

Lecture notes on classical electrodynamics

Ming-Che Chang

Department of Physics, National Taiwan Normal University, Taipei, Taiwan

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I. ELECTRIC MULTIPOLES

A. Multipole expansion

Electric multipoles are useful if 1). the charge distribution $\rho(\mathbf{r})$ is localized within a finite region, and 2). the location of observation is far away (Fig. 1).

In general, the electric potential is given as,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int dv' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (1.1)$$

If the condition above is satisfied, $r \gg r'$, then we can use the binomial expansion to have

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \simeq \frac{1}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{r^2} + \frac{1}{2r^3} [3(\hat{\mathbf{r}} \cdot \mathbf{r}')^2 - |\mathbf{r}'|^2]. \quad (1.2)$$

It follows that,

$$\begin{aligned} \phi(\mathbf{r}) &\simeq \frac{1}{4\pi\epsilon_0} \left[\int dv' \rho(\mathbf{r}') \right] \frac{1}{r} \\ &+ \frac{1}{4\pi\epsilon_0} \left[\int dv' \rho(\mathbf{r}') \mathbf{r}' \right] \cdot \frac{\mathbf{r}}{r^3} \\ &+ \frac{1}{4\pi\epsilon_0} \left[\frac{1}{2} \int dv' \rho(\mathbf{r}') (3r'_i r'_j - r'^2 \delta_{ij}) \right] \frac{r_i r_j}{r^5}. \end{aligned} \quad (1.3)$$

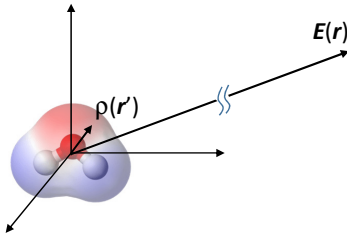


FIG. 1 Observe a localized charge distribution at a distance far away.

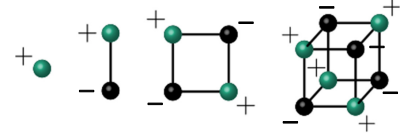


FIG. 2 From left to right, sets of point charges with electric monopole, dipole, quadrupole, and octupole.

The Einstein summation convention has been used. Inside the square brackets are electric monopole moment (electric charge), **electric dipole moment**, and **electric quadrupole moment**,

$$Q = \int dv' \rho(\mathbf{r}'), \quad (1.4)$$

$$\mathbf{p} = \int dv' \rho(\mathbf{r}') \mathbf{r}', \quad (1.5)$$

$$\Theta_{ij} = \frac{1}{2} \int dv' \rho(\mathbf{r}') (3r'_i r'_j - r'^2 \delta_{ij}). \quad (1.6)$$

Hence,

$$\phi(\mathbf{r}) \simeq \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} + \Theta_{ij} \frac{r_i r_j}{r^5} \right). \quad (1.7)$$

Note that Q , p_i and Θ_{ij} are simply sets of numbers, not functions of \mathbf{r} . Once these numbers are known for the charge distribution of interest, then its potential everywhere can be easily obtained from Eq. (1.7). The potentials of monopole, dipole, and quadrupole decrease with distance as $1/r$, $1/r^2$, and $1/r^3$. At large distance, higher multipoles can be neglected.

The quadrupole moment Θ_{ij} is a 3×3 matrix. It is not difficult to see from Eq. (1.6) that

$$\Theta_{ji} = \Theta_{ij} \quad (1.8)$$

$$\text{tr } \Theta_{ij} \equiv \Theta_{ii} = 0. \quad (1.9)$$

That is, it is a *traceless, symmetric* matrix. Hence it has only 5 independent matrix elements. Thus, the multipole moments Q , p_i and Θ_{ij} have 1, 3, and 5 independent components respectively.

For a set of point charges $\{q_\alpha, \alpha = 1, \dots, N\}$, their charge density is (see Chap 2),

$$\rho(\mathbf{r}) = \sum_{\alpha=1}^N q_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha). \quad (1.10)$$

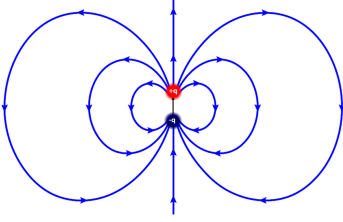


FIG. 3 The electric field of an electric dipole.

Substitute this to Eqs. (1.4), (1.5), and (1.6), we will get

$$Q = \sum_{\alpha} q_{\alpha}, \quad (1.11)$$

$$p_i = \sum_{\alpha} q_{\alpha} r_{\alpha i}, \quad (1.12)$$

$$\Theta_{ij} = \frac{1}{2} \sum_{\alpha} q_{\alpha} (3r_{\alpha i} r_{\alpha j} - r_{\alpha}^2 \delta_{ij}) \quad (1.13)$$

See Fig. 2 for examples of multipoles with point charges.

B. Electric dipole

From the dipole potential,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}, \quad (1.14)$$

we can derive its electric field,

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi = \frac{1}{4\pi\epsilon_0} \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}) - \mathbf{p}}{r^3}. \quad (1.15)$$

The field weakens as $1/r^3$ and has the distribution shown in Fig. 3.

Example:

Suppose there are charges $\rho(\mathbf{r})$ inside a ball V with volume V , show that the average of the electric field over the ball,

$$\langle \mathbf{E}(\mathbf{r}) \rangle_V \equiv \frac{1}{V} \int_V dv \mathbf{E}(\mathbf{r}) = -\frac{1}{3\epsilon_0} \frac{\mathbf{p}}{V}, \quad (1.16)$$

where \mathbf{p} is the electric dipole moment due to the charges (see Fig. 4(a)). On the other hand, if the charges $\rho(\mathbf{r})$ are outside V , then the averaged field is equal to the field at the center of the sphere (Fig. 4(b)),

$$\langle \mathbf{E}(\mathbf{r}) \rangle_V = \mathbf{E}(\mathbf{0}). \quad (1.17)$$

The latter is analogous to the mean value theorem of electrostatic potential (Chap 3).

Pf: The Coulomb integral for electric field is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int dv' \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (1.18)$$

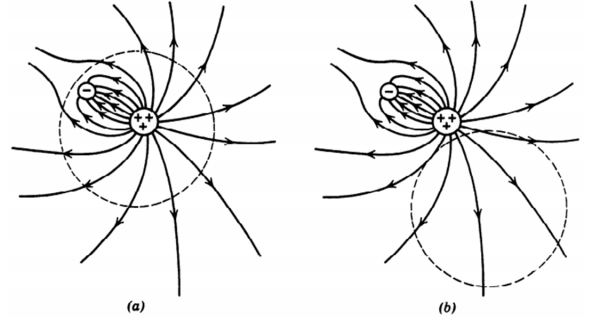


FIG. 4 Charges are inside a sphere (a), and outside a sphere (b). (Fig. from Jackson)

Thus,

$$\begin{aligned} \frac{1}{V} \int_V dv \mathbf{E}(\mathbf{r}) &= \frac{1}{V} \int_V dv \frac{1}{4\pi\epsilon_0} \int_{\rho \neq 0} dv' \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \\ &= -\frac{1}{V} \int_{\rho \neq 0} dv' \rho(\mathbf{r}') \underbrace{\frac{1}{4\pi\epsilon_0} \int_V dv \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3}}_{=\tilde{\mathbf{E}}(\mathbf{r}')}, \end{aligned}$$

where $\tilde{\mathbf{E}}(\mathbf{r}')$ is the electric field of a *fictitious* ball V with charge density $\tilde{\rho} = 1$.

Note that \mathbf{r}' is inside V if all of the charges are inside V . To get the electric field inside, choose a sphere S with radius r' and use

$$\int_S ds \cdot \tilde{\mathbf{E}} = \frac{\tilde{Q}(r')}{\epsilon_0}, \quad (1.19)$$

$$\rightarrow \tilde{E} 4\pi r'^2 = \frac{1}{\epsilon_0} \frac{4}{3} \pi r'^3, \quad (1.20)$$

$$\rightarrow \tilde{\mathbf{E}}(\mathbf{r}') = \frac{1}{3\epsilon_0} \mathbf{r}'. \quad (1.21)$$

Thus,

$$\langle \mathbf{E} \rangle_V = -\frac{1}{V} \int dv' \rho(\mathbf{r}') \frac{1}{3\epsilon_0} \mathbf{r}' \quad (1.22)$$

$$= -\frac{1}{3\epsilon_0} \frac{\mathbf{p}}{V}. \quad (1.23)$$

Why there is a minus sign in front of \mathbf{p} ? From Fig. 4(a), you can see that \mathbf{p} points to the center, but most of the field lines inside the sphere point *away* from the center. This is why $\langle \mathbf{E} \rangle_V$ is anti-parallel to \mathbf{p} .

If $\rho(\mathbf{r})$ is on the outside of the sphere V , then the first two steps of the proof above are the same, but now $\tilde{\mathbf{E}}(\mathbf{r}')$ is the field *outside* the uniformly charged ball. It follows that

$$\mathbf{E}(\mathbf{r}') = \frac{V}{4\pi\epsilon_0} \frac{\mathbf{r}'}{r'^3} \quad (\tilde{\rho} = 1). \quad (1.24)$$

Thus,

$$\langle \mathbf{E} \rangle_V = -\frac{1}{V} \int dv' \rho(\mathbf{r}') \frac{V}{4\pi\epsilon_0} \frac{\mathbf{r}'}{r'^3} \quad (1.25)$$

$$= \frac{1}{4\pi\epsilon_0} \int dv' \rho(\mathbf{r}') \frac{0 - \mathbf{r}'}{|\mathbf{r}'|^3} \quad (1.26)$$

$$= \mathbf{E}(0). \quad (1.27)$$

QED.

In general, when there are charges both inside and outside of the sphere, then

$$\langle \mathbf{E}(\mathbf{r}) \rangle_V = -\frac{1}{3\epsilon_0} \frac{\mathbf{p}_{in}}{V} + \mathbf{E}_{out}(0), \quad (1.28)$$

where \mathbf{p}_{in} is due to the charges inside, and $\mathbf{E}_{out}(0)$ is due to the charges outside.

1. Point electric dipole

Consider the electric dipole shown in Fig. 3. The two charges $\pm q/s$ are separated by $s\mathbf{b}$ and has an electric dipole moment $\mathbf{p} = q\mathbf{b}$. In the limit $s \rightarrow 0$, it becomes a **point electric dipole**, but the dipole moment is not changed. Thus, the dipole field remains the same,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}) - \mathbf{p}}{r^3}. \quad (1.29)$$

As we have explained at the beginning of this chapter, the multipole expansion is valid when $r \gg r'$. For a point dipole, $r' \rightarrow 0$, thus the range of validity of the dipole field above extends down to the region close to the point $r \rightarrow 0$.

However, if you integrate the field in Eq. (1.29) over a ball V centered at $\mathbf{r} = 0$, then

$$\int_V dv \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V dv \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}) - \mathbf{p}}{r^3} = 0. \quad (1.30)$$

It is zero due to angular integration, no matter if the ball is large or small. This contradicts the result in Eq. (1.16).

Eq. (1.29) is valid almost everywhere, except at $\mathbf{r} = 0$, where the field diverges. In order to resolve the contradiction and ensure that

$$\frac{1}{V} \int_V dv \mathbf{E}(\mathbf{r}) = -\frac{1}{3\epsilon_0} \frac{\mathbf{p}}{V}, \quad (1.31)$$

we can add a delta function to Eq. (1.29), so that

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}) - \mathbf{p}}{r^3} - \frac{\mathbf{p}}{3\epsilon_0} \delta(\mathbf{r}). \quad (1.32)$$

The delta function can only be non-zero (in fact, infinite) when $\mathbf{r} = 0$. Now the equation is valid everywhere, *including* the origin.

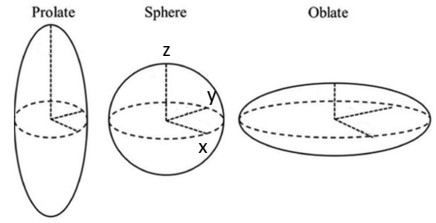


FIG. 5 From left to right, a prolate ellipsoid, a sphere, and an oblate ellipsoid.

C. Electric quadrupole

Recall that the quadrupole moment is a traceless, symmetric matrix. Like the **moment of inertia** in classical mechanics, we can always find a coordinate so that the matrix is diagonalized. Under this circumstance, the coordinate axes are called **principle axes**. If the charge distribution has certain symmetry, then the principle axes are along the symmetry axes.

For example, for the ellipsoids in Fig. 5, the principle axes are along the dotted lines. There is no distinction between x -axis and y -axis, so we expect $\Theta_{xx} = \Theta_{yy}$. Furthermore, since the quadrupole moment matrix is traceless,

$$\Theta_{xx} + \Theta_{yy} + \Theta_{zz} = 0, \quad (1.33)$$

so it must be of the form

$$\Theta = \begin{pmatrix} -\Theta_{zz}/2 & 0 & 0 \\ 0 & -\Theta_{zz}/2 & 0 \\ 0 & 0 & \Theta_{zz} \end{pmatrix}, \quad (1.34)$$

where

$$\Theta_{zz} = \int dv \rho(\mathbf{r}) (3z^2 - r^2) \quad (1.35)$$

$$= \int dv \rho(\mathbf{r}) (2z^2 - x^2 - y^2). \quad (1.36)$$

If the prolate ellipsoid is uniformly charged, then it's not difficult to see that $\Theta_{zz} > 0$. On the other hand, the oblate ellipsoid has $\Theta_{zz} < 0$. A uniformly charged ball has no quadrupole moment, $\Theta_{zz} = 0$.

D. Potential energy and force

The potential energy of a charge distribution $\rho(\mathbf{r})$ in an external potential $\phi(\mathbf{r})$ is,

$$V_E = \int dv \rho(\mathbf{r}) \phi(\mathbf{r}). \quad (1.37)$$

Assume that the potential varies smoothly compared to the charge distribution, then we can expand it with respect to a point 0 near the charges,

$$\phi(\mathbf{r}) \simeq \phi(0) + \mathbf{r} \cdot \nabla \phi(0) + \frac{1}{2} (\mathbf{r} \cdot \nabla)^2 \phi(0) \quad (1.38)$$

$$= \phi(0) - \mathbf{r} \cdot \mathbf{E}(0) - \frac{1}{2} r_i r_j \frac{\partial E_j}{\partial r_i}(0). \quad (1.39)$$

Since $\nabla \cdot \mathbf{E} = 0$ for the external field near the charges, we can add $\frac{1}{6}r^2\nabla \cdot \mathbf{E}(0)$ to the last term and get

$$\phi(\mathbf{r}) = \phi(0) - \mathbf{r} \cdot \mathbf{E}(0) - \frac{1}{6}(3r_i r_j - r^2 \delta_{ij}) \frac{\partial E_j}{\partial r_i}(0). \quad (1.40)$$

Thus, with Eqs. (1.4), (1.5), and (1.6), we have

$$V_E = q\phi(0) - \mathbf{p} \cdot \mathbf{E}(0) - \frac{1}{3}\Theta_{ij} \frac{\partial E_j}{\partial r_i}(0). \quad (1.41)$$

It is composed of monopole energy, dipole energy, and quadrupole energy (higher order terms are neglected). Note that the monopole energy depends on the potential, the dipole energy depends on the field, while the quadrupole energy depends on the field gradient. Hence, if the field is *uniform*, then there is *no* quadrupole energy.

In particular, a dipole \mathbf{p}_1 in the field of another dipole \mathbf{p}_2 (see Eq. (1.15)) has the dipole-dipole interaction energy,

$$V_{12} = -\mathbf{p}_1 \cdot \mathbf{E}_2(\mathbf{r}_1) \quad (1.42)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\hat{\mathbf{R}} \cdot \mathbf{p}_1)(\hat{\mathbf{R}} \cdot \mathbf{p}_2)}{R^3}, \quad (1.43)$$

which can either be repulsive or attractive, and decreases as $1/R^3$, $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$.

The forces on the multipoles are given by the gradient of potential energy. Thus,

$$\mathbf{F} = -\nabla V_E \quad (1.44)$$

$$= q\mathbf{E}(0) - \nabla(\mathbf{p} \cdot \mathbf{E}) - \frac{1}{3}\Theta_{ij} \frac{\partial^2 \mathbf{E}}{\partial r_i \partial r_j}. \quad (1.45)$$

That is, you need a field gradient to have a dipole force, and a non-zero second order derivative of field to have a quadrupole force.

E. Macroscopic polarizable medium

Consider a polarizable medium that is composed of polarizable atoms or molecules. If the dipole moment of the i -th atom is \mathbf{p}_i , then we can define the **electric polarization** as,

$$\mathbf{P}(\mathbf{r}') = \frac{\sum_{i \text{ in } \Delta V} \mathbf{p}_i}{\Delta V}, \quad (1.46)$$

where ΔV is a volume element around \mathbf{r}' (Fig. 6(a)). The volume element is microscopically large but macroscopically small (e.g., $1 \mu\text{m}$ in size), so that there are many atoms in ΔV , but \mathbf{r}' remains a point from human's point of view.

Since the volume element ΔV has charge $q = \rho\Delta V$ and dipole $\mathbf{p} = \mathbf{P}\Delta V$, it produces a potential at \mathbf{r} far away,

$$\begin{aligned} \Delta\phi(\mathbf{r}) &\simeq \frac{1}{4\pi\epsilon_0} \left(\frac{q}{R} + \frac{\mathbf{p} \cdot \mathbf{R}}{R^3} \right), \quad \mathbf{R} = \mathbf{r} - \mathbf{r}' \quad (1.47) \\ &= \frac{1}{4\pi\epsilon_0} \left[\frac{\rho(\mathbf{r}')\Delta V}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{P}(\mathbf{r}')\Delta V \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right]. \end{aligned}$$

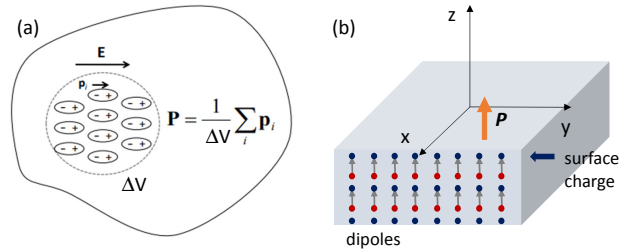


FIG. 6 (a) A volume element with many dipoles in a polarizable medium. (b) A semi-infinite dielectric below the $x - y$ plane.

After integration, we have the total potential,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left[\int dv' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \int dv' \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right]. \quad (1.48)$$

Since (see Chap 1)

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|}, \quad (1.49)$$

the second term of Eq. (1.48), after integration by parts, can be written as

$$\int dv' \mathbf{P}(\mathbf{r}') \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} = - \int dv' \nabla' \cdot \mathbf{P}(\mathbf{r}') \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \quad (1.50)$$

It follows that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int dv' \frac{\rho(\mathbf{r}') - \nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \quad (1.51)$$

The numerator can be considered as an effective charge density $\rho_{eff} = \rho + \rho_P$, where

$$\rho_P(\mathbf{r}) \equiv -\nabla \cdot \mathbf{P}(\mathbf{r}) \quad (1.52)$$

is the **polarization charge density**. Since the charge density in the integral above directly links with the one in Gauss's law (see Chap 2), so we have

$$\nabla \cdot \mathbf{E} = \frac{\rho_{eff}}{\epsilon_0} = \frac{1}{\epsilon_0} (\rho - \nabla \cdot \mathbf{P}). \quad (1.53)$$

Define the **electric displacement field**,

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad (1.54)$$

$$\text{then } \nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}). \quad (1.55)$$

This is Gauss's law in material (rather than in vacuum).

A side note: Maxwell coined the term *displacement*, which might be based on his (now out-of-date) mechanical model of *ether*. This field can be dispensed with, since we can just use \mathbf{E} and \mathbf{P} instead. According to Purcell, this quantity is sometimes treated "with more respect than it deserves" (Purcell, 2004).

If the polarization is proportional to the electric field,

$$\mathbf{P}(\propto \mathbf{E}) = \epsilon_0 \chi_e \mathbf{E}, \quad (1.56)$$

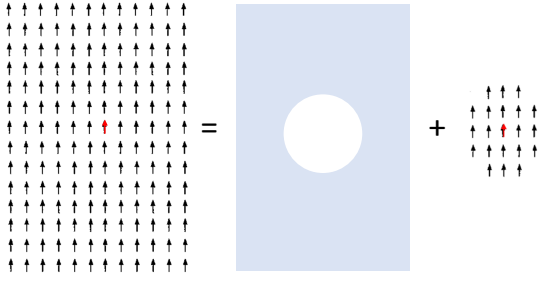


FIG. 7 The local field \mathbf{E}_{loc} at the red arrow is approximately the sum of the field from the continuous medium with a hole, plus the field from dipoles inside the hole.

then

$$\mathbf{D} = \varepsilon_0(1 + \chi_e)\mathbf{E} = \varepsilon\mathbf{E}, \quad (1.57)$$

where χ_e is **electric susceptibility**, and $\varepsilon \equiv \varepsilon_0(1 + \chi_e)$ is **electric permittivity**.

1. Polarization charge

Non-uniform polarization generates effective charge, $\rho_P = -\nabla \cdot \mathbf{P}$. We'll use a simple example to illustrate this: In Fig. 6(b) there is a semi-infinite dielectric with uniform polarization,

$$\mathbf{P} = P_0\theta(-z)\hat{z}, \quad (1.58)$$

in which θ is the step function. Its polarization charge density is,

$$\rho_P = -\nabla \cdot \mathbf{P} = P_0\delta(z)\hat{z}. \quad (1.59)$$

We can see from the figure that the bulk is charge-neutral, and only the outer-most electrons can be exposed. So it's reasonable for the polarization charges to reside on the surface of the dielectric.

Note that the polarization charges are bounded to molecules. They cannot move away like free electrons in metals.

F. Local field and electric permittivity

Apply an electric field \mathbf{E}_{ex} to a polarizable medium, then the medium is polarized with $\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}_{loc}$. For a rarefied medium, \mathbf{E}_{loc} is just the applied field \mathbf{E}_{ex} . However, for a dense medium, it is the applied field *plus* the induced field \mathbf{E}_p due to polarization,

$$\mathbf{E}_{loc} = \mathbf{E}_{ex} + \mathbf{E}_p. \quad (1.60)$$

An atom or a molecule inside the material is polarized by the **local field** \mathbf{E}_{loc} . Instead of adding up the dipolar fields from other molecules, we use the following trick to calculate \mathbf{E}_{loc} : Divide the medium into two regions, a

spherical region with radius R and a region *without* the sphere. The molecule (or atom) of interest is inside the sphere that is macroscopically small but microscopically large.

For the charge outside the sphere, we can adopt the coarse-grained, macroscopic electric field \mathbf{E} . Inside the sphere near the molecule, the material is not treated as a continuous spherical medium that produces \mathbf{E}_{sph} , but as a collection of dipoles that produces \mathbf{E}_{near} . Thus,

$$\mathbf{E}_{loc} = \mathbf{E} - \mathbf{E}_{sph} + \mathbf{E}_{near}. \quad (1.61)$$

That is, we remove the field \mathbf{E}_{sph} from \mathbf{E} and fill in the field \mathbf{E}_{near} (see Fig. 7).

We can estimate \mathbf{E}_{sph} with Eq. (1.16),

$$\mathbf{E}_{sph} \simeq \langle \mathbf{E} \rangle_V = -\frac{1}{3\varepsilon_0} \frac{\mathbf{p}_V}{V} = -\frac{1}{3\varepsilon_0} \mathbf{P}, \quad (1.62)$$

where \mathbf{p}_V is the total dipole moments inside V , $\mathbf{p}_V = \mathbf{P}V$. The field \mathbf{E}_{near} depends on crystal symmetry. For a regular lattice, or a random distribution of dipoles, $\mathbf{E}_{near} \simeq 0$ due to the cancellation from dipoles at symmetric positions. Thus,

$$\mathbf{E}_{loc} = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}. \quad (1.63)$$

This is the **Lorentz relation**.

The molecule is polarized by the local field,

$$\mathbf{p} = \varepsilon_0\gamma_m\mathbf{E}_{loc}, \quad (1.64)$$

where γ_m is the **molecular polarizability**. If the density of the number of dipoles is n , then

$$\mathbf{P} = n\mathbf{p} = n\varepsilon_0\gamma_m \left(\mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0} \right), \quad (1.65)$$

which gives

$$\mathbf{P} = \varepsilon_0 \frac{n\gamma_m}{1 - \frac{1}{3}n\gamma_m} \mathbf{E} = \varepsilon_0\chi_e\mathbf{E}. \quad (1.66)$$

The **relativity permittivity** $\varepsilon_r \equiv \varepsilon/\varepsilon_0 = 1 + \chi_e$. Thus,

$$\varepsilon_r = 1 + \frac{n\gamma_m}{1 - \frac{1}{3}n\gamma_m}, \quad (1.67)$$

or

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{\gamma_m}{3}n. \quad (1.68)$$

This is the **Clausius-Mossotti relation**, which links the macroscopic quantity ε_r with the microscopic quantity γ_m . We also see that, for a given material, $(\varepsilon_r - 1)/(\varepsilon_r + 2)$ is proportional to the density of molecules.

For example, the molecular polarizability of methane (CH_4) is $\gamma_m = 4\pi \times 2.6 \times 10^{-30} \text{ m}^3$. At freezing point

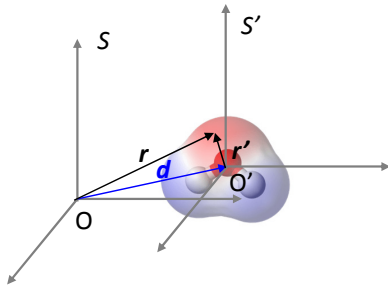


FIG. 8 The same charge distribution viewed from two different coordinates.

and 1 atm, there are 2.8×10^{25} molecules per cubic meter, hence

$$\varepsilon_r = 1.00091, \quad (1.69)$$

which is close to the dielectric constant measured $\varepsilon_r = 1.00088$ (Purcell, 2004).

Note that according to Eq. (1.63), the local field

$$\mathbf{E}_{loc} = \left(1 + \frac{\chi_e}{3}\right) \mathbf{E} \quad (1.70)$$

$$= \frac{1}{1 - \frac{1}{3}n\gamma_m} \mathbf{E} = \frac{\varepsilon_r + 2}{3} \mathbf{E}. \quad (1.71)$$

It is larger than the macroscopic field if $\varepsilon_r > 1$.

Problem:

1. From the third term of the binomial expansion in Eq. (1.2), we get the quadrupole potential in Eq. (1.3). Show that the quadrupole potential can also be written as

$$\phi_{quad}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} Q_{ij} \frac{3r_i r_j - \delta_{ij} r^2}{r^5}, \quad (1.72)$$

$$\text{where } Q_{ij} \equiv \frac{1}{2} \int dv' \rho(\mathbf{r}') r'_i r'_j. \quad (1.73)$$

2. The magnitude of an electric *charge* is independent of the choice of coordinate (Fig. 8). However, in general p_i and Θ_{ij} do. Show that

(a) if a system is neutral ($Q = 0$), then \mathbf{p} is independent of the choice of coordinate.

(b) If both Q and \mathbf{p} vanish, then Θ_{ij} is independent of the choice of coordinate.

References

Purcell, E. M., 2004, *Electricity and Magnetism*, 2nd edition.