Lecture notes on classical electrodynamics

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CONTENTS

I. MAGNETIC MULTIPOLES

A. Multipole expansion

Recall that in Chap 4, given an electric potential,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{1.1}$$

if $r \gg r'$, then we can expand

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \simeq \frac{1}{r} + \frac{\hat{\mathbf{r}}}{r^2}\cdot\mathbf{r}' + \frac{1}{2r^3}\left[3(\hat{\mathbf{r}}\cdot\mathbf{r}')^2 - |\mathbf{r}'|^2\right]. \tag{1.2}$$

Each term contributes to the potential of a certain electric multipole.

Similar approximation can be applied to the vector potential,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int dv' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.3}$$

If $r \gg r'$, that is, the current source is localized and the observer is far away (Fig. 1), then we can use the expansion in Eq. (1.2) and keep terms to the first order to get

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &\simeq \frac{\mu_0}{4\pi r}\int dv'\mathbf{J}(\mathbf{r}') + \frac{\mu_0}{4\pi r^3}\mathbf{r}\cdot\mathbf{r}'\int dv'\mathbf{J}(\mathbf{r}') & (1.4)\\ &\simeq 0 + \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}, & (1.5)\end{aligned}$$

where $\mathbf{m}$ is the magnetic dipole moment. The magnetic quadrupole potential from the second-order term is not considered here.

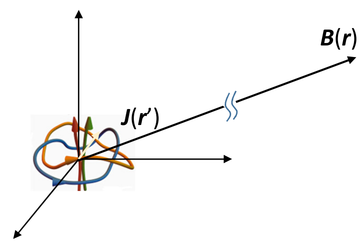

FIG. 1 An observation point is far away from a localized current distribution.

We now explain how Eq. (1.5) is obtained. First, two identities are required. For a steady, localized current distribution,

$$\int dv' J_i(\mathbf{r}') = 0, \quad i = x, y, z \tag{1.6}$$

$$\int dv' \left[r_i' J_j(\mathbf{r}') + r_j' J_i(\mathbf{r}')\right] = 0. \tag{1.7}$$

Pf: From the equation of continuity, for a steady current,

$$\begin{aligned}\nabla\cdot\mathbf{J} &= 0, & (1.8)\\ \nabla\cdot(r_i\mathbf{J}) &= J_i + r_i\cancel{\nabla\cdot\mathbf{J}}, & (1.9)\\ \nabla\cdot(r_i r_j\mathbf{J}) &= r_iJ_j + r_jJ_i + r_ir_j\cancel{\nabla\cdot\mathbf{J}}. & (1.10)\end{aligned}$$

The integration of Eq. (1.9) over the whole space gives,

$$\begin{aligned}\int dv' J_i &= \int dv'\nabla'\cdot(r_i'\mathbf{J}) & (1.11)\\ &= \int d\mathbf{s}'\cdot(r_i'\mathbf{J}) = 0. & (1.12)\end{aligned}$$

The integral is zero since the current is localized while the surface of integration is at infinity. Thus, the monopole term in Eq. (1.4) vanishes.

The integration of Eq. (1.10) over the whole space gives,

$$\int dv'(r_i'J_j + r_j'J_i) = \int d\mathbf{s}'\cdot(r_i'r_j'\mathbf{J}) = 0. \tag{1.13}$$

Thus, we can write the integral of the dipole term in

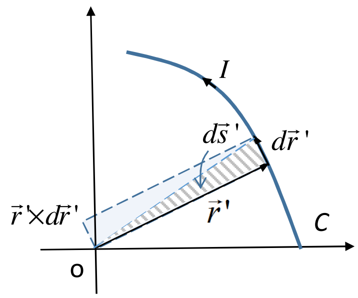

FIG. 2 A planar loop with current I.

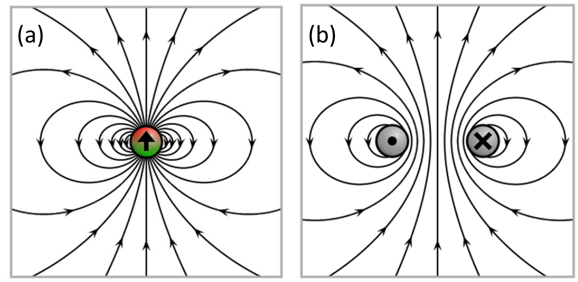

FIG. 3 The fields from (a) an electric dipole and (b) a current loop.

Eq. (1.4) as,

$$r_i \int dv' r_i' J_j = \frac{r_i}{2} \int dv' \underbrace{(r_i' J_j - r_j' J_i)}_{=\epsilon_{ijk}(\mathbf{r}'\times\mathbf{J})_k} \quad (1.14)$$

$$= \frac{1}{2} \int dv' \left[(\mathbf{r}' \times \mathbf{J}) \times \mathbf{r} \right]_j \quad (1.15)$$

$$= (\mathbf{m} \times \mathbf{r})_j, \quad (1.16)$$

where

$$\mathbf{m} \equiv \frac{1}{2} \int dv' \mathbf{r}' \times \mathbf{J}(\mathbf{r}'). \quad (1.17)$$

Hence, up to the first order,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \quad (1.18)$$

Eq. (1.17) is the most general form of the **magnetic dipole moment**. It reduces to other forms under special circumstances:

1. Thin wire:

For the current carried by a thin wire of loop C, just replace $dv'\mathbf{J}$ with $Id\mathbf{r}'$ to get

$$\mathbf{m} \equiv \frac{I}{2} \oint_C \mathbf{r}' \times d\mathbf{r}'. \quad (1.19)$$

If furthermore, C is a *planar loop*, then (see Fig. 2),

$$\frac{1}{2} \mathbf{r}' \times d\mathbf{r}' = d\mathbf{s}'. \quad (1.20)$$

Hence, after integration,

$$\mathbf{m} = I \oint d\mathbf{s}' = I\mathbf{S}. \quad (1.21)$$

The magnetic moment is proportional to the surface area of the loop. The direction of $\mathbf{S}$ is determined by the right-hand rule.

2. Point charges:

A set of moving charges has the current density,

$$\mathbf{J}(\mathbf{r}) = \sum_{k=1}^{N} q_k \mathbf{v}_k \delta(\mathbf{r} - \mathbf{r}_k). \quad (1.22)$$

Substitute it to Eq. (1.17) and get

$$\mathbf{m} = \sum_{k=1}^{N} \frac{q_k}{2} \int dv' \mathbf{r}' \times \mathbf{v}_k \delta(\mathbf{r}' - \mathbf{r}_k) \quad (1.23)$$

$$= \frac{1}{2} \sum_k q_k (\mathbf{r}_k \times \mathbf{v}_k) \quad (1.24)$$

$$= \sum_k \frac{q_k}{2m_k} \mathbf{L}_k, \quad \mathbf{L}_k \equiv m_k \mathbf{r}_k \times \mathbf{v}_k. \quad (1.25)$$

If q_k/m_k is a constant, then the **orbital magnetic moment**

$$\mathbf{m} = \frac{q}{2m} \mathbf{L}, \quad (1.26)$$

where $\mathbf{L}$ is the total angular momentum of these charges.

B. Magnetic dipole

From the vector potential of a magnetic dipole (valid for $r \gg r'$),

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}, \quad (1.27)$$

we can calculate its magnetic field,

$$\mathbf{B}(\mathbf{r}) = \nabla \times \left(\frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} \right) \quad (1.28)$$

$$\cdots = \frac{\mu_0}{4\pi} \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{r^3}. \quad (1.29)$$

The field decreases as $1/r^3$ and has the distribution shown in Fig. 3, which is similar to the electric dipole field (Chap 4) when $r \gg r'$.

Example:

Suppose current distribution $\mathbf{J}(\mathbf{r})$ flows inside a ball V with volume V, show that the average of the magnetic field over the ball,

$$\langle \mathbf{B}(\mathbf{r}) \rangle_V \equiv \frac{1}{V} \int_V dv \mathbf{B}(\mathbf{r}) = \frac{2\mu_0}{3} \frac{\mathbf{m}}{V}, \quad (1.30)$$

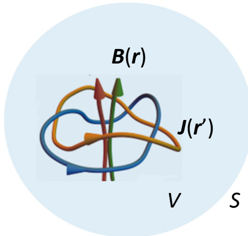

FIG. 4 The current is confined within a sphere.

where $\mathbf{m}$ is the magnetic dipole moment due to the current (see Fig. 4).
Pf: Start from the Biot-Savart law,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{J\neq 0} dv'\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{1.31}$$

then

$$\begin{aligned}\int_V dv\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V dv\int_{J\neq 0} dv'\mathbf{J}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\\ &= -\frac{\mu_0}{4\pi}\int_{J\neq 0} dv'\mathbf{J}(\mathbf{r}')\times\underbrace{\int_V dv\frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}}_{=\tilde{\mathbf{E}}(\mathbf{r}')},\end{aligned}$$

where $\tilde{\mathbf{E}}(\mathbf{r}')$ is the *fictitious* "electric" field of a ball V with charge density $\tilde{\rho} = 4\pi\varepsilon_0$. According to the analysis in Chap 4,

$$\tilde{\mathbf{E}}(\mathbf{r}') = \frac{4\pi}{3}\mathbf{r}'. \tag{1.32}$$

Thus,

$$\begin{aligned}\langle\mathbf{B}\rangle_V &= -\frac{\mu_0}{V}\int dv'\mathbf{J}(\mathbf{r}')\times\frac{1}{3}\mathbf{r}' && (1.33)\\ &= +\frac{2\mu_0}{3}\frac{\mathbf{m}}{V}. && (1.34)\end{aligned}$$

Similar to the case of the electric dipole, if the current is outside of the sphere, then

$$\langle\mathbf{B}(\mathbf{r})\rangle_V = \mathbf{B}(0). \tag{1.35}$$

Its proof is similar to the case of electric dipole and will not be repeated here.

1. Point magnetic dipole

When a magnetic dipole is produced by the current in a tiny region (say a nucleus), we have a **point magnetic dipole**. The formula in Eq. (1.29) remains valid as long as $r \neq 0$.

However, if you integrate the field in Eq. (1.29) over a ball V centered at $\mathbf{r} = 0$, then

$$\int_V dv\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V dv\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m})-\mathbf{m}}{r^3} = 0. \tag{1.36}$$

It is zero due to angular integration, no matter if the ball is large or small. This contradicts the result in Eq. (1.34). To fix this discrepancy, we can add a delta function to Eq. (1.29), so that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m})-\mathbf{m}}{r^3} + \frac{2\mu_0}{3}\mathbf{m}\delta(\mathbf{r}). \tag{1.37}$$

The added term is important in the calculation of hyperfine structure (more later).

2. Magnetic dipole layer

In Fig. 5(a), there is a continuous distribution of magnetic dipoles on surface S. Suppose these dipole moments are from orbital motion of charges (not from electron spins), and are *perpendicular* to the surface. If S is an open surface, then the magnetic field from these dipoles is equal to the $\mathbf{B}$ field produced by a current flowing around the boundary C of S. This **Ampère's theorem**.
Pf: Each magnetic dipole is produced by a small current loop,

$$d\mathbf{m} = Id\mathbf{s}, \tag{1.38}$$

where $d\mathbf{s}$ is an area element. The dipole at $\mathbf{r}'$ on the surface generates a vector potential,

$$d\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}d\mathbf{m}\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{1.39}$$

Using the identity,

$$\int_S d\mathbf{s}\times\nabla f(\mathbf{r}) = \oint_C d\mathbf{r}f(\mathbf{r}), \tag{1.40}$$

where C is the boundary of surface S, one then has

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int d\mathbf{m}\times\underbrace{\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}}_{=\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}} && (1.41)\\ &= \frac{\mu_0}{4\pi}I\int_S d\mathbf{s}'\times\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} && (1.42)\\ &= \frac{\mu_0}{4\pi}I\oint_C d\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}. && (1.43)\end{aligned}$$

The line integral above equals the vector potential produced by a loop C carrying current I. QED.

If you're familiar with Stoke's theorem, then Ampère's theorem is simply a variant of Stokes theorem: The sum of the circulation of current loops packed together equals the circulation around the outer boundary of these loops (Fig. 5(a)).

Example:
A long magnetic tape with width d is lying along the x-axis, as shown in Fig. 5(b). The magnetic dipoles on the tape stand straight up, and the magnetic moment per

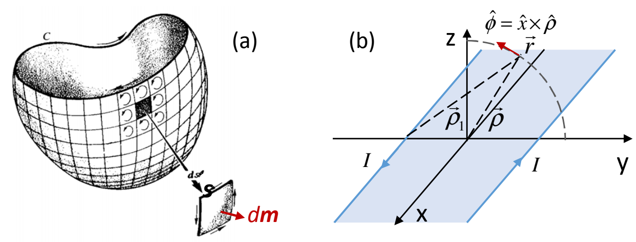

FIG. 5 (a) Magnetic dipole moments are standing on an open surface. (b) A magnetic tape on the x-y plane.

unit area is M. Find out the magnetic field around this magnetic tape.
Sol'n: According to Ampère's theorem, we only need to calculate the $\mathbf{B}$ field produced by the current flowing along the boundary of the tape. Since $d\mathbf{m} = Id\mathbf{s}$, so

$$I = \frac{dm}{ds} = M. \tag{1.44}$$

We need to calculate the magnetic field of two long straight wires with current I.

If the wire is lying on the x-axis, then for a point $\mathbf{r}$ on y-z plane,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{2\pi}\frac{I}{\rho}\hat{\boldsymbol{\phi}}, \tag{1.45}$$

where $\boldsymbol{\rho}$ and $\hat{\boldsymbol{\phi}} = \hat{\mathbf{x}} \times \hat{\boldsymbol{\rho}}$ are shown in Fig. 5(b).

For a wire lying along $y = -d/2$,

$$\mathbf{B}_1 = \frac{\mu_0 I}{2\pi}\frac{\hat{\mathbf{x}} \times \hat{\boldsymbol{\rho}}_1}{\rho_1}, \tag{1.46}$$

where (Fig. 5(b))

$$\boldsymbol{\rho}_1 = \boldsymbol{\rho} + \frac{d}{2}\hat{\mathbf{y}} = \left(y + \frac{d}{2}\right)\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{1.47}$$

Similarly, for the other wire with current flowing along the opposite direction,

$$\mathbf{B}_2 = -\frac{\mu_0 I}{2\pi}\frac{\hat{\mathbf{x}} \times \hat{\boldsymbol{\rho}}_2}{\rho_2}, \tag{1.48}$$

and

$$\boldsymbol{\rho}_2 = \boldsymbol{\rho} - \frac{d}{2}\hat{\mathbf{y}} = \left(y - \frac{d}{2}\right)\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{1.49}$$

Finally, the total magnetic field

$$\begin{aligned} \mathbf{B} &= \frac{\mu_0 I}{2\pi}\left(\frac{\hat{\mathbf{x}} \times \boldsymbol{\rho}_1}{\rho_1^2} - \frac{\hat{\mathbf{x}} \times \boldsymbol{\rho}_2}{\rho_2^2}\right) \qquad (1.50)\\ \text{or} \quad &= \frac{\mu_0 I}{2\pi}\left[\frac{\left(y+\frac{d}{2}\right)\hat{\mathbf{z}} - z\hat{\mathbf{y}}}{\left(y+\frac{d}{2}\right)^2 + z^2} - \frac{\left(y-\frac{d}{2}\right)\hat{\mathbf{z}} - z\hat{\mathbf{y}}}{\left(y-\frac{d}{2}\right)^2 + z^2}\right]. \end{aligned}$$

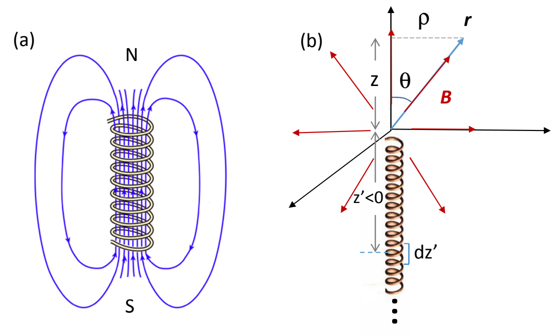

FIG. 6 (a) The magnetic field of a solenoid. (b) A semi-infinite solenoid along the negative z-axis.

C. Magnetic monopole

We have shown in Sec. A that the monopole potential of a localized current distribution is zero. Also, no magnetic monopole has been observed so far. Nevertheless, theory itself does not forbid the existence of magnetic monopole, as we'll show now.

The magnetic field produced by a finite solenoid is similar to that of a bar of magnetic (Fig. 6(a)). If a solenoid is very long, then its N-pole and S-pole are far away from each other. For a semi-infinite solenoid that extends from the origin to $z = -\infty$ (Fig. 6(b)), its S-pole is pushed to infinity and all of the magnetic field emanates from the N-pole — the opening at the origin. We can use it to simulate a magnetic monopole.

Example:
A semi-infinite solenoid along negative z-axis carries a current I. The cross section area is s, and the number of coils per unit length is n. Find out its vector potential and magnetic field.
Sol'n:
First, the vector potential of a current loop on the x-y plane with magnetic moment $\mathbf{m} = Is\hat{\mathbf{z}}$ is,

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}, \quad \mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}} \qquad (1.51)\\ &= \frac{\mu_0}{4\pi}m\frac{\rho}{(\rho^2 + z^2)^{3/2}}\hat{\boldsymbol{\phi}}. \qquad (1.52) \end{aligned}$$

Now, the number of loops within dz' at position z' (< 0) is ndz'. Its vector potential,

$$d\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}m(ndz')\frac{\rho}{\left[\rho^2 + (z - z')^2\right]^{3/2}}\hat{\boldsymbol{\phi}}. \tag{1.53}$$

After integration,

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}mn\int_{-\infty}^{0} dz'\frac{\rho}{\left[\rho^2 + (z - z')^2\right]^{3/2}}\hat{\boldsymbol{\phi}}\\ &= \frac{\mu_0}{4\pi}g\underbrace{\int_{-\infty}^{-z} dz'\frac{\rho}{(\rho^2 + z'^2)^{3/2}}}_{\equiv I(z)}\hat{\boldsymbol{\phi}}, \; g \equiv mn. \qquad (1.54) \end{aligned}$$

Let $z' = -\rho\tan\varphi$, then $dz' = -\rho\sec^2\varphi d\varphi$, the integral becomes

$$\begin{aligned} I(z) &= \int_{\tan^{-1}\frac{z}{\rho}}^{\frac{\pi}{2}} d\varphi \frac{1}{\rho\sec\varphi} && (1.55)\\ &= \frac{1}{\rho}\sin\varphi\Big|_{\tan^{-1}\frac{z}{\rho}}^{\frac{\pi}{2}} && (1.56)\\ &= \frac{1}{\rho}\left(1-\frac{z}{\sqrt{z^2+\rho^2}}\right). && (1.57)\end{aligned}$$

If we choose spherical coordinate ($\rho = r\sin\theta$), then

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{g}{r}\frac{1-\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}}. \quad (1.58)$$

Its magnetic field,

$$\begin{aligned} \mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{\mathbf{r}} + \cdots \\ &= \frac{\mu_0 g}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}. && (1.59)\end{aligned}$$

This is valid as long as $\mathbf{r}$ is away from the solenoid. Finally, let $s \to 0$, and $n \to \infty$, such that $g = Isn$ remains fixed. Then Eqs. (1.58) and (1.59) are valid everywhere, except along the negative z-axis.

The monopole field $\mathbf{B}(\mathbf{r})$ is the same as the Coulomb field for a point charge and decreases as $1/r^2$. If magnetic monopole exists, then the divergence of $\mathbf{B}$ is no longer zero, but (for the example above)

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = \mu_0 g\delta(\mathbf{r}). \quad (1.60)$$

Recall that the divergence of curl is always zero, so how can $\nabla\cdot\nabla\times\mathbf{A}(\mathbf{r})$ be non-zero here? In fact, $\nabla\cdot\nabla\times\mathbf{V}(\mathbf{r}) = 0$ is valid *only if* $\mathbf{V}(\mathbf{r})$ has no singularity, which is not the case for the $\mathbf{A}(\mathbf{r})$ here. The vector potential is singular along the negative z-axis, when $\theta = \pi$.

This string of singularity, called **Dirac string**, is an artifact of theory and cannot be detected in experiment if the monopole charge is quantized (Jackson, 1998). It's possible to simulate a monopole using a semi-infinite solenoid along the *positive* z-axis (or other places), then

$$\mathbf{A}'(\mathbf{r}) = -\frac{\mu_0}{4\pi}\frac{g}{r}\frac{1+\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}}, \quad (1.61)$$

which produces the same monopole field $\mathbf{B}(\mathbf{r})$. In this case, the Dirac string is along the positive z-axis. You may check that $\mathbf{A}$ and $\mathbf{A}'$ differ by a gauge transformation. That is, the position of the Dirac string is gauge dependent.

D. Force and energy

Consider a distribution of current in an external magnetic field $\mathbf{B}(\mathbf{r})$. Suppose the current is "rigid". That is, the external magnetic field cannot alter the distribution of current, then it feels a force,

$$\mathbf{F} = \int dv\mathbf{J}(\mathbf{r})\times\mathbf{B}(\mathbf{r}). \quad (1.62)$$

Assume the magnetic field varies slowly across the current, then we can expand it with respect to a point 0 near the current,

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(0) + (\mathbf{r}\cdot\nabla)\mathbf{B}(0) + \cdots. \quad (1.63)$$

Thus,

$$\mathbf{F} \simeq \underbrace{\left(\int dv\mathbf{J}(\mathbf{r})\right)}_{=0}\times\mathbf{B}(0) + \int dv\mathbf{J}(\mathbf{r})\times(\mathbf{r}\cdot\nabla)\mathbf{B}(0). \quad (1.64)$$

The first integral is zero, as has been shown in Eq. (1.12). When written in components, one has

$$F_i = \epsilon_{ijk}\int dv J_j r_\ell \nabla_\ell B_k. \quad (1.65)$$

Before moving on, recall that (Chap 1)

$$\begin{aligned} (\mathbf{u}\times\mathbf{v})_j &= \epsilon_{jkl}u_k v_l, && (1.66)\\ \epsilon_{klj}\epsilon_{kmn} &= \begin{vmatrix} \delta_{lm} & \delta_{ln} \\ \delta_{jm} & \delta_{jn}\end{vmatrix}. && (1.67)\end{aligned}$$

Also, for an arbitrary vector $\mathbf{w}$,

$$w_\ell\int dv r_\ell J_j = \frac{1}{2}\int dv\left[(\mathbf{r}\times\mathbf{J})\times\mathbf{w}\right]_j. \quad (1.68)$$

Pf:

$$\begin{aligned} \int dv\left[(\mathbf{r}\times\mathbf{J})\times\mathbf{w}\right]_j &= \int dv\epsilon_{jkl}(\mathbf{r}\times\mathbf{J})_k w_l && (1.69)\\ &= \int dv \underbrace{\epsilon_{jkl}\epsilon_{kmn}}_{=\delta_{lm}\delta_{jn}-\delta_{ln}\delta_{jm}} r_m J_n w_l \\ &= \int dv(r_l J_j w_l - r_j J_l w_l) && (1.70)\\ &= 2\int dv w_l r_l J_j, && (1.71)\end{aligned}$$

where we have switched the subscripts of $r_j J_l$ in the second term and used Eq. (1.13). Hence Eq. (1.68) follows. QED.

Replace w_l by $\nabla_l B_k$ (with a fixed k), then

$$\begin{aligned} \nabla_\ell B_k\int dv r_\ell J_j &= \frac{1}{2}\int dv\left[(\mathbf{r}\times\mathbf{J})\times\nabla B_k\right]_j \\ &= (\mathbf{m}\times\nabla)_j B_k. && (1.72)\end{aligned}$$

Thus Eq. (1.65) becomes

$$\mathbf{F} = (\mathbf{m}\times\nabla)\times\mathbf{B}. \quad (1.73)$$

With the help of

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\nabla\cdot\mathbf{b}+\mathbf{b}\nabla\cdot\mathbf{a} + (\mathbf{a}\times\nabla)\times\mathbf{b}+(\mathbf{b}\times\nabla)\times\mathbf{a}, \quad (1.74)$$

we have

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) \quad (1.75)$$
$$= -\nabla U, \quad (1.76)$$

where

$$U \equiv -\mathbf{m}\cdot\mathbf{B} \quad (1.77)$$

is the **magnetic dipole energy**.

1. Hyperfine structure

In an atom, such as the hydrogen atom, from the point of view of an orbiting electron, the nucleus is nearly a point since it is about 10^5 times smaller than the radius of the electron orbital. The magnetic field produced by the nucleus magnetic dipole moment $\mathbf{m}_N$ is (Eq. (1.37)),

$$\mathbf{B}_N(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}_N)-\mathbf{m}_N}{r^3}+\frac{2\mu_0}{3}\mathbf{m}_N\delta(\mathbf{r}). \quad (1.78)$$

An electron with dipole moment $\mathbf{m}_e$ would interact with $\mathbf{B}_N$. The Hamiltonian of the interaction is,

$$H_{HFS} = -\mathbf{m}_e\cdot\mathbf{B}_N(\mathbf{r}) \quad (1.79)$$
$$= -\frac{\mu_0}{4\pi}\frac{3(\hat{\mathbf{r}}\cdot\mathbf{m}_e)(\hat{\mathbf{r}}\cdot\mathbf{m}_N)-\mathbf{m}_e\cdot\mathbf{m}_N}{r^3} -\frac{2\mu_0}{3}\mathbf{m}_e\cdot\mathbf{m}_N\delta(\mathbf{r}). \quad (1.80)$$

The first term is the typical *dipole-dipole interaction*, and the second term is a *contact interaction*.

For an s-orbital $\psi(\mathbf{r})$, which is non-zero at the origin (not so for a p-orbital or other non-s-orbitals, which vanishes at the origin), the second term causes an energy shift,

$$\Delta E_{HFS} = \langle\psi|H_{HFS}|\psi\rangle \quad (1.81)$$
$$= -\frac{2\mu_0}{3}\mathbf{m}_e\cdot\mathbf{m}_N|\psi(0)|^2. \quad (1.82)$$

The expectation of the first term in H_{HFS} is zero since s-orbital is spherical. As a result of this contact interaction, spin-up and spin-down electrons have slightly different energy levels (Fig. 7(a)). This is the **hyperfine structure** in atomic spectroscopy.

For an electron in the $1s$ orbital of H atom, $\Delta E_{HFS} \simeq 5.89\times10^{-6}$ eV. The electron transition between this two energy levels emits a radio wave with wavelength 21 cm. This is the famous **21-centimeter line** in astrophysics that can help scientists mapping out the structure of the Galaxy (Fig. 7(b)).

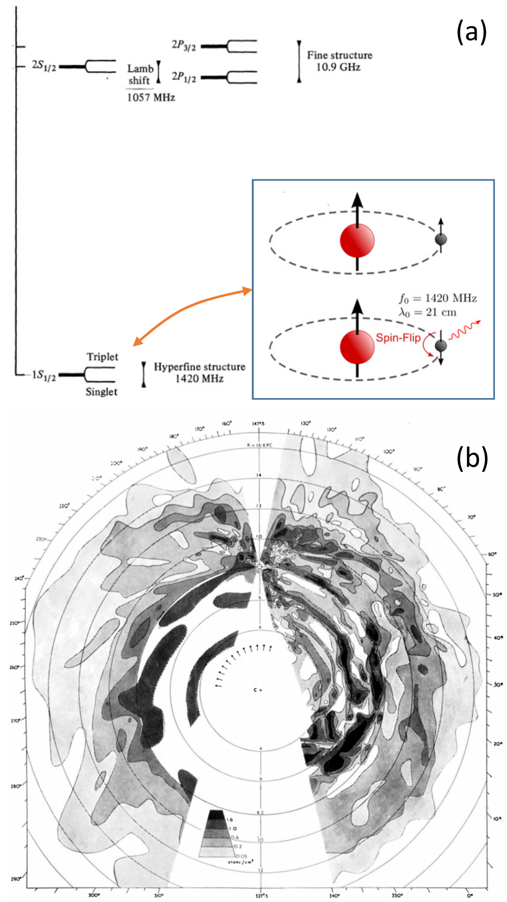

FIG. 7 (a) The hyperfine structure in hydrogen spectrum. (b) The global structure of the Galaxy determined by the 21 cm line. Fig. from Unknown, 1958.

In early times, some scientists thought that the magnetic moments in magnetic materials could be due to point magnetic charges, instead of tiny current loops (see Fig. 8). If so, then instead of magnetization current around the side surface, we should have magnetic charges on top and bottom surfaces, as in electric polarization.

However, from the calculation of hyperfine structure, we know that if the magnetic dipole is due to point charges, then the contact term should be $-\frac{\mu_0}{3}\mathbf{m}_N\delta(\mathbf{r})$, as in the case of electric dipole, instead of $+\frac{2\mu_0}{3}\mathbf{m}_N\delta(\mathbf{r})$. This would leads to a hyperfine splitting *half* of the present value, and in turn produces 42-cm hydrogen line (which is not observed). Thus, there are no magnetic monopoles hidden inside tiny magnetic dipoles.

E. Macroscopic magnetizable medium

Consider a magnetic medium that is composed of small current loops. If the magnetic moment of the i-th element

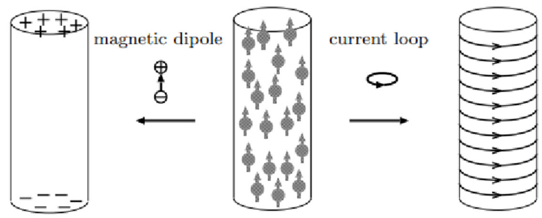

FIG. 8 Two possible scenarios of magnetic dipoles. Fig. from Kitano, 2006.

is $\mathbf{m}_i$, then we can define the **magnetization** as,

$$\mathbf{M}(\mathbf{r}') = \frac{\sum\limits_{i \text{ in } \Delta V} \mathbf{m}_i}{\Delta V}, \tag{1.83}$$

where ΔV is a volume element around $\mathbf{r}'$. The volume element is microscopically large but macroscopically small, so that there are many elements in ΔV, but it remains a point from human's point of view.

A volume element ΔV has magnetic moment $\mathbf{m} = \mathbf{M}\Delta V$ and produces a vector potential,

$$\Delta\mathbf{A}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi}\left[\frac{\mathbf{J}(\mathbf{r}')\Delta V}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}')\Delta V \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right].$$

After integration, we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int_V dv' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \int_V dv' \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right], \tag{1.84}$$

where V is the volume of the material. Write

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}, \tag{1.85}$$

use

$$\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f\nabla \times \mathbf{v}, \tag{1.86}$$

and integrate by parts, the second integral can be written as

$$\begin{aligned}\int_V dv' \mathbf{M}(\mathbf{r}') \times \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \int_V dv' \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &- \int_V dv' \nabla' \times \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right).\end{aligned}$$

We then use

$$\int_V dv \nabla \times \mathbf{v} = \int_S d\mathbf{s} \times \mathbf{v}, \tag{1.87}$$

where S is the boundary of V, and write

$$\int_V dv' \nabla' \times \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) = \int_S d\mathbf{s}' \times \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.88}$$

If follows that,

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V dv' \frac{\mathbf{J}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &+ \frac{\mu_0}{4\pi}\int_S ds' \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}}{|\mathbf{r}-\mathbf{r}'|}.\end{aligned} \tag{1.89}$$

The numerator of the first integral can be considered as an effective current density $\mathbf{J}_{eff} = \mathbf{J} + \mathbf{J}_m$, where

$$\mathbf{J}_m(\mathbf{r}) \equiv \nabla \times \mathbf{M}(\mathbf{r}) \tag{1.90}$$

is the **magnetization current density**. The numerator of the second integral is the **magnetization surface current density**,

$$\mathbf{K}_m(\mathbf{r}) \equiv \mathbf{M}(\mathbf{r}) \times \hat{\mathbf{n}}, \tag{1.91}$$

where $\mathbf{r}$ is located on the surface.

Note that instead of integrating over the material body V, we can also integrate over the whole space, then S is the surface at infinity, and the surface integral vanishes since $M = 0$ at infinity. These two choices of V give the same result of $\mathbf{A}$, since in the second choice, $\mathbf{J}_m$ would automatically pick up the surface current on boundary (see next subsection). In what follows, we prefer to integrate over the whole space, so that the surface integral in Eq. (1.89) can be dispensed with.

Now, since the current density in the volume integral above directly links with the one in Ampère's law (see Chap 2), we have

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{eff} = \mu_0 (\mathbf{J} + \nabla \times \mathbf{M}). \tag{1.92}$$

Introduce an H field,

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}, \tag{1.93}$$

$$\text{then} \quad \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}). \tag{1.94}$$

This is Ampère's law in material. The official term for $\mathbf{B}$ is **magnetic flux density**, which is in units of Tesla (N s/C m). The H field is called **magnetic field strength**, which is in units of A/m. For simplicity, we'll call them B field and H field, or simply magnetic field (for both) if the context is clear.

Note that

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \tag{1.95}$$

We call a magnet *simple* if it is isotropic and linear. That is, its $\mathbf{M}$ is proportional to and aligns with $\mathbf{H}$,

$$\mathbf{M} = \chi_m \mathbf{H}, \tag{1.96}$$

and

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H}, \tag{1.97}$$

where χ_m is the **magnetic susceptibility**, and μ the **magnetic permeability** of material.

For **paramagnetic material** ,

$$\mathbf{M} \parallel \mathbf{H}, \quad \text{and } \chi_m > 0. \tag{1.98}$$

For **diamagnetic material**,

$$\mathbf{M} \parallel -\mathbf{H}, \quad \text{and } \chi_m < 0. \tag{1.99}$$

The magnitude of χ_m is typically of the order of 10^{-5}. A simple magnet such as soft iron can have $\chi_m \sim 10^4$. The χ_m of hard ferromagnet materials can be as large as 10^6, but they are not simple magnets.

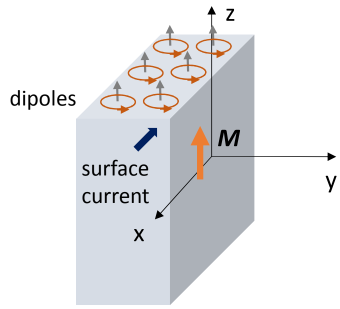

FIG. 9 Distribution of magnetic dipoles in the half-space $y < 0$.

1. Magnetization current

Non-uniform magnetization generates effective current, $\mathbf{J}_m = \nabla \times \mathbf{M}$. We'll use a simple example to illustrate this: In Fig. 9 there is a semi-infinite magnet with uniform magnetization,

$$\mathbf{M} = M_0 \theta(-y)\hat{z}. \tag{1.100}$$

Its magnetization current density is,

$$\mathbf{J}_m = \nabla \times \mathbf{M} = -M_0 \delta(y)\hat{x}. \tag{1.101}$$

That is, magnetization current flows only on the surface of the magnet. In the figure, molecular currents generate magnetic dipoles. Near the interface between neighboring current loops, the currents flow along opposite directions. Thus, there is no current inside the bulk, and only the outer-most current exposed.

Note that the magnetization currents in the example are bounded to molecules. They cannot flow away like conduction current in metals.

Since the magnetization current flows on a surface, we can describe it with surface current density $\mathbf{K}_m$,

$$\begin{aligned} \mathbf{K}_m = \int dy \mathbf{J}_m &= -\int dy M_0 \delta(y)\hat{\mathbf{x}} & (1.102)\\ &= -M_0\hat{\mathbf{x}} = \mathbf{M}_s \times \hat{\mathbf{n}}, & (1.103) \end{aligned}$$

where $\mathbf{M}_s$ is the magnetization on the surface.

2. Boundary condition

In previous chapter, we have learned about the boundary condition for magnetic field,

$$\begin{aligned} \hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) &= 0, & (1.104)\\ \hat{\mathbf{n}} \times (\mathbf{B}_2 - \mathbf{B}_1) &= \mu_0 \mathbf{K}. & (1.105) \end{aligned}$$

In the presence of magnetic materials, the boundary condition would depend on magnetization and needs be re-derived. Let's start from the integral form of the Maxwell equations,

$$\begin{aligned} \int_S d\mathbf{s} \cdot \mathbf{B} &= 0, & (1.106)\\ \oint_C d\mathbf{r} \cdot \mathbf{H} &= I, & (1.107) \end{aligned}$$

where I is the current flowing through C, *not including the magnetization current.*

As shown in Fig. 10(a), near the boundary surface, we can choose the S in Eq. (1.106) to be a small pillar box with area ds and *nearly zero thickness*, then

$$\int_S d\mathbf{s} \cdot \mathbf{B} \simeq \mathbf{B}_1 \cdot ds(-\hat{\mathbf{n}}) + \mathbf{B}_2 \cdot ds\hat{\mathbf{n}} = 0, \tag{1.108}$$

where $\hat{\mathbf{n}}$ points from region 1 to region 2. Hence, the normal components

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0, \tag{1.109}$$

which is the same as Eq. (1.104).

Choose the C in Eq. (1.107) to be a small rectangular loop perpendicular to the current flow. Suppose the loop has width d and *nearly zero height*, then

$$\oint_C d\mathbf{r} \cdot \mathbf{H} \simeq \mathbf{H}_1 \cdot (-\mathbf{d}) + \mathbf{H}_2 \cdot \mathbf{d} = I, \tag{1.110}$$

where $\mathbf{d} = d\hat{\mathbf{d}}$ and $\hat{\mathbf{d}} = \hat{\mathbf{J}} \times \hat{\mathbf{n}}$, as shown in figure. Hence

$$\begin{aligned} (\mathbf{H}_2 - \mathbf{H}_1) \cdot \hat{\mathbf{d}} &= K, & (1.111)\\ \text{or} \quad \hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) &= \mathbf{K}, & (1.112) \end{aligned}$$

which replaces Eq. (1.105).

F. Magnetostatic energy

The magnetostatic energy of a current distribution equals the total work required to assemble the current, starting from the state when there is no current. The increase of current leads to the increase of magnetic field, which induces an electric field $\mathbf{E}$ that interacts with the current.

For the reason above, even though we are discussing magneto*static* energy, Faraday's law needs be used,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.113}$$

It is assumed that the current builds up slowly so the process is *quasi-static*. The electromagnetic energy in a *dynamic* system will be discussed in Chap 15.

Suppose charge $\rho\Delta V$ is displaced by $\Delta\mathbf{r}$ due to fields, the mechanical work done *by electromagnetic field* on charged particles is,

$$\Delta w_m = \rho \Delta V (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \Delta\mathbf{r}. \tag{1.114}$$

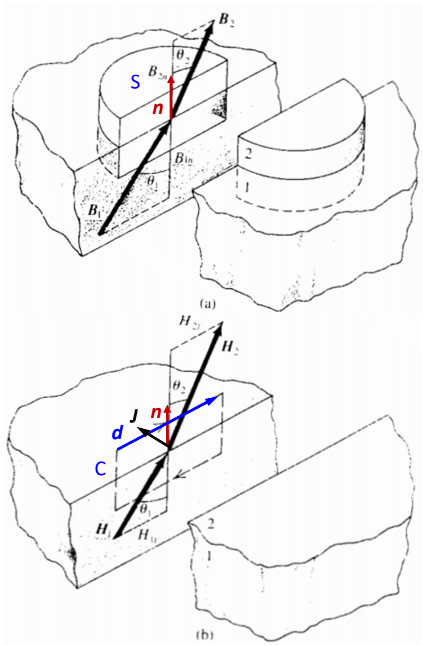

FIG. 10 (a) Gaussian surface for the flux of **B**. (b) Ampère loop for the circulation of **H**. Fig. from Lorrain and Corson, 1970.

The rate of total work done is

$$\frac{dW_m}{dt} = \int dv\rho(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\mathbf{v} \qquad (1.115)$$

$$= \int dv\mathbf{J}\cdot\mathbf{E}, \quad \mathbf{J}=\rho\mathbf{v}. \qquad (1.116)$$

Note that *the magnetic force does no work.*

According to **Lenz's law**, the induced field **E** *opposes* the increase of current. To build up the current, an external agent must provide W_{ext} to work *against* W_m. It is *this* external work that increases the energy U_B of the system.

For simplicity, consider a thin wire of loop C. Replace $dv\mathbf{J}$ with $Id\mathbf{r}$, then

$$\frac{dW_m}{dt} = I\oint_C d\mathbf{r}\cdot\mathbf{E} = I\mathcal{E}, \qquad (1.117)$$

where $\mathcal{E}$ is the **electromotive force** around C. The rate of external work is,

$$\frac{dW_{ext}}{dt} = -I\int_C d\mathbf{r}\cdot\mathbf{E} \qquad (1.118)$$

$$= -I\int_S d\mathbf{s}\cdot\nabla\times\mathbf{E} \qquad (1.119)$$

$$= I\int_S d\mathbf{s}\cdot\frac{\partial\mathbf{B}}{\partial t} \qquad (1.120)$$

$$= I\frac{d\Phi}{dt}, \qquad (1.121)$$

where S is a surface (not moving) bounded by C, and Φ is the magnetic flux through S. Finally, in a time δt,

$$\delta W_{ext} = \frac{dW_{ext}}{dt}\delta t = I\delta\Phi, \qquad (1.122)$$

hence

$$\delta U_B = \delta W_{ext} = I\delta\Phi. \qquad (1.123)$$

This is the first form of δU_B.

With Stoke's theorem, the magnetic flux can be written as,

$$\Phi = \int_S d\mathbf{s}\cdot\nabla\times\mathbf{A} = \oint_C d\mathbf{r}\cdot\mathbf{A}. \qquad (1.124)$$

The current is held fixed in δt, hence

$$\delta U_B = I\oint_C d\mathbf{r}\cdot\delta\mathbf{A}. \qquad (1.125)$$

For a general current distribution, replace $Id\mathbf{r}$ with $dv\mathbf{J}$, then

$$\delta U_B = \int dv\mathbf{J}\cdot\delta\mathbf{A}. \qquad (1.126)$$

This is the second form of δU_B.

We can also write U_B in terms of magnetic field. Recall that $\nabla\times\mathbf{H}$ equals J (not including the magnetization current), thus

$$\delta U_B = \int dv(\nabla\times\mathbf{H})\cdot\delta\mathbf{A}. \qquad (1.127)$$

Using

$$\nabla\cdot(\mathbf{u}\times\mathbf{v}) = (\nabla\times\mathbf{u})\cdot\mathbf{v} - (\nabla\times\mathbf{v})\cdot\mathbf{u}, \qquad (1.128)$$

then

$$\delta U_B = \int dv(\nabla\times\delta\mathbf{A})\cdot\mathbf{H} + \int dv\nabla\cdot(\mathbf{H}\times\delta\mathbf{A})$$

$$= \int dv\ \delta\mathbf{B}\cdot\mathbf{H}. \qquad (1.129)$$

The second integral can be converted to an integral over a boundary surface at infinity and vanishes when the field distribution is localized. This is the third form of δU_B.

Back to the first form of δU_B. Suppose the current (flux) increases from 0 to a final value I (Φ). In an intermediate state,

$$I(\lambda) = \lambda I,\ \text{and}\ \delta\Phi(\lambda) = \delta\lambda\Phi \quad (0\le\lambda\le 1), \qquad (1.130)$$

then

$$U_B = \int I(\lambda)\delta\Phi(\lambda) \qquad (1.131)$$

$$= \int_0^1 d\lambda\lambda\ I\Phi \qquad (1.132)$$

$$= \frac{1}{2}I\Phi. \qquad (1.133)$$

Similarly, the second form also has the factor 1/2,

$$U_B = \frac{1}{2}\int dv\mathbf{J}\cdot\mathbf{A}. \tag{1.134}$$

For the third form, in a simple magnet (such as a paramagnetic or a diamagnetic material), $\mathbf{B}$ is proportional to $\mathbf{H}$, and we can also have

$$U_B = \frac{1}{2}\int dv\mathbf{B}\cdot\mathbf{H}. \tag{1.135}$$

The integrand is the **energy density** for magnetic field,

$$u_B = \frac{1}{2}\mathbf{B}\cdot\mathbf{H} = \frac{\mu}{2}H^2. \tag{1.136}$$

However, for non-simple magnet (such as a ferromagnet), the original form in Eq. (1.129) needs be used to compute the change of energy step by step.
Problem:
1. Suppose a current distribution *outside* a sphere with volume V produces a magnetic field $\mathbf{B}(\mathbf{r})$. Show that the magnetic field averaged over the sphere (which has no current inside) equals the field at the center of the sphere,

$$\langle\mathbf{B}(\mathbf{r})\rangle_V \equiv \frac{1}{V}\int_V dv\mathbf{B}(\mathbf{r}) = \mathbf{B}(0).$$

2. A point magnetic dipole at the origin produces a magnetic field,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m})-\mathbf{m}}{r^3}, \quad r>0.$$

Suppose $\mathbf{m} = m\hat{\mathbf{z}}$. Show that the field averaged over a sphere centered at the origin is zero,

$$\int_V dv\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V dv\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m})-\mathbf{m}}{r^3} = 0.$$

REFERENCES

Jackson, J D (1998), *Classical Electrodynamics*, 3rd ed. (Wiley).

Kitano, Masao (2006), "Derivation of magnetic coulomb's law for thin, semi-infinite solenoids," arXiv:physics/0611099 [physics.class-ph].

Lorrain, Paul, and Dale Corson (1970), *Electromagnetic fields and waves* (W. H. Freeman).

Unknown, (1958), "Reports on the Progress of Astronomy the Galactic System as a Spiral Nebula," Monthly Notices of the Royal Astronomical Society **118** (4), 379–389.