\mathbf{B} &= 0, &(1.87)\\ \nabla\times\mathbf{E}_1 &= -\frac{\partial\mathbf{B}}{\partial t},\ (\nabla\times\mathbf{E}_0=0), &(1.88)\\ \nabla\times\mathbf{B} &= \mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{E}_0+\mathbf{E}_1). &(1.89)\end{aligned}$$

One can write $\mathbf{B}$ as $\mathbf{B}_0+\mathbf{B}_1$, but that doesn't alter the following analysis.

Order-of-magnitude estimate gives

$$\begin{aligned} E_1 &\sim \omega\ell B, &(1.90)\\ B &\sim \mu_0 J\ell+\frac{\omega\ell}{c^2}(E_0+E_1) &(1.91)\\ &\sim \mu_0 J\ell+\frac{\omega\ell^2}{c^2}\frac{\rho}{\varepsilon_0}+\left(\frac{\omega\ell}{c}\right)^2 B. &(1.92)\end{aligned}$$

Note that the Eq. of continuity gives,

$$\frac{J}{\ell}\sim\omega\rho. \tag{1.93}$$

Thus, the 2nd term on the RHS of Eq. (1.92) has the same magnitude as the first term, and

$$B\sim\mu_0\omega\ell^2\rho+\left(\frac{\omega\ell}{c}\right)^2 B. \tag{1.94}$$

This gives,

$$B\sim\frac{\mu_0\omega\ell^2\rho}{1-\left(\frac{\omega\ell}{c}\right)^2}. \tag{1.95}$$

It follows that

$$\frac{E_1}{E_0}\sim\frac{\omega\ell B}{\rho\ell/\varepsilon_0}\sim\frac{(\omega\ell/c)^2}{1-(\omega\ell/c)^2}. \tag{1.96}$$

Thus, E_1 is of the second order: $E_1 \ll E_0$ if $\omega\ell \ll c$. Note: Even though the 2nd term on the RHS of Eq. (1.92) has the same magnitude as the first term here, this is not always so. For the static case, the second term from $\partial\mathbf{E}_0/\partial t$ would vanish. So it cannot be taken as the zeroth order in the *static* case.

Now,

$$\frac{\partial B}{\partial t}\sim\mu_0 J\omega\ell\sim\frac{\rho}{\varepsilon_o}\left(\frac{\omega\ell}{c}\right)^2. \tag{1.97}$$

Thus, the inductive term $\partial\mathbf{B}/\partial t$ is also of the second order. This is the only term in the Maxwell equations whose first non-trivial order is of the second order. Hence it can be ignored if we are only interested in the first-order correction. Therefore, under the **quasi-electrostatic (QES) approximation**, we have

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad (1.98)$$
$$\nabla\cdot\mathbf{B} = 0, \quad (1.99)$$
$$\nabla\times\mathbf{E} = 0, \quad (1.100)$$
$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}, \quad (1.101)$$

The first and the third equations are the same as those of electrostatic field, the only difference being $\rho(\mathbf{r},t)$ is time-dependent here. Thus, the quasi-electrostatic field is,

$$\mathbf{E}(\mathbf{r},t) = -\nabla\phi(\mathbf{r},t), \quad \phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}. \quad (1.102)$$

Similarly (but hard to prove), the magnetic field is,

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t), \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int dv'\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}. \quad (1.103)$$

You may check that they satisfy Ampere's law.

For example, consider a point charge moving with constant velocity $\mathbf{v}$ ($v \ll c$). The charge and current densities are,

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{v}t), \quad (1.104)$$
$$\mathbf{J}(\mathbf{r},t) = q\mathbf{v}\delta(\mathbf{r}-\mathbf{v}t). \quad (1.105)$$

Thus,

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{r}-\mathbf{v}t|}, \quad (1.106)$$
$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\frac{q\mathbf{v}}{|\mathbf{r}-\mathbf{v}t|}. \quad (1.107)$$

It follows that,

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}-\mathbf{v}t}{|\mathbf{r}-\mathbf{v}t|^3}, \quad (1.108)$$
$$\mathbf{B}(\mathbf{r},t) = \nabla\times\frac{\mathbf{v}}{c^2}\frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{r}-\mathbf{v}t|} \quad (1.109)$$
$$= \frac{\mathbf{v}}{c^2}\times\mathbf{E}(\mathbf{r},t). \quad (1.110)$$

With a small modification, the potentials in Eqs. (1.102) and (1.103) can be universally valid: Just change the t in the sources to be the **retarded time** t_r,

$$t \to t_r \equiv t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}. \quad (1.111)$$

That is,

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}, \quad (1.112)$$
$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int dv'\frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}. \quad (1.113)$$

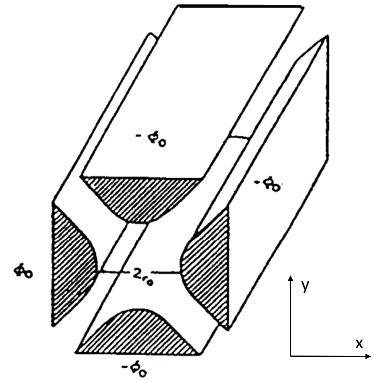

FIG. 9 Paul trap as a quadrupole mass spectrometer.

We will prove this in a later chapter. The physics behind is simple: whatever happens to the source at $\mathbf{r}'$, the disturbance takes time $|\mathbf{r}-\mathbf{r}'|/c$ to reach $\mathbf{r}$.

After having the electromagnetic potentials, one can proceed to calculate the electromagnetic fields. However, the dummy variable $\mathbf{r}'$ hidden inside the time-dependence of ρ and $\mathbf{J}$ makes it very difficult to carry out the integrations. This will be left to a later chapter on the generation of EM radiation.

Note that the relation between the *fields* and their source is required to be causal. But the relation between the *potentials* and its source actually needs not to be *explicitly* causal (as the ones above). We will come back to this in next chapter when talking about the Coulomb gauge.

• Paul trap and quadrupole mass spectrometer

A **Paul trap** consists of four parallel metal rods. Two opposing rods are held at positive potential, while the other two negative (Fig. 9). Near the central axis of the four rods, the potential is

$$\phi(x,y) = \phi_0\frac{x^2-y^2}{R^2} + \text{higher order terms}, \quad (1.114)$$

where R is roughly the scale of the trap. The electrostatic potential has a saddle point at the origin. A charge is trapped along x-axis, but it can roll away along y-axis. Nevertheless, a charged particle can be trapped with an ingenious modification: Let the potential be oscillatory,

$$\phi_0 \to \phi_0(t) = u_0 + v_0\cos\omega t. \quad (1.115)$$

After half of a period, the polarities of the potential are reversed, and a charge is trapped along y-axis instead. An anaogy is that, you can trap a bunch of small fishes with only two hands, as long as you swap hands quickly and alternatively between two orthogonal directions.

With a slow oscillation, we can use the quasi-electrostatic approximation: $\nabla\times\mathbf{E} = 0$, so $\mathbf{E} = -\nabla\phi$.

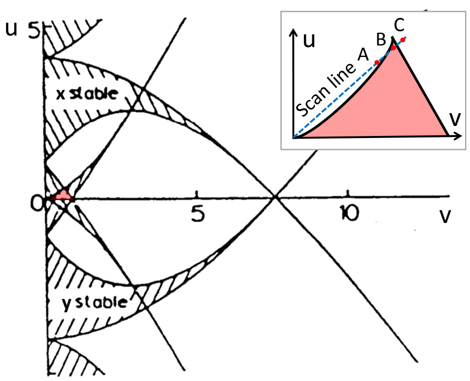

FIG. 10 Stable solutions for Eq. (1.120) belong to the upper half of the shaded regions, while those for $y(t)$ belong to the lower half. Part of the overlap region is indicated in red and is blown up in the inset. Along the scan line, only point B is stable. Fig. from Paul, 1990

The electric force on a charged particle is,

$$\mathbf{F} = m\ddot{\mathbf{r}} = -q\frac{\partial\phi}{\partial\mathbf{r}}. \tag{1.116}$$

When written in components, one has

$$\begin{aligned} m\ddot{x} + \frac{2q}{R^2}\phi_0(t)x &= 0, &(1.117)\\ m\ddot{y} - \frac{2q}{R^2}\phi_0(t)y &= 0, &(1.118)\\ m\ddot{z} &= 0. &(1.119)\end{aligned}$$

For the x component (similarly for the y component), we have

$$\ddot{x} + (u + v\cos\omega t)x = 0, \tag{1.120}$$

$$u = \frac{2q}{mR^2}u_0, v = \frac{2q}{mR^2}v_0. \tag{1.121}$$

This is the **Mathieu differential equation**, which is also used in, e.g., the problem of inverted pendulum, or the Schrödinger equation with periodic potential.

The solution can be stable or unstable (diverge with time) depending on the parameters u, v (Fig. 10). A particle can be trapped only if the parameters (u, v) are located in the intersection of x-stable region and y-stable region.

In an experiment, one can scan u, v, but keep u/v fixed (inset of Fig. 10), so that only particular values of u, v can be stable. As a result, only particles with certain q/m ratio can be stable. They would stay near the central axis while flying along z-axis and be selected. Such a device is called a **quadrupole mass spectrometer**.

2. Slowly time-varying current in vacuum

We now show that when current varies slowly, the $\partial\mathbf{E}/\partial t$ term can be ignored. First, we get the zeroth order fields, $\mathbf{E}_0$ and $\mathbf{B}_0$, from the Maxwell equations if all of the terms with time-derivative are dropped. Next, consider the first-order correction of the magnetic field. We have

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{\rho}{\varepsilon_0}, &(1.122)\\ \nabla\cdot(\mathbf{B}_0+\mathbf{B}_1) &= 0, &(1.123)\\ \nabla\times\mathbf{E} &= -\frac{\partial(\mathbf{B}_0+\mathbf{B}_1)}{\partial t}, &(1.124)\\ \nabla\times\mathbf{B}_1 &= \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t},\ (\nabla\times\mathbf{B}_0 = \mu_0\mathbf{J}). &(1.125)\end{aligned}$$

Faraday's law gives

$$E \sim \omega\ell(B_0 + B_1). \tag{1.126}$$

Since $\nabla\times\mathbf{B}_0 = \mu_0\mathbf{J}$, one has

$$B_0 \sim \mu_0 J\ell,\ B_1 \sim \frac{\omega\ell}{c^2}E. \tag{1.127}$$

It follows that,

$$\begin{aligned} E &\sim \omega\ell\mu_0 J\ell + \left(\frac{\omega\ell}{c}\right)^2 E, &(1.128)\\ \rightarrow E &\sim \frac{\mu_0\omega\ell^2 J}{1-\left(\frac{\omega\ell}{c}\right)^2}. &(1.129)\end{aligned}$$

Eventually,

$$\frac{B_1}{B_0} \sim \frac{\frac{\omega\ell}{c^2}E}{\mu_0 J\ell} \sim \frac{\left(\frac{\omega\ell}{c}\right)^2}{1-\left(\frac{\omega\ell}{c}\right)^2}. \tag{1.130}$$

Thus, $B_1 \ll B_0$ if $\omega\ell \ll c$. It's not difficult to see that, among all of the terms in the Maxwell equations, only the $\partial\mathbf{E}/\partial t$ term is of the second order and can be dropped. Under the **quasi-magnetostatic (QMS) approximation**, we then have

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{\rho}{\varepsilon_0}, &(1.131)\\ \nabla\cdot\mathbf{B} &= 0, &(1.132)\\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}, &(1.133)\\ \nabla\times\mathbf{B} &= \mu_0\mathbf{J}. &(1.134)\end{aligned}$$

The second and the last equations are the same as those of magnetostatic field, the only difference being $\mathbf{J}(\mathbf{r}, t)$ is time-dependent here.

Taking the divergence of the last equation, one has

$$\nabla\cdot\nabla\times\mathbf{B} = \mu_0\nabla\cdot\mathbf{J} = 0. \tag{1.135}$$

This leads to a constraint on charge density through the Eq. of continuity,

$$\frac{\partial\rho}{\partial t} = 0 \ \rightarrow\ \rho(\mathbf{r}, t) = \rho(\mathbf{r}). \tag{1.136}$$

If the system is charge neutral at the beginning, then it stays neutral and ρ can be ignored from the Maxwell equations.

• A solenoid with time-varying current

Consider an infinite solenoid along the z-axis with current $I(t)$ and radius a. There are n turns of coils per unit length. Suppose the variation of the current, $I(t) = I_0 \cos \omega t$, satisfies the QMS approximation. We'd like to find out the EM field inside and outside the solenoid.

We'll calculate the EM field in the following order: First, the zero-th order magnetic field satisfies

$$\nabla \times \mathbf{B}_0 = \mu_0 \mathbf{J}. \tag{1.137}$$

From $\mathbf{B}_0$ we can get the first-order electric field,

$$\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_0}{\partial t}. \tag{1.138}$$

From $\mathbf{E}_1$ we can get the first-order correction of the magnetic field,

$$\nabla \times \mathbf{B}_1 = \frac{1}{c^2}\frac{\partial \mathbf{E}_1}{\partial t}. \tag{1.139}$$

When written in integral form, these equations become

$$\oint_C \mathbf{B}_0 \cdot d\mathbf{r} = \mu_0 I, \tag{1.140}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d\Phi_B}{dt}, \tag{1.141}$$

$$\oint_C \mathbf{B}_1 \cdot d\mathbf{r} = \frac{1}{c^2}\frac{d\Phi_E}{dt}. \tag{1.142}$$

From Eq. (1.140), one gets

$$\mathbf{B}_0(t) = \begin{cases} \mu_0 n I(t)\hat{\mathbf{z}} & \text{for } \rho < a, \\ 0 & \text{for } \rho > a. \end{cases} \tag{1.143}$$

This is similar to the result for the magnetostatic case.

The changing magnetic field would induce a circular electric field around the solenoid, $\mathbf{E} = E_\phi(\rho, t)\hat{\boldsymbol{\phi}}$. From Eq. (1.141), it's not difficult to obtain

$$E_\phi(\rho, t) = \begin{cases} -\mu_0 n \dot{I} \frac{\rho}{2} & \text{for } \rho < a, \\ -\mu_0 n \dot{I} \frac{a^2}{2\rho} & \text{for } \rho > a. \end{cases} \tag{1.144}$$

Finally, to find the $\mathbf{B}_1$ outside the solenoid with Eq. (1.142), one can choose a rectangular path C shown in Fig. 11. Suppose the left and right legs of C are located at radii ρ_1 and ρ_2 respectively, then Eq. (1.142) gives

$$[B_1(\rho_1, t) - B_1(\rho_2, t)]\ell = \frac{1}{c^2}\frac{d}{dt}\int_{\rho_1}^{\rho_2} E_\phi d\rho \ell \tag{1.145}$$

$$= -\mu_0 n \frac{a^2}{c^2}\int_{\rho_1}^{\rho_2} \frac{\ddot{I}}{2\rho} d\rho \ell \tag{1.146}$$

$$= -\mu_0 n \frac{a^2}{2c^2}\omega^2 I \ell \ln \frac{\rho_1}{\rho_2}, \tag{1.147}$$

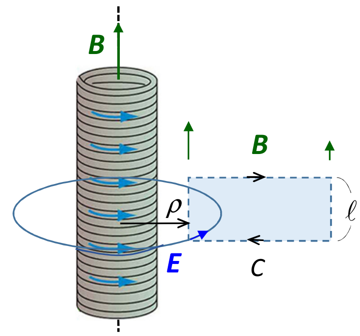

FIG. 11 A rectangular path C near the solenoid.

in which $\ddot{I} = -\omega^2 I$ has been used. There are two unknowns, $B_1(\rho_1, t)$ and $B_1(\rho_2, t)$ in the equation. We cannot choose $\rho_1 = a^+$ since the magnetic field there is not known. We cannot choose $\rho_2 = \infty$ either – even though it removes $B_1(\rho_2, t)$, since $\ln \rho_2$ diverges. Furthermore, the QMS approximation works only if the scale of the system is smaller than $\lambda \simeq c/\omega$. So we will set $\rho_2 = c/\omega$, and let $B_1(\rho_2, t) \simeq 0$. It follows that

$$B_1(\rho, t) \simeq -\mu_0 n \frac{a^2}{2c^2}\omega^2 I \ln \frac{\omega\rho}{c}. \tag{1.148}$$

This is very close to the result obtained from a rigorous analysis (see p. 718 of Zangwill),

$$B_1(\rho, t) = -\mu_0 n \frac{a^2}{2c^2}\omega^2 I_0 \left(\cos \omega t \ln \frac{\omega\rho}{c} - \frac{\pi}{2}\sin \omega t\right). \tag{1.149}$$

F. Quasi-static fields in matter

If a material is isotropic, and that $\mathbf{P}$ is proportional to $\mathbf{E}$, $\mathbf{M}$ is proportional to $\mathbf{B}$, then we say that it is *simple*. A simple material has

$$\mathbf{D} = \varepsilon \mathbf{E}, \;\; \mathbf{H} = \mathbf{B}/\mu. \tag{1.150}$$

The Maxwell equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho_f}{\varepsilon}, \tag{1.151}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.152}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.153}$$

$$\nabla \times \mathbf{B} = \mu \mathbf{J}_f + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{1.154}$$

in which $1/c^2 = \varepsilon\mu$ for the velocity of light c in matter. The quasi-static condition becomes $\omega\ell \ll c$.

• Charge relaxation inside a conductor

Suppose we put a charge inside a conductor, then it would move to the surface quickly. This can be studied with the Eq. of continuity,

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0. \tag{1.155}$$

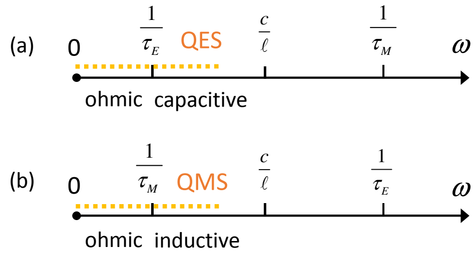

FIG. 12 Comparison between different rates. The valid ranges of (a) QES approximation and (b) QMS approximation are indicated by dotted lines.

Since

$$\mathbf{J}_f = \sigma \mathbf{E}, \tag{1.156}$$
$$\nabla \cdot \mathbf{E} = \frac{\rho_f}{\varepsilon}, \tag{1.157}$$

one has

$$\sigma \frac{\rho_f}{\varepsilon} + \frac{\partial \rho_f}{\partial t} = 0. \tag{1.158}$$

Thus,

$$\rho_f(\mathbf{r}, t) = \rho_f(\mathbf{r}, 0) e^{-t/\tau_E}, \ \tau_E \equiv \varepsilon/\sigma. \tag{1.159}$$

The **relaxation time** τ_E is inversely proportional to the conductivity.

For example, for distilled water, the conductivity $\sigma \sim 2\times 10^{-4}$ $1/(\Omega\text{m})$, and $\varepsilon \simeq 80\ \varepsilon_0$. Therefore, $\tau_E \sim 10^{-6}$ s. For copper, the conductivity $\sigma \sim 6\times 10^{7}$ $1/(\Omega\text{m})$, and $\varepsilon \simeq \varepsilon_0$. Therefore, $\tau_E \sim 10^{-19}$ s. In the second example (for good conductors), this result is not accurate. In reality, $\tau_E \sim 10^{-14}$ s. The problem is not with the QES approximation (since $\nabla\times\mathbf{E} = 0$ has not been used above), but with the assumption that σ is a constant. A better analysis needs to take into account the frequency dependence of conductivity, $\sigma(\omega)$ (see p. 634 of Zangwill, 2013).

1. Poor conductors: Quasi-electrostatics

Inside a matter, for the quasi-static approximation to be valid, we need $\omega\ell \ll c$. Charge relaxation time τ_E offers another characteristic time, which can be larger or smaller than ℓ/c. Assume $\tau_E > \ell/c$, then under the QES condition, one can still have $\omega\tau_E > 1$ or $\omega\tau_E < 1$ (Fig. 11(a)). In the former, energy is mostly stored in electric field (called capacitive regime). In the latter, most of the energy is dissipated through current (called ohmic region). See p. 205 of Orlando and Delin, 1991.

2. Good conductors: Quasi-magnetostatics

Suppose a system is charge neutral so that $\nabla\cdot\mathbf{E} = 0, \nabla\cdot\mathbf{B} = 0$. Before any approximation, we have

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.160}$$
$$\nabla\times\mathbf{B} = \mu_0 \mathbf{J}_f + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{1.161}$$

Let's consider the possibility that the induced electric field $\mathbf{E}$ drives a current (such as the eddy current), so that

$$\mathbf{J}_f = \mathbf{J}_{ext} + \sigma\mathbf{E}. \tag{1.162}$$

$\mathbf{J}_{ext}$ is the current without induction. To ignore the displacement current $\mathbf{J}_D$ (as in the QMS approximation), one needs

$$\mathbf{J}_D \left(= \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) \ll \mathbf{J}_{ext}, \sigma\mathbf{E}. \tag{1.163}$$

First, to the 0-th order,

$$\nabla\times\mathbf{B}_0 = \mu_0 \mathbf{J}_f, \tag{1.164}$$
$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}_0}{\partial t}. \tag{1.165}$$

Therefore,

$$B_0 \sim \ell\mu J_{ext}, \tag{1.166}$$
$$E \sim \omega\ell B_0, \tag{1.167}$$

and

$$J_D \sim \varepsilon\omega E. \tag{1.168}$$

It follows that

$$\frac{J_D}{J_{ext}} \sim \varepsilon\omega \frac{E}{J_{ext}} \sim \varepsilon\mu\omega^2\ell^2, \tag{1.169}$$

and $J_D \ll J_{ext}$ if $\omega\ell \ll c$.

Second,

$$\frac{J_D}{\sigma E} \sim \frac{\varepsilon\omega E}{\sigma E} = \omega\tau_E. \tag{1.170}$$

Thus, $J_D \gg \sigma E$ if $\omega \gg 1/\tau_E$. This would be called a *poor* conductor. A *good* conductor would have $J_D \ll \sigma E$ and $\omega \ll 1/\tau_E$. In order to meet the QMS condition and drop the displacement current, one needs $\omega \ll c/\ell$ *and* $\omega \ll 1/\tau_E$. Several interesting phenomena occur in this range, such as skin effect, eddy current, and magnetic diffusion (see Zangwill, 2013 for more).

We can also compare the magnitudes of induced current and J_{ext},

$$\frac{\sigma E}{J_{ext}} \sim \frac{\sigma E}{B/\ell\mu} \sim \frac{\sigma E}{E/\omega\ell^2\mu} \tag{1.171}$$
$$= \omega\tau_M, \ \tau_M \equiv \sigma\mu\ell^2. \tag{1.172}$$

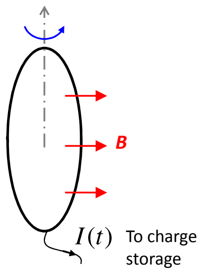

FIG. 13 Flip a coil with respect to the vertical axis generates a current.

This gives another time scale τ_M (Fig. 11(b)). When $\omega \ll 1/\tau_M$, magnetic induction is negligible (called ohmic regime). When $\omega \gg 1/\tau_M$, magnetic induction is dominant (called inductive regime). Finally, note that

$$\tau_E \tau_M = \varepsilon \mu \ell^2 = (\ell/c)^2. \tag{1.173}$$

Homework:

1. The flip coil shown in figure is a simple device to measure the magnitude of magnetic field. Suppose the radius of the coil is a, and a uniform magnetic field is perpendicular to the plane of the coil. Flip it around a radial axis would generate a transient current $I(t)$ around the coil. Connect the coil to an external charge storage and collect the accumulated charge Q (from a 180 degrees flip). Determine the magnetic field B from the charge Q.

2. The scalar and the vector potentials of a slowly moving charge are given in Eqs.(1.106) and (1.107). Confirm that the electric and the magnetic fields are those shown in Eqs. (1.108) and (1.110).

REFERENCES

Feynman,, Leighton, and Sands (2010), *The Feynman Lectures on Physics*, new millennium ed. (California Institute of Technology) www.feynmanlectures.caltech.edu/info/.

Kelly, A G (1998), "Faraday's final riddle; does the field rotate with a magnet?" (Institution of Engineers of Ireland) monographs 5 and 6.

Orlando, Terry, and Kevin A. Delin (1991), *Foundations of Applied Superconductivity* (Addison-Wesley).

Paul, Wolfgang (1990), "Electromagnetic traps for charged and neutral particles," Rev. Mod. Phys. **62**, 531–540.

Zangwill, Andrew (2013), *Modern electrodynamics* (Cambridge Univ. Press, Cambridge).