

Lecture notes on classical electrodynamics

Ming-Che Chang

Department of Physics,
National Taiwan Normal University, Taipei,
Taiwan

(Dated: November 21, 2022)

CONTENTS

I. Magnetostatics	1
A. Introduction	1
1. Magnetic force	1
2. Thomson's theorem	2
B. Biot-Savart law	2
1. Solenoid	3
2. Solenoid with non-circular cross section	3
C. Ampère's law	4
D. Boundary condition for \mathbf{B}	6
1. Force on current sheet	6
E. Vector potential	6
F. Magnetic scalar potential	8
1. Potential of a current loop	8
2. Multi-valuedness of ψ	9
References	9

I. MAGNETOSTATICS

A. Introduction

There are several ways to find out a magnetic field. Given a current distribution, we can always use the Biot-Savart law,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V dv' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (1.1)$$

where $\mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2$. Alternatively, we can find out the vector potential using

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dv' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \quad (1.2)$$

then take its curl to find the field, $\mathbf{B} = \nabla \times \mathbf{A}$.

Two of the Maxwell equations govern the magnetostatic field,

$$\int_S d\mathbf{s} \cdot \mathbf{B}(\mathbf{r}) = 0, \quad (1.3)$$

$$\oint_C d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = \mu_0 I, \quad (1.4)$$

where I is the current flowing through loop C . If the current distribution has certain symmetry, then it is convenient to find out \mathbf{B} using the Ampère law in Eq. (1.4).

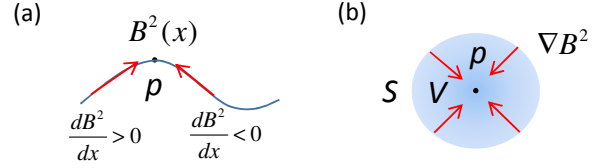


FIG. 1 (a) $B^2(x)$ and its slope in one dimension. (b) $B^2(\mathbf{r})$ and its gradient in three dimension.

The differential form of the Maxwell equations are,

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0, \quad (1.5)$$

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}. \quad (1.6)$$

Since a field without divergence can be written as a curl, the first equation implies $\mathbf{B} = \nabla \times \mathbf{A}$. Substitute it to the second equation, and recall that $\nabla \cdot \mathbf{A} = 0$ for steady current (see Chap 2), we have the **vector Poisson equation**,

$$\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r}). \quad (1.7)$$

It needs to be solved together with the boundary condition. Again, as in electrostatics, we will not use this approach in this course.

1. Magnetic force

A point charge q moving in a magnetic field \mathbf{B} feels a magnetic force, called the **Lorentz force**,

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \quad (1.8)$$

For a thin wire carrying a current I , each line element $d\mathbf{r}$ feels a Lorentz force,

$$d\mathbf{F} = I d\mathbf{r} \times \mathbf{B}. \quad (1.9)$$

For the whole wire, just integrate to have the total force,

$$\mathbf{F} = I \int_C d\mathbf{r} \times \mathbf{B}(\mathbf{r}). \quad (1.10)$$

For a general current distribution $\mathbf{J}(\mathbf{r})$, just replace $I d\mathbf{r}$ with $\mathbf{J}(\mathbf{r}) dv$, so that

$$\mathbf{F} = \int dv \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}). \quad (1.11)$$

Note that the field \mathbf{B} in the equation is an *external* one, not including the field produced by \mathbf{J} itself.

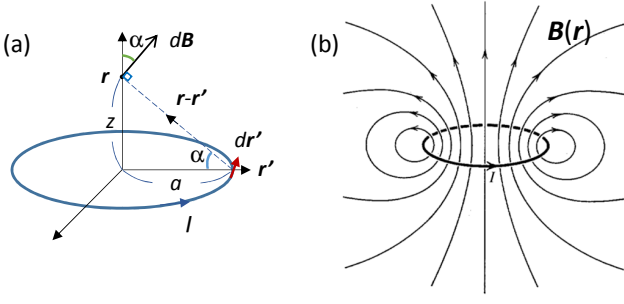


FIG. 2 (a) A ring with radius a and current I . (b) Distribution of magnetic field near a ring.

2. Thomson's theorem

In a region V without any current, a magnetic field $|\mathbf{B}(\mathbf{r})|$ can have local minimum, but *not* local maximum. *Pf.* We'll prove this by contradiction. Suppose $|\mathbf{B}(\mathbf{r})|$, or $B^2(\mathbf{r})$, has local maximum at a point p , then near the point, ∇B^2 points toward p (Fig. 1). Therefore, if we integrate it over a spherical surface S surrounding p , then

$$\int_S d\mathbf{s} \cdot \nabla B^2 < 0. \quad (1.12)$$

Using the divergence theorem,

$$\int_S d\mathbf{s} \cdot \mathbf{V} = \int_V dv \nabla \cdot \mathbf{V}, \quad (1.13)$$

we have

$$\int_S d\mathbf{s} \cdot \nabla B^2 = \int_S dv \nabla^2 B^2 \quad (1.14)$$

$$= \int_V \nabla_i \nabla_i B_j B_j. \quad (1.15)$$

The integrand

$$\nabla_i (\nabla_i B_j B_j) = \nabla_i [2B_j (\nabla_i B_j)] \quad (1.16)$$

$$= 2(\nabla_i B_j)^2 + 2B_j (\nabla^2 B_j). \quad (1.17)$$

The integral of the second term is zero (proved later), thus

$$\int_S d\mathbf{s} \cdot \nabla^2 B^2 = \int_V dv 2(\nabla_i B_j)^2 > 0. \quad (1.18)$$

This contradicts with Eq. (1.12). Thus the premise that $|\mathbf{B}(\mathbf{r})|$ has local maximum can't be valid. QED.

We now prove that the integral of the second term in Eq. (1.17) is zero. First, since there is no current inside V , $\nabla \times \mathbf{B} = 0$, thus

$$\nabla_i B_j = \nabla_j B_i. \quad (1.19)$$

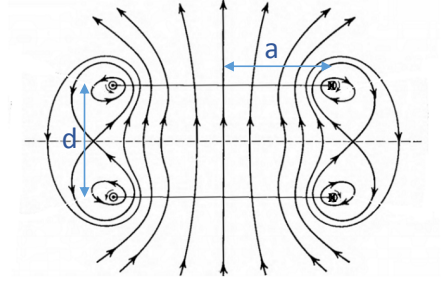


FIG. 3 The magnetic field at the center of a Helmholtz coil is nearly uniform. Fig. from Zangwill, 2013.

It follows that

$$\int_V dv B_j (\nabla^2 B_j) = \int_V dv B_j \nabla_i \underbrace{\nabla_i B_j}_{=\nabla_j B_i} \quad (1.20)$$

$$= \int_V dv B_j \nabla_j (\nabla \cdot \mathbf{B}) \quad (1.21)$$

$$= 0. \quad (1.22)$$

B. Biot-Savart law

Let's start with a classic example:

Find out the magnetic field along the central axis of a circular wire with radius a and current I .

Sol'n:

According to the Biot-Savart law, a line element $I d\mathbf{r}'$ generates

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I d\mathbf{r}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \quad (1.23)$$

$$= \frac{\mu_0}{4\pi} I d\mathbf{r}' \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \text{ along } d\mathbf{B}. \quad (1.24)$$

Note that $d\mathbf{r}' \perp \mathbf{r} - \mathbf{r}'$, and $d\mathbf{B}$ is shown in Fig. 2(a).

When one integrates over the circle, the horizontal component of $d\mathbf{B}$ vanishes, but $dB_z = dB \cos \alpha$ survives. Therefore,

$$\begin{aligned} B_z(z) &= \frac{\mu_0}{4\pi} \oint_C I d\mathbf{r}' \frac{\cos \alpha}{z^2 + a^2}, \quad \cos \alpha = \frac{a}{\sqrt{z^2 + a^2}} \\ &= \frac{\mu_0 I}{2} \frac{a^2}{(z^2 + a^2)^{3/2}}, \end{aligned} \quad (1.25)$$

and $\mathbf{B}(z) = B_z(z)\hat{\mathbf{z}}$. The field decreases as $1/z^3$ at large distance. The distribution of field lines is shown in Fig. 2(b).

A **Helmholtz coil** consists of two rings with the same radius, and the same magnitude and direction of current (Fig. 3). Along the central axis,

$$B_z(z) = \frac{\mu_0}{2} \left\{ \frac{Ia^2}{[(z - d/2)^2 + a^2]^{3/2}} + \frac{Ia^2}{[(z + d/2)^2 + a^2]^{3/2}} \right\}. \quad (1.26)$$

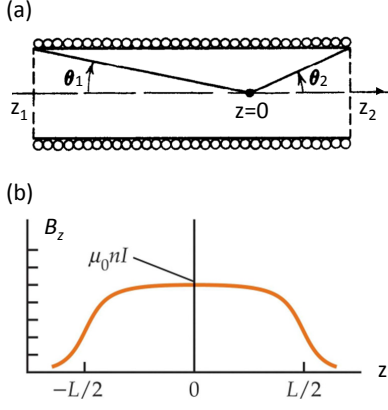


FIG. 4 (a) A solenoid with finite length. (b) The magnetic field along the central axis of the solenoid.

It can be shown that $dB_z(z)/dz = 0$ at the center ($z = 0$). Actually, from the symmetry of the Helmholtz coil, one can argue that the derivatives of odd orders at $z = 0$ should be zero. It is left as an exercise to show that when the separation between rings $d = a$, then $d^2 B_z(z)/dz^2|_{z=0} = 0$. Thus, the first non-zero derivative is of the fourth order, $d^4 B_z(z)/dz^4$. As a result, the magnetic field is nearly uniform at the center of the Helmholtz coil with $d = a$.

1. Solenoid

Consider a solenoid with finite length L (Fig. 4(a)). It has a uniform **surface current density** (current per unit length) $K = nI$, where n is the number of coils per unit length. Let's find out the magnetic field $\mathbf{B}(z)$ along the central axis inside the solenoid. The observation point is set as the origin of the coordinate. A slice of the solenoid with width dz has current $dI = Kdz$, which produces a magnetic field at the origin (see Eq. (1.25)),

$$dB_z = \frac{\mu_0}{2} K dz \frac{a^2}{(z^2 + a^2)^{3/2}}. \quad (1.27)$$

Let $z = a \cot \theta$, then $dz = -a \csc^2 \theta d\theta$. Integrate over the whole solenoid to get,

$$B_z(z=0) = \frac{\mu_0}{2} K a^2 \int_{z_1}^{z_2} \frac{dz}{(z^2 + a^2)^{3/2}} \quad (1.28)$$

$$= -\frac{\mu_0}{2} K \int_{\pi-\theta_1}^{\theta_2} \sin \theta d\theta \quad (1.29)$$

$$= \frac{\mu_0}{2} K (\cos \theta_1 + \cos \theta_2). \quad (1.30)$$

The dependence of B_z on z is shown in Fig. 4(b). If the solenoid has infinite length, then $\theta_1, \theta_2 \rightarrow 0$, and

$$B_z(z) = \mu_0 K = \mu_0 n I, \quad (1.31)$$

where n is the density of coils per unit length.

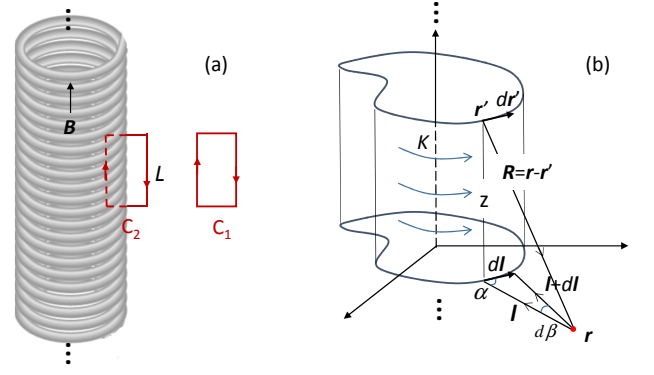


FIG. 5 (a) A solenoid with infinite length. (b) An infinite solenoid with a non-circular cross section that is uniform along its length.

For the infinite solenoid, due to the translation symmetry along the z -axis, we expect the magnetic field to be uniform along z and directs along the z -direction, $\mathbf{B}(\mathbf{r}) = B_z(\rho)\hat{\mathbf{z}}$, both inside and outside the solenoid. We can find out the magnetic field easily using Ampère's law. First, choose the loop C_1 in Fig. 5(a). Since there is no current flowing through C_1 , hence

$$\oint_{C_1} d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = 0. \quad (1.32)$$

Because the choice of C_1 is arbitrary (as long as it is outside), the magnetic field outside must be a constant and can only be zero.

Next choose the loop C_2 , then

$$\oint_{C_2} d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = B_z(\rho)L = \mu_0 I. \quad (1.33)$$

Thus, $B_z(\rho) = \mu_0 K$ is independent of ρ inside the solenoid. Note that the derivation that leads to Eq. (1.31) applies only to the magnetic field along the central axis, while the derivation here applies to any location inside the solenoid.

2. Solenoid with non-circular cross section

Consider a solenoid with infinite length but arbitrary cross section, as shown in Fig. 5(b). The cross section is uniform along its length. The surface current density $\mathbf{K}(\mathbf{r})$ is uniform and flows horizontally. Consider a point \mathbf{r} outside or inside the solenoid. From the Biot-Savart law,

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} K dz d\mathbf{r}' \times \frac{\mathbf{R}}{R^3}, \quad \mathbf{R} = \mathbf{r} - \mathbf{r}' \quad (1.34)$$

in which we have replaced $Id\mathbf{r}'$ with $(Kdz)d\mathbf{r}'$. From the geometry in Fig. 5(b), we can see that

$$d\mathbf{r}' = d\boldsymbol{\ell}, \quad (1.35)$$

$$\mathbf{R} + \boldsymbol{\ell} = -z\hat{\mathbf{z}}, \quad (1.36)$$

$$\text{and } R^2 = z^2 + \ell^2. \quad (1.37)$$

Thus,

$$d\mathbf{r}' \times \frac{\mathbf{R}}{R^3} = -\frac{d\boldsymbol{\ell} \times \boldsymbol{\ell}}{R^3} - \frac{d\boldsymbol{\ell} \times z\hat{\mathbf{z}}}{R^3}. \quad (1.38)$$

After integration,

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= -\frac{\mu_0}{4\pi} K \int_{-\infty}^{\infty} dz \oint \left(\frac{\boldsymbol{\ell} \times d\boldsymbol{\ell}}{R^3} + \frac{z\hat{\mathbf{z}} \times d\boldsymbol{\ell}}{R^3} \right) \quad (1.39) \\ &= -\frac{\mu_0}{4\pi} K \oint \left(\boldsymbol{\ell} \times d\boldsymbol{\ell} \int_{-\infty}^{\infty} \frac{dz}{R^3} + \hat{\mathbf{z}} \times d\boldsymbol{\ell} \int_{-\infty}^{\infty} dz \frac{z}{R^3} \right). \end{aligned}$$

The second integral is zero; the first integral equals $2/\ell^2$. Thus,

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{2\pi} K \oint \frac{\boldsymbol{\ell} \times d\boldsymbol{\ell}}{\ell^2}. \quad (1.40)$$

Note that

$$|\boldsymbol{\ell} \times d\boldsymbol{\ell}| = \ell d\ell \sin \alpha, \quad (1.41)$$

and

$$d\ell \sin \alpha = |\boldsymbol{\ell} + d\boldsymbol{\ell}| \sin d\beta \simeq \ell d\beta. \quad (1.42)$$

Hence,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{2\pi} K \oint d\beta \hat{\mathbf{z}} \quad (1.43)$$

$$= \begin{cases} \mu_0 K \hat{\mathbf{z}} & \text{inside the solenoid} \\ 0 & \text{outside the solenoid} \end{cases} \quad (1.44)$$

Note that in a real solenoid, the current cannot be purely azimuthal since as a whole it needs to flow forward along the central axis. When we take this into account, the magnetic field would have certain azimuthal component B_ϕ .

C. Ampère's law

If the distribution of current has a simple symmetry, then we can use the integral form of the Ampère's law to find out the magnetic field.

Example:

Suppose there is a straight wire with infinite length lying along the z -axis. It has a cylindrical shape with radius a and carries a uniform current I . Find out the magnetic field generated by this wire.

Sol'n:

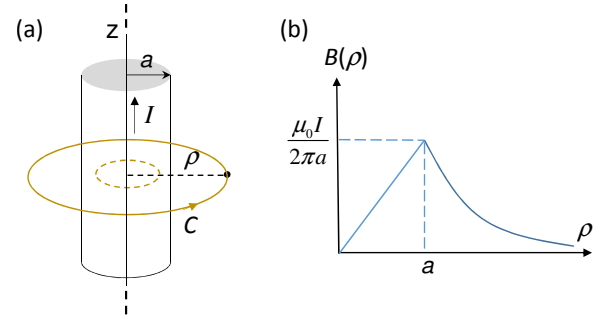


FIG. 6 (a) A cylindrical wire along the z -axis. (b) The magnetic field inside and outside the wire.

Let's choose the cylindrical coordinate. The system is invariant if you rotate around z -axis, or translate along z -axis, so the magnetic field cannot depend on ϕ, z . It follows that,

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\rho) = B_\rho(\rho)\hat{\boldsymbol{\rho}} + B_\phi(\rho)\hat{\boldsymbol{\phi}} + B_z(\rho)\hat{\mathbf{z}}. \quad (1.45)$$

From Ampère's right-hand rule, we expect the magnetic field to be along $\hat{\boldsymbol{\phi}}$, so

$$\mathbf{B}(\mathbf{r}) = B_\phi(\rho)\hat{\boldsymbol{\phi}}. \quad (1.46)$$

You may reach the same conclusion with a more detailed analysis of the Biot-Savart integral.

Choose a loop C with radius ρ around the wire (Fig. 6(a)), then

$$\oint_C d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = \mu_0 I(\rho), \quad (1.47)$$

$$\rightarrow 2\pi\rho B_\phi(\rho) = \mu_0 I(\rho), \quad (1.48)$$

where $I(\rho)$ is the current passing through the circle C ,

$$I(\rho) = \begin{cases} I \frac{\rho^2}{a^2} & \text{if } \rho < a \\ I & \text{if } \rho > a \end{cases}. \quad (1.49)$$

Thus (Fig. 6(b)),

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \frac{\mu_0 I \rho}{2\pi a} \hat{\boldsymbol{\phi}} & \text{if } \rho < a \\ \frac{\mu_0 I}{2\pi \rho} \hat{\boldsymbol{\phi}} & \text{if } \rho > a \end{cases} \quad (1.50)$$

Example:

In Fig. 7(a), a hollow cylindrical can with radius R and height L has a wire at its center. A current I flows up the wire, spreads out, flows down, converges at the bottom of the wire and flows up again.

(a) Using cylindrical coordinate, argue that the magnetic field has the following form everywhere, both inside and outside the can,

$$\mathbf{B}(\mathbf{r}) = B(\rho, z)\hat{\boldsymbol{\phi}}. \quad (1.51)$$

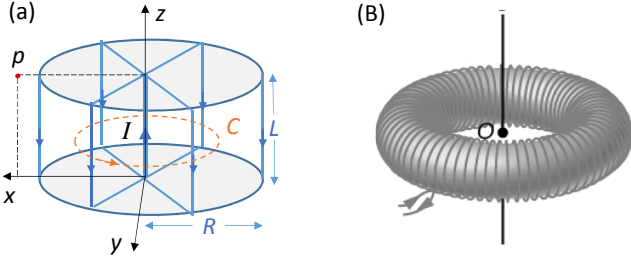


FIG. 7 (a) A hollow can with a wire inside along its central axis. (b) A toroidal solenoid.

(b) Find out $B(\rho, z)$.

Sol'n:

(a) Since the system is invariant with respect to the rotation around the wire (z -axis), so the magnetic field cannot depend on ϕ ,

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\rho, z) \quad (1.52)$$

$$= B_\rho(\rho, z)\hat{\rho} + B_\phi(\rho, z)\hat{\phi} + B_z(\rho, z)\hat{z}. \quad (1.53)$$

There is no obvious reason to rule out certain component of \mathbf{B} . But from a detailed analysis of the Biot-Savart law, we can show that the field \mathbf{B} is circular and has only the $\hat{\phi}$ component:

First, align the x -axis with the direction of observation point p , which can be inside or outside the can (Fig. 7(a)). In general,

$$\mathbf{J}(\mathbf{r}') = J_x\hat{x} + J_y\hat{y} + J_z\hat{z}, \quad (1.54)$$

$$\mathbf{r} - \mathbf{r}' = (x - x')\hat{x} + (0 - y')\hat{y} + (z - z')\hat{z}. \quad (1.55)$$

The distribution of current has a mirror symmetry with respect to the x - z plane. So for a current element $\mathbf{J}(\mathbf{r}')dv'$, there is a mirror counterpart $\tilde{\mathbf{J}}(\tilde{\mathbf{r}}')dv'$, with

$$\tilde{\mathbf{J}} = (J_x, -J_y, J_z), \text{ and } \tilde{\mathbf{r}}' = (x', -y', z'). \quad (1.56)$$

The magnetic field produced by this pair of current elements is

$$d\mathbf{B} \sim \mathbf{J} \times (\mathbf{r} - \mathbf{r}') + \tilde{\mathbf{J}} \times (\mathbf{r} - \tilde{\mathbf{r}}') \quad (1.57)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ J_x & J_y & J_z \\ x - x' & -y' & z - z' \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ J_x & -J_y & J_z \\ x - x' & +y' & z - z' \end{vmatrix} \\ = \begin{vmatrix} \hat{x} & 2\hat{y} & \hat{z} \\ J_x & 0 & J_z \\ x - x' & 0 & z - z' \end{vmatrix} \sim \hat{y}. \quad (1.58)$$

Thus, after integration, $\mathbf{B} \sim \hat{y} = \hat{\phi}$.

(b) After the form of $\mathbf{B}(\mathbf{r})$ has been narrowed down, it's easy to evaluate the Ampère integral. Choose the path C to be a horizontal circle with radius ρ , then

$$\oint_C d\mathbf{r} \cdot \mathbf{B} = 2\pi\rho B(\rho, z) = \begin{cases} \mu_0 I & \text{if } C \text{ is inside the can} \\ 0 & \text{if } C \text{ is outside the can} \end{cases} \quad (1.59)$$

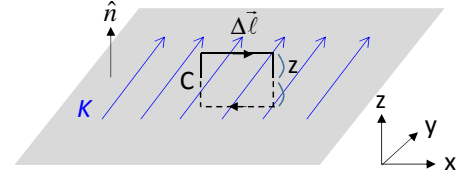


FIG. 8 An infinite plate on the x - y plane with a uniform sheet of current $\mathbf{K} = K\hat{y}$.

Thus, inside the can,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi\rho} \hat{\phi}. \quad (1.60)$$

There is no magnetic field outside the can.

Note that the same argument applies to other systems with azimuthal symmetry and radial current flow, and their magnetic fields must be circular. For example, for the toroidal solenoid in Fig. 7(b), the magnetic field inside is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 N I}{2\pi\rho} \hat{\phi}, \quad (1.61)$$

where N is the number of coils. There is no magnetic field outside the solenoid.

Example:

There is a thin plate on the x - y plane with a uniform current. Its current per unit length, or surface current density $\mathbf{K} = K\hat{y}$ (Fig. 8). Find out the magnetic field on both sides of the plate.

Sol'n:

Since the plate can be considered as a collection of wires along y direction, according to Ampère's right-hand rule, we expect the magnetic field to be along the x -axis (see Zangwill, 2013 for a detailed analysis based on symmetry),

$$\mathbf{B}(\mathbf{r}) = B(z)\hat{x}. \quad (1.62)$$

Choose a small rectangular loop C with a surface normal parallel to \mathbf{K} , as shown in Fig. 8. The current passing through C is $I_\square = K\Delta\ell$. Thus, the circulation

$$\oint_C d\mathbf{r} \cdot \mathbf{B} = \mathbf{B}_+ \cdot \Delta\ell + \mathbf{B}_- \cdot (-\Delta\ell) = \mu_0 I_\square, \quad (1.63)$$

where \mathbf{B}_+ (\mathbf{B}_-) is the field above (below) the plane. We expect $\mathbf{B}_- = -\mathbf{B}_+$, thus

$$\mathbf{B}_+ = +\frac{\mu_0}{2} K\hat{x} \text{ or } \frac{\mu_0}{2} \mathbf{K} \times \hat{n}, \quad K = \frac{I_\square}{\Delta\ell} \quad (1.64)$$

$$\mathbf{B}_- = -\frac{\mu_0}{2} K\hat{x} \text{ or } -\frac{\mu_0}{2} \mathbf{K} \times \hat{n}, \quad (1.65)$$

in which \hat{n} points up. The magnetic field is uniform and does not decrease with distance z .

D. Boundary condition for \mathbf{B}

In general, the magnetic fields on opposite sides of a current sheet are not the same. Their difference is caused by the current on the surface. Suppose a surface has surface current density $\mathbf{K}(\mathbf{r})$. At a point \mathbf{r} on the surface, the magnetic fields on opposite sides are $\mathbf{B}_1(\mathbf{r})$ and $\mathbf{B}_2(\mathbf{r})$ (Fig. 9). What's the relation between these two magnetic fields?

First, divide the surface S into a small rectangle \square and a surface S' (S with \square removed),

$$S = \square + S'. \quad (1.66)$$

The rectangle is microscopically large, but macroscopically small (say, with a size of $1 \mu\text{m}$). It can be considered as flat since it is just a small part of the smooth surface S . The field, $\mathbf{B}_1(\mathbf{r})$ or $\mathbf{B}_2(\mathbf{r})$, is the superposition of the fields produced by \square and S' .

When one infinitesimally approaches the center of the rectangle, the field is close to the field of an infinite plane, $\mathbf{B}(\mathbf{r}) = \pm \frac{\mu_0}{2} \mathbf{K} \times \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points from region 1 to region 2. Suppose the field produced by S' is \mathbf{B}_S , then

$$\mathbf{B}_1 = \mathbf{B}_S - \frac{\mu_0}{2} \mathbf{K} \times \hat{\mathbf{n}}, \quad (1.67)$$

$$\mathbf{B}_2 = \mathbf{B}_S + \frac{\mu_0}{2} \mathbf{K} \times \hat{\mathbf{n}}. \quad (1.68)$$

Even though \mathbf{B}_S remains unknown, we can subtract the field to get

$$\mathbf{B}_2(\mathbf{r}) - \mathbf{B}_1(\mathbf{r}) = \mu_0 \mathbf{K}(\mathbf{r}) \times \hat{\mathbf{n}}. \quad (1.69)$$

This is the BC for fields near a current sheet. Sometimes it is written as,

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0, \quad (1.70)$$

$$\hat{\mathbf{n}} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mu_0 \mathbf{K}. \quad (1.71)$$

1. Force on current sheet

To find out the magnetic force on a current sheet, divide the surface S into \square and S' , as in previous section. The rectangle exerts no force on itself. So the force is due to the magnetic field produced by S' ,

$$\mathbf{F}_\square = I_\square \Delta \mathbf{L} \times \mathbf{B}_S \quad (1.72)$$

$$= (\mathbf{K} \Delta \ell) \Delta L \times \mathbf{B}_S, \quad (1.73)$$

where I_\square is the current passing through \square (Fig. 9). Since

$$\mathbf{B}_S = \frac{1}{2}(\mathbf{B}_1 + \mathbf{B}_2), \quad (1.74)$$

so the force density (or pressure)

$$\mathbf{f}_\square \equiv \frac{\mathbf{F}_\square}{\Delta \ell \Delta L} = \frac{1}{2} \mathbf{K} \times (\mathbf{B}_1 + \mathbf{B}_2). \quad (1.75)$$

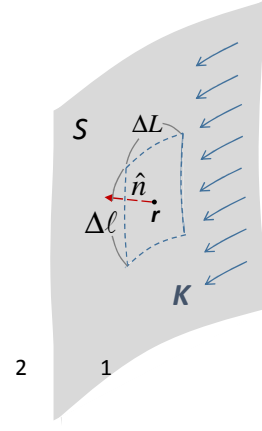


FIG. 9 A surface S is divided into a small rectangle and a surface without the rectangle, S' .

For example, for a long solenoid, the magnetic field inside and outside are (see Eq. (1.44))

$$\mathbf{B}_1 = \mu_0 K \hat{\mathbf{z}}, \quad \mathbf{B}_2 = 0. \quad (1.76)$$

So the magnetic pressure on the wall of the solenoid is

$$\mathbf{f} = \mu_0 \frac{K^2}{2} \hat{\mathbf{n}}, \quad (1.77)$$

where $\hat{\mathbf{n}}$ points outward.

For example, for a solenoid with $I = 0.1 \text{ A}$ and coil density $a = 10^3/\text{m}$, its surface current density $K = 100 \text{ A/m}$. The magnetic pressure on the wall $\mathbf{f} = 2\pi \times 10^{-3} \text{ N/m}^2$.

E. Vector potential

Assume that two vector potentials differ by a gradient $\nabla \chi(\mathbf{r})$,

$$\mathbf{A}' = \mathbf{A} + \nabla \chi. \quad (1.78)$$

Since $\nabla \times \nabla \chi(\mathbf{r}) = 0$ for any scalar function $\chi(\mathbf{r})$ without singularity, so \mathbf{A}' and \mathbf{A} yield the same magnetic field \mathbf{B} . That is, one magnetic field can have different vector potentials. This is called **gauge degree of freedom**.

For example, given $\mathbf{B}(\mathbf{r}) = B_0 \hat{\mathbf{z}}$, then its vector potential can be

$$\mathbf{A}(\mathbf{r}) = B_0(0, x, 0), \quad (1.79)$$

$$\text{or } \mathbf{A}(\mathbf{r}) = \frac{B_0}{2}(-y, x, 0). \quad (1.80)$$

They differ by the gradient in Eq. (1.78) with $\chi(\mathbf{r}) = -B_0 xy/2$.

With the help of χ , one can demand the vector potential to satisfy the **Coulomb gauge**,

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = 0. \quad (1.81)$$

Pf: Suppose $\nabla \cdot \mathbf{A} \neq 0$, then we can choose a $\chi(\mathbf{r})$ such that

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} + \nabla^2 \chi = 0. \quad (1.82)$$

What we need is a χ that satisfies

$$\nabla^2 \chi(\mathbf{r}) = -\nabla \cdot \mathbf{A}(\mathbf{r}). \quad (1.83)$$

The RHS is like the source term of the Poisson equation in electrostatics, and in principle a solution $\chi(\mathbf{r})$ always exists. Thus, we can always have $\nabla \cdot \mathbf{A}' = 0$. QED.

Usually, all we need to know is that χ exists. It is not necessary to actually find out $\chi(\mathbf{r})$.

Note: In Chap 2, we have shown that Eq. (1.81) is always valid for steady current. But its validity extends to dynamic field, as we will show in a later chapter.

We can write Ampère's law in terms of the vector potential,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (1.84)$$

$$\rightarrow \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \quad (1.85)$$

With the Coulomb gauge $\nabla \cdot \mathbf{A}(\mathbf{r}) = 0$, we have the vector Poisson equation,

$$\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r}). \quad (1.86)$$

Each component of Eq. (1.86) is a scalar Poisson equation. Thus, it has the formal solution,

$$A_i = \frac{\mu_0}{4\pi} \int dv' \frac{J_i(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \quad i = x, y, z \quad (1.87)$$

$$\text{or } \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dv' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \quad (1.88)$$

which is consistent with Eq. (1.2). For a thin wire, just replace $dv' \mathbf{J}$ with $I d\mathbf{r}'$, and

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}. \quad (1.89)$$

Example:

Find out the vector potential of a wire that is straight, infinite, and carries a current I .

Sol'n:

Adopt the cylindrical coordinate, and lay the wire along z -axis (Fig. 10(a)). In the integral,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}, \quad (1.90)$$

$d\mathbf{r}' = dz \hat{\mathbf{z}}$, thus $\mathbf{A}(\mathbf{r}) \parallel \hat{\mathbf{z}}$.

With the help of

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(2x + 2\sqrt{x^2 + a^2}), \quad (1.91)$$

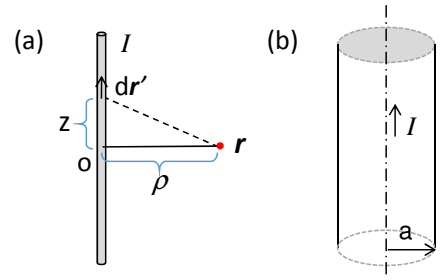


FIG. 10 (a) A thin wire with a current. (b) A cylindrical wire with a uniform current inside.

for a wire with length $2L$, we have

$$A_z(\rho) = \frac{\mu_0 I}{4\pi} \int_{-L}^L \frac{dz}{\sqrt{z^2 + \rho^2}} \quad (1.92)$$

$$= \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{1 + (\rho/L)^2} + 1}{\sqrt{1 + (\rho/L)^2} - 1}. \quad (1.93)$$

If $L \gg \rho$, then

$$\ln \frac{\sqrt{1 + (\rho/L)^2} + 1}{\sqrt{1 + (\rho/L)^2} - 1} \simeq \ln 4 - 2 \ln \frac{\rho}{L}, \quad (1.94)$$

hence

$$A_z(\rho) \simeq -\frac{\mu_0 I}{2\pi} \ln \rho + \text{const.} \quad (1.95)$$

It diverges when $\rho \rightarrow 0$ (and at infinity). Finally, it's not difficult to show that,

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 I}{2\pi \rho} \hat{\phi}. \quad (1.96)$$

Example:

Find out the vector potential of a straight, infinite cylindrical wire with radius a (Fig. 10(b)). The wire carries a uniform current I .

Sol'n:

When the wire has a finite radius, the divergence of \mathbf{A} as $\rho \rightarrow 0$ can be avoided. However, it's no longer convenient to use the integral formula in Eq. (1.88). Thus we will use the vector Poisson equation instead.

First, since $\mathbf{J}(\mathbf{r}) = J_0 \hat{\mathbf{z}}$, where $J_0 = I/\pi a^2$ is a constant, Eq. (1.88) tells us that $\mathbf{A}(\mathbf{r}) = A_z(\mathbf{r}) \hat{\mathbf{z}}$. Furthermore, we expect $A_z(\mathbf{r}) = A_z(\rho)$, thus $\nabla \cdot \mathbf{A} = 0$ is automatically satisfied, and

$$\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r}). \quad (1.97)$$

Since

$$\nabla^2 A_z(\mathbf{r}) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial A_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2}, \quad (1.98)$$

it follows that

$$\frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{dA_z}{d\rho} \right) = -\mu_0 J_0. \quad (1.99)$$

Direct integration gives

$$A_z(\rho) = -\frac{\mu_0}{4}J_0\rho^2 + C\ln\rho + \text{constant} \quad (1.100)$$

$$= \begin{cases} -\frac{\mu_0}{4}J_0\rho^2 + D & \text{for } \rho \leq a, \\ +C\ln\rho + D' & \text{for } \rho \geq a. \end{cases} \quad (1.101)$$

Some terms have been dropped to avoid unphysical divergence.

The vector potential needs be continuous at $\rho = a$ (otherwise the magnetic field would diverge there). This gives

$$A_z(\rho) = \begin{cases} -\frac{\mu_0}{4}J_0\rho^2 + D & \text{for } \rho \leq a, \\ -\frac{\mu_0}{4}J_0a^2 + C\ln\frac{\rho}{a} + D & \text{for } \rho \geq a. \end{cases} \quad (1.102)$$

We can ignore the constant D , but C is still unknown.

To find C , we require that the curl of \mathbf{A} be continuous across the boundary. That is,

$$\mathbf{B}_{out} - \mathbf{B}_{in} = 0. \quad (1.103)$$

This so because the surface current density is zero, $\mathbf{K} = 0$, for a boundary layer that is infinitely thin. Now

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = -\frac{dA_z}{d\rho}\hat{\phi}. \quad (1.104)$$

Match \mathbf{B}_{in} and \mathbf{B}_{out} at the boundary to get $C = -\frac{\mu_0}{2}J_0a^2$.

Finally, drop D to get

$$A_z(\rho) = \begin{cases} -\frac{\mu_0}{4}J_0\rho^2 & \text{for } \rho \leq a, \\ -\frac{\mu_0}{4}J_0a^2[1 + 2\ln(\rho/a)] & \text{for } \rho \geq a. \end{cases} \quad (1.105)$$

It's not difficult to see that

$$B_\phi(\rho) = \begin{cases} \frac{\mu_0 I}{2\pi} \frac{\rho}{a^2} & \text{for } \rho \leq a, \\ \frac{\mu_0 I}{2\pi\rho} & \text{for } \rho \geq a. \end{cases} \quad (1.106)$$

This agrees with the result in Eq. (1.50), which was obtained by a simpler approach.

F. Magnetic scalar potential

Since a vector field with zero curl can be written as a gradient, for a static magnetic field *in vacuum* with $\nabla \times \mathbf{B} = 0$, one can write

$$\mathbf{B}(\mathbf{r}) = -\nabla\psi(\mathbf{r}), \quad (1.107)$$

where ψ is the magnetic scalar potential. Combined with the equation $\nabla \cdot \mathbf{B} = 0$, we have

$$\nabla^2\psi(\mathbf{r}) = 0. \quad (1.108)$$

Unlike the vector potential, the magnetic scalar potential is not applicable to dynamic magnetic field.

1. Potential of a current loop

Suppose a magnetic field is generated from a loop of thin wire C with current I . From Biot-Savart law,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}I \oint_C d\mathbf{r}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (1.109)$$

With the help of the identity,

$$\oint_C d\mathbf{r}' \times \mathbf{V} = \int_S ds_k \nabla V_k - \int_S d\mathbf{s} \nabla \cdot \mathbf{V}, \quad (1.110)$$

where C is the boundary of S , we can write

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_S ds_k \nabla' \cdot \frac{(\mathbf{r} - \mathbf{r}')_k}{|\mathbf{r} - \mathbf{r}'|^3} - \frac{\mu_0 I}{4\pi} \int_S d\mathbf{s} \nabla' \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (1.111)$$

The integrand of the second term,

$$\nabla' \cdot \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \nabla' \cdot \nabla' \cdot \frac{1}{|\mathbf{r} - \mathbf{r}'|} \quad (1.112)$$

$$= -4\pi\delta(\mathbf{r} - \mathbf{r}'). \quad (1.113)$$

Thus the second term is zero as long as the observation point \mathbf{r} is not *on* the surface S . For the first term, switch ∇' to ∇ (getting a minus sign), then

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi} \nabla \int_S d\mathbf{s} \cdot \underbrace{\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}}_{=-d\Omega_S} \quad (1.114)$$

$$= \frac{\mu_0 I}{4\pi} \nabla \Omega_S(\mathbf{r}), \quad (1.115)$$

where $\Omega_S(\mathbf{r})$ is the solid angle of S with respect to the observation point \mathbf{r} . Therefore, the magnetic scalar potential

$$\psi(\mathbf{r}) = -\frac{\mu_0 I}{4\pi} \Omega_S(\mathbf{r}). \quad (1.116)$$

Take the ring in Fig. 2(a) as an example. Note that the current flows counter-clockwise, hence the normal vector of S points up, instead of pointing down, away from the observation point. As a result, there is an extra minus sign in Ω_S , and

$$\begin{aligned} \Omega_S &= -2\pi \left[1 - \cos\left(\frac{\pi}{2} - \alpha\right) \right], \quad \sin\alpha = \frac{z}{\sqrt{z^2 + a^2}} \\ &= -2\pi \left(1 - \frac{z}{\sqrt{z^2 + a^2}} \right) \end{aligned} \quad (1.117)$$

Taking the gradient of Ω_S to obtain

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \frac{d}{dz} \Omega_S(z) \hat{\mathbf{z}} = \frac{\mu_0 I}{2} \frac{a^2}{(z^2 + a^2)^{3/2}} \hat{\mathbf{z}}. \quad (1.118)$$

This agrees with the result in Eq. (1.25).

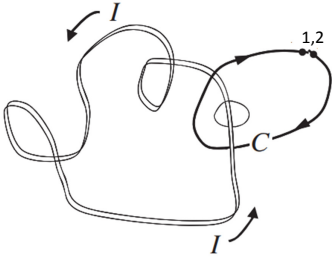


FIG. 11 A current pierces through a surface S bounded by C . Fig. from Zangwill, 2013

2. Multi-valuedness of ψ

It is known that for static electric field,

$$\oint_C d\mathbf{r} \cdot \mathbf{E}(\mathbf{r}) = 0. \quad (1.119)$$

This implies that the potential difference,

$$\psi(\mathbf{r}_2) - \psi(\mathbf{r}_1) = - \int_1^2 d\mathbf{r} \cdot \mathbf{E}(\mathbf{r}), \quad (1.120)$$

is independent of the path of the integral from point-1 to point-2.

For a static magnetic field, however, the loop integral of \mathbf{B} may not be zero. Thus, if one moves from \mathbf{r}_1 to $\mathbf{r}_2 =$

\mathbf{r}_1 around a loop C that encloses a current I (Fig. 11), then

$$\psi(\mathbf{r}_2) - \psi(\mathbf{r}_1) = - \oint_C d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = \pm \mu_0 I. \quad (1.121)$$

That is, ψ is not single-valued. To prevent it from having multiple values at the same location, we can refrain the path C from crossing the surface bounded by the current loop (Zangwill, 2013).

Problem:

1. Along the central axis of a Helmholtz coils (Fig. 3),

$$B_z(z) = \frac{\mu_0}{2} \left\{ \frac{Ia^2}{[(z - d/2)^2 + a^2]^{3/2}} + \frac{Ia^2}{[(z + d/2)^2 + a^2]^{3/2}} \right\}.$$

- (a) Show that $dB_z(z)/dz = 0$ at the center ($z = 0$).
- (b) Argue that the derivatives of odd orders at $z = 0$ should be zero.
- (c) Show that when the separation between rings $d = a$, $d^2 B_z(z)/dz^2|_{z=0} = 0$.

REFERENCES

Zangwill, Andrew (2013), *Modern electrodynamics* (Cambridge Univ. Press, Cambridge).