Lecture notes on classical electrodynamics

Ming-Che Chang

Department of Physics, National Taiwan Normal University, Taipei, Taiwan

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Contents

I. ELECTROSTATICS

A. Introduction

There are several ways to find out an electric field. First, if we have the complete information of charge distribution $\rho(\mathbf{r})$, then one just needs to evaluate the Coulomb integral,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{1.1}$$

where $\varepsilon_0 = 8.8542 \times 10^{-12}$ C^2/Nm^2. Or, one may calculate the electric potential first,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{1.2}$$

then take its gradient to get $\mathbf{E} = -\nabla\phi$.
Note: when we write $\int$ instead of $\int_V$, an integration over the whole space is often implied.

The problem with the method above is that the charge distribution is not always known. For example, when you place a point charge near a grounded metal sphere, the induced charge is not known beforehand (Fig. 1). Or, a metal box is grounded for five of its surface, except that the top surface is maintained at potential ϕ_0. The charges on metal box redistribute themselves to meet this condition, but their distribution unknown. For these cases, we need Gauss's law,

$$\nabla\cdot\mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0}, \tag{1.3}$$

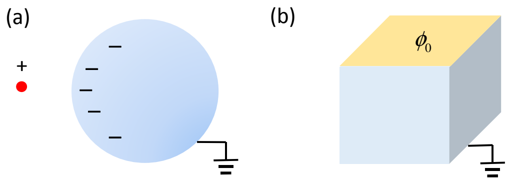

FIG. 1 (a) When a positive charge is near a grounded metal sphere, there are negative induced charges on the surface of the sphere. (b) The metal box is grounded, except the top surface, which is maintained at potential ϕ_0.

or Poisson equation,

$$\nabla^2\phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}. \tag{1.4}$$

We need to solve it, given the boundary condition (BC) for ϕ. Afterwards, we can get the electric field $\mathbf{E} = -\nabla\phi$. The distribution of charges can be determined after the field is known.

When a system has simple geometry, such as a cylinder or a sphere, it is convenient to find $\mathbf{E}$ using the integral form of Gauss's law,

$$\int_S d\mathbf{s}\cdot\mathbf{E}(\mathbf{r}) = \frac{Q}{\varepsilon_0}. \tag{1.5}$$

In this course, we avoid using the second method. Not because it's not important or less used, but because we'd like to focus more on physics, less on solving partial differential equations and wielding special functions.

B. Coulomb's law

Let's practice the first, direct integration method with an example.
Example:
Find the electric field along the central axis of (a) a charged ring, (b) a charged disk, and (c) a charged plane. All of them uniformly charged.
Sol'n:
(a) Suppose a ring with radius r has charge Q, then its charge density per unit length $\lambda = Q/2\pi r$. A short segment $d\ell$ with charge $dQ = \lambda d\ell$ produces an electric field $d\mathbf{E}$ (Fig. 2(a)). Along the central axis at a distance z away,

$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r^2+z^2}\cos\alpha,\ \cos\alpha = \frac{z}{\sqrt{r^2+z^2}}. \tag{1.6}$$

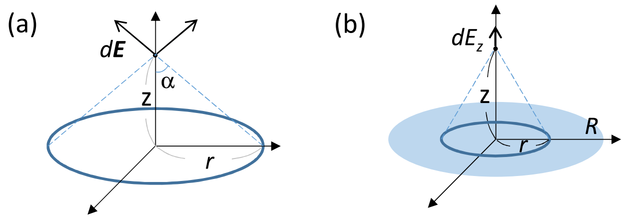

FIG. 2 (a) A charged ring. (b) A charged disk.

After integration,

$$E_z(z) = \frac{1}{4\pi\varepsilon_0}\oint_C \frac{\lambda d\ell}{r^2+z^2}\cos\alpha \quad (1.7)$$
$$= \frac{Q}{4\pi\varepsilon_0}\frac{z}{(r^2+z^2)^{3/2}}. \quad (1.8)$$

The components $E_{x,y}$ cancels away after integration, thus $\mathbf{E}(z) = E_z(z)\hat{\mathbf{z}}$. If you are interested in the potential away from the central axis, which is a more difficult problem, see Chap 3 of Jackson, 1998.

(b) A disk can be considered as a collection of rings (Fig. 2(b)). Suppose it has radius R and charge Q, then its surface charge density $\sigma = Q/\pi R^2$. A ring with radius r and width dr has charge

$$dQ = \sigma\ 2\pi r dr. \quad (1.9)$$

According to Eq. (1.8), along the central axis at a distance z away,

$$dE_z = \frac{dQ}{4\pi\varepsilon_0}\frac{z}{(r^2+z^2)^{3/2}}. \quad (1.10)$$

Integrate along the radial direction to get

$$E_z(z) = \frac{1}{2\varepsilon_0}\int_0^R \sigma r dr \frac{z}{(r^2+z^2)^{3/2}} \quad (1.11)$$
$$= \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right). \quad (1.12)$$

Finally, $\mathbf{E}(z) = E_z(z)\hat{\mathbf{z}}$.

(c) To get the electric field of an infinite charged plane, just let the R in Eq. (1.12) be infinite,

$$\mathbf{E}(z>0) = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}. \quad (1.13)$$

On the other side of the plane, obviously we have

$$\mathbf{E}(z<0) = -\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}. \quad (1.14)$$

The electric field is discontinuous across the plate,

$$\mathbf{E}(0^+) - \mathbf{E}(0^-) = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{z}}. \quad (1.15)$$

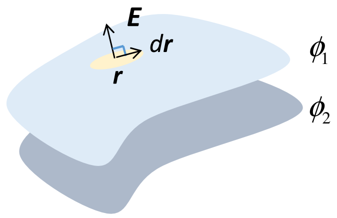

FIG. 3 Electric field is perpendicular to equipotential surface.

C. Electric potential

The following three equations state the same fact about the electrostatic field,

1. $\mathbf{E} = -\nabla\phi$, (1.16)
2. $\nabla\times\mathbf{E} = 0$, (1.17)
3. $\oint d\mathbf{r}\cdot\mathbf{E} = 0$, (1.18)

1 implies 2 since the curl of gradient is zero. Conversely, 2 implies 1 since if a vector field is curless, then it can be written as a gradient (see Chap 1). Also, 2 and 3 are simply the differential form and the integral form of the same Maxwell equation (see Chap 2).

1. Equipotential surface

The equation $\phi(\mathbf{r}) = \phi_0$, where ϕ_0 is a constant, defines an **equipotential surface** S_0. If $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ are both located on S_0, then moving a charge from $\mathbf{r}$ to $\mathbf{r}+d\mathbf{r}$ requires no work,

$$dW = q\mathbf{E}\cdot d\mathbf{r} = 0. \quad (1.19)$$

This is valid for any tangent vector $d\mathbf{r}$ emanating from $\mathbf{r}$. Thus, $\mathbf{E}(\mathbf{r})$ is perpendicular to the tangent plane of S_0 at $\mathbf{r}$ (Fig. 3). That is, *the steepest descent $-\nabla\phi$ is perpendicular to the equipotential surface.*

Example:
Find out the electric potential of a uniformly charged wire with length $2L$ and linear charge density λ.
Sol'n:
Suppose the wire is lying on the z-axis, as in Fig. 4(a). Since there is rotational symmetry around the wire, it is convenient to use the cylindrical coordinate. The potential at a point with coordinate z,ρ is,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{wire}}\frac{\lambda dr'}{|\mathbf{r}-\mathbf{r}'|} \quad (1.20)$$
$$= \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{\lambda dz'}{\sqrt{(z'-z)^2+\rho^2}} \quad (1.21)$$
$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{\sqrt{(L-z)^2+\rho^2}+L-z}{\sqrt{(L+z)^2+\rho^2}-L-z}\right), \quad (1.22)$$

where we have used

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \ln\left(\sqrt{x^2+a^2}+x\right). \quad (1.23)$$

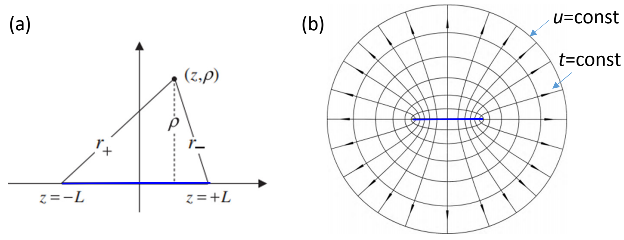

FIG. 4 (a) A charged wire. (b) Equipotential surfaces and field lines of a charged wire. (Fig. from Zangwill)

When the observation point is far away from the wire, $z, \rho \gg L$, and $r = \sqrt{z^2+\rho^2} \gg z, L$, one has

$$\phi(\mathbf{r}) \simeq \frac{1}{4\pi\varepsilon}\frac{Q}{r}, \; Q = 2L\lambda. \tag{1.24}$$

It is similar to the potential of a point charge.

On the other hand, if the observation point is close to the center of the wire, $\rho \ll L, z = 0$, then expand the potential to the second order of ρ/L to get

$$\phi(\rho) \simeq -\frac{\lambda}{2\pi\varepsilon_0}\ln\rho + \frac{\lambda}{2\pi\varepsilon_0}\ln(2L). \tag{1.25}$$

Note that it diverges if $\rho \to 0$. Its gradient gives the electric field,

$$\mathbf{E}(\rho) = -\nabla\phi \simeq \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\rho}. \tag{1.26}$$

Further analysis of the result:
Instead of z, ρ, we can use r_+, r_- as coordinates (see Fig. 4(a)),

$$r_\pm \equiv \sqrt{(L\pm z)^2+\rho^2}. \tag{1.27}$$

Note that

$$r_+^2 - r_-^2 = 4Lz \quad \to \quad z = \frac{1}{4L}\left(r_+^2 - r_-^2\right). \tag{1.28}$$

With the two relations above, we can write the potential in new coordinate $\phi(r_+, r_-)$.

The third choice of coordinate is u, t, where

$$\left\{\begin{array}{l} u = \frac{1}{2}(r_- + r_+), \\ t = \frac{1}{2}(r_- - r_+) \end{array}\right. \quad \leftrightarrow \quad \left\{\begin{array}{l} r_- = u + t, \\ r_+ = u - t \end{array}\right. \tag{1.29}$$

Note that the equation $u =$constant draws out an ellipse, and $t =$constant an hyperbola. Thus the new coordinate is called **elliptic coordinate**, which is an *orthogonal* coordinate since at the intersection of coordinate curves, the tangents are perpendicular to each other.

Now,

$$ut = -zL \quad \to \quad z = -\frac{ut}{L}. \tag{1.30}$$

Thus,

$$\begin{aligned} \phi(u,t) &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{u+t+L-z}{u-t-L-z}\right) && (1.31) \\ &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{u+L}{u-L}\right), && (1.32) \end{aligned}$$

which is independent of t. Hence, the potential is a constant when u is fixed. That is, the equipotential surface is an ellipse (Fig. 4(b)), or an ellipsoid after revolving around the z-axis. Furthermore, since the curves of fixed t's describe electric field lines, since they are perpendicular to the equipotential surfaces.

2. Earnshaw's theorem

Inside a region V without any charge, the electric potential *cannot* have any local minimum or local maximum. This is called **Earnshaw's theorem**, which is true for *electrostatic* field.

Pf: We'll prove this by contradiction. Suppose the potential $\phi(\mathbf{r})$ has a local minimum at point P inside V. Then, when one moves away from P, the potential increases (Fig. 5).

Surround the point P with a small spherical surface S. Then on surface S, the gradient $\nabla\phi$, which is along the steepest ascent, points outward. That is, if $\hat{\mathbf{n}}$ is the normal vector of S (pointing outward), then

$$\hat{\mathbf{n}}\cdot\nabla\phi > 0. \tag{1.33}$$

for *every* point on S.

Thus, after integration,

$$\int_S ds\hat{\mathbf{n}}\cdot\nabla\phi > 0. \tag{1.34}$$

With the help of divergence theorem, the LHS can be written as,

$$\int_S d\mathbf{s}\cdot\nabla\phi = -\int_V dv\nabla\cdot\mathbf{E} = 0, \tag{1.35}$$

It is zero because there is no charge inside V. Thus, we have a contradiction. The same contradiction occurs if P is a local maximum. Hence, neither local minimum nor maximum is allowed inside V. QED.

Alternatively speaking, the location of local max or local min of ϕ always hosts positive or negative charges.

D. Gauss's law

As we have mentioned in Sec. I.A, when a system has a simple geometry, we can use the integral form of the Gauss's law to find electric field,

$$\int_S d\mathbf{s}\cdot\mathbf{E}(\mathbf{r}) = \frac{Q}{\varepsilon_0}. \tag{1.36}$$

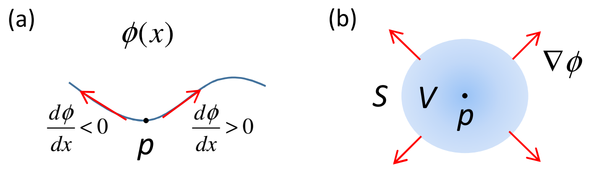

FIG. 5 (a) Potential and its slope in one dimension. (b) Potential and its gradient in three dimension.

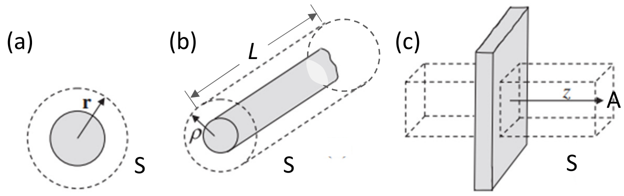

FIG. 6 Charge distribution with (a) spherical symmetry, (b) cylindrical symmetry, and (c) planar symmetry. (Fig. from Zangwill)

Example:
Find out the electric field for systems with (Fig. 6)
(a) Spherical symmetry: $\rho(r,\theta,\phi) = \rho(r)$.
(b) Cylindrical symmetry: $\rho_e(\rho,\phi,z) = \rho_e(\rho)$.
(c) Planar symmetry: $\rho(x,y,z) = \rho(z)$. Furthermore, assume $\rho(-z) = \rho(z)$.
Sol'n:
(a) We expect the electric field to be radial and depend only on r, $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$. Choose S to be a spherical surface with radius r, then Eq. (1.36) gives

$$\int_S d\mathbf{s}\cdot\mathbf{E}(\mathbf{r}) = 4\pi r^2 E(r) = \frac{Q(r)}{\varepsilon_0}, \tag{1.37}$$

where $Q(r)$ is the charge inside the surface S. Hence,

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r^2}. \tag{1.38}$$

If all of the charges Q_0 are confined within radius R, then when $r \geq R$,

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{r^2}, \tag{1.39}$$

same as the field of a point charge Q_0 at the origin.

(b) We expect the electric field to be radial and depend only on ρ, $\mathbf{E}(\mathbf{r}) = E(\rho)\hat{\boldsymbol{\rho}}$. Choose S to be a cylindrical surface with radius ρ and height L, then Eq. (1.36) gives

$$\int_S d\mathbf{s}\cdot\mathbf{E}(\mathbf{r}) = 2\pi\rho L E(\rho) = \frac{Q(\rho)}{\varepsilon_0}, \tag{1.40}$$

where $Q(\rho)$ is the charge inside the surface S. Hence,

$$E(\rho) = \frac{1}{2\pi\varepsilon_0}\frac{Q(\rho)/L}{\rho}. \tag{1.41}$$

(c) We expect the electric field to be along z and depend only on z,

$$\mathbf{E}(\mathbf{r}) = \begin{cases} E(z)\hat{\mathbf{z}}, & \text{for } z > 0 \\ -E(z)\hat{\mathbf{z}}, & \text{for } z < 0 \end{cases} \tag{1.42}$$

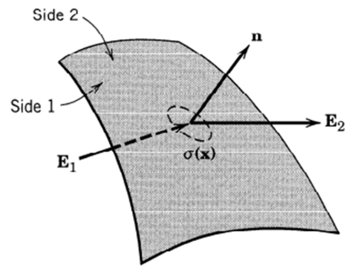

FIG. 7 A charge surface has different electric fields on two sides. (Fig. from Jackson, 1998)

Choose S to be a box surface (bisected by the x-y plane) with area A and height $2z$, then Eq. (1.36) gives

$$\int_S d\mathbf{s}\cdot\mathbf{E}(\mathbf{r}) = 2AE(z) = \frac{Q(z)}{\varepsilon_0}, \tag{1.43}$$

where $Q(z)$ is the charge inside the box S. Hence,

$$E(z) = \frac{Q(z)/A}{2\varepsilon_0}. \tag{1.44}$$

If all of the charges are confined within $|z| < Z$, then when $z \geq Z$,

$$E(z) = \frac{\sigma_0}{2\varepsilon_0}, \tag{1.45}$$

where $\sigma_0 = Q(Z)/A$ is the surface charge density. In general, for $|z| \geq Z$

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma_0}{2\varepsilon_0}\text{sgn}(z)\hat{\mathbf{z}}. \tag{1.46}$$

E. Boundary condition for E

In general, the electric fields on opposite sides of a charged surface are not the same. Their difference is caused by the charges on the surface. Suppose a surface has surface charge density $\sigma(\mathbf{r})$. At a point $\mathbf{r}$ on the surface, the electric fields on opposite sides are $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{E}_2(\mathbf{r})$ (Fig. 7). What's the relation between this two electric fields?

First, divide the surface S into a small disk $\circ$ and a surface S' (S with $\circ$ removed),

$$S = \circ + S'. \tag{1.47}$$

The disk is microscopically large, but macroscopically small (say, with a radius of 1 μm). The field, $\mathbf{E}_1(\mathbf{r})$ or $\mathbf{E}_2(\mathbf{r})$, is the superposition of the fields produced by $\circ$ and S'.

When one approaches the center of the disk, the field is close to the field of an infinite plane, $\mathbf{E}(\mathbf{r}) = \frac{\sigma}{2\varepsilon_0}\text{sgn}(z)\hat{\mathbf{z}}$. Suppose the field produced by S' is $\mathbf{E}_S$, then

$$\mathbf{E}_1 = \mathbf{E}_S - \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}, \tag{1.48}$$

$$\mathbf{E}_2 = \mathbf{E}_S + \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}, \tag{1.49}$$

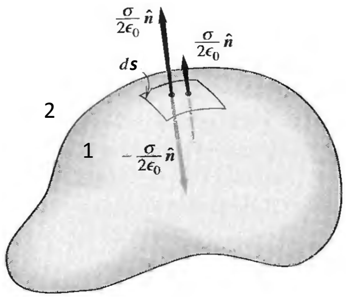

FIG. 8 The electric fields produced by an area element $d\mathbf{s}$ and the surface with a hole (at $d\mathbf{s}$). (Fig. from Lorrain and Corson)

where $\hat{\mathbf{n}}$ is the normal vector pointing *from region 1 to region 2.*

Even though $\mathbf{E}_S$ remains unknown, we can substrate the field to get

$$\mathbf{E}_2(\mathbf{r}) - \mathbf{E}_1(\mathbf{r}) = \frac{\sigma(\mathbf{r})}{\varepsilon_0}\hat{\mathbf{n}}. \tag{1.50}$$

This is the BC for fields near a charged surface. Sometimes it is written as,

$$\hat{\mathbf{n}} \cdot (\mathbf{E}_2 - \mathbf{E}_1) = \frac{\sigma}{\varepsilon_0}, \tag{1.51}$$

$$\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0. \tag{1.52}$$

1. Force on charged surface

Following the example above, the force $d\mathbf{F}$ on disk $\circ$ is due to the charges on S'. *The disk exerts no force on itself.* If the disk has area ds, then

$$d\mathbf{F} = (\sigma ds)\mathbf{E}_S. \tag{1.53}$$

The force per unit area (or pressure), is

$$\mathbf{f} \equiv \frac{d\mathbf{F}}{ds} = \sigma \mathbf{E}_S. \tag{1.54}$$

Since $\mathbf{E}_S = (\mathbf{E}_1 + \mathbf{E}_2)/2$, we have

$$\mathbf{f} = \frac{\sigma}{2}(\mathbf{E}_1 + \mathbf{E}_2). \tag{1.55}$$

For example, for a closed metallic surface, the electric fields on the inside and outside are (Fig. 8),

$$\mathbf{E}_1 = 0, \quad \mathbf{E}_2 = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}. \tag{1.56}$$

Hence, according to Eq. (1.55),

$$\mathbf{f} = \frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{n}}, \tag{1.57}$$

where $\hat{\mathbf{n}}$ points out of the sphere.

Note that if one calculates the force via

$$\mathbf{f} = \sigma \mathbf{E}_2 = \frac{\sigma^2}{\varepsilon_0}\hat{\mathbf{n}}, \tag{1.58}$$

then the result is wrong by a factor of two, since it has wrongly included the force exerted by the disk on itself.

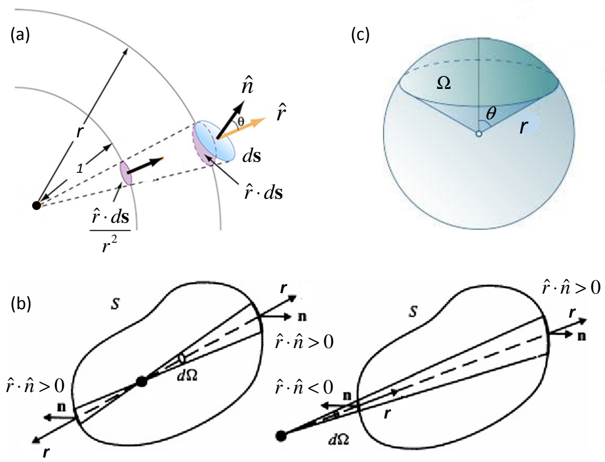

FIG. 9 (a) The areas $d\mathbf{s}$, $\hat{\mathbf{r}} \cdot d\mathbf{s}$, and $\hat{\mathbf{r}} \cdot d\mathbf{s}/r^2$. (b) The origin is inside (left) or outside (right) S. When it's outside, two projected areas with equal magnitude but opposite signs cancel with each other. (c) A spherical cap.

F. Solid angle

The solid angle spanned by an area element $d\mathbf{s} = ds\hat{\mathbf{n}}$ located at $\mathbf{r}$ with respect to the origin (Fig. 9(a)) is defined as,

$$d\Omega \equiv \frac{\hat{\mathbf{r}} \cdot d\mathbf{s}}{r^2} = \frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{n}} ds}{r^2}. \tag{1.59}$$

Since $\hat{\mathbf{r}} \cdot d\mathbf{s}$ is the area of ds projected onto a sphere with radius r, so $d\Omega$ equals *the projected area on a unit sphere* centered at $\mathbf{r} = 0$. The solid angle $d\Omega$ can be negative if $\hat{\mathbf{r}} \cdot \hat{\mathbf{n}} < 0$.

In spherical coordinate,

$$d\mathbf{s} = r^2 \sin\theta d\theta d\phi \; \hat{\mathbf{n}}, \tag{1.60}$$

hence

$$d\Omega = \hat{\mathbf{r}} \cdot \hat{\mathbf{n}} \sin\theta d\theta d\phi. \tag{1.61}$$

For a sphere with radius r, the solid angle extended by an area $ds\hat{\mathbf{n}}$ ($\hat{\mathbf{n}} = \hat{\mathbf{r}}$) on its surface is,

$$d\Omega = \frac{ds}{r^2} = \sin\theta d\theta d\phi, \tag{1.62}$$

$$\text{or} \quad ds = r^2 d\Omega. \tag{1.63}$$

The total solid angle of a sphere is

$$\Omega = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = 4\pi. \tag{1.64}$$

If S is a *closed* surface surrounding the origin, then its projected image covers the unit sphere centered at the origin once. If the origin is outside S, then part of S has "positive" image on the unit sphere, the other part has negative image, and the two parts cancel with each other. Thus, the total solid angle Ω is zero (Fig. 9(b)). That is,

$$\Omega = \begin{cases} 4\pi \text{ if the origin is } \textit{inside } S, \\ 0 \text{ if the origin is } \textit{outside } S. \end{cases} \tag{1.65}$$

In general, for a surface S described by coordinate $\mathbf{r}$, its solid angle with respect to a point $\mathbf{r}_s$ is,

$$\Omega = \int d\Omega = \int_S \frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3} \cdot d\mathbf{s}. \tag{1.66}$$

We have just replaced the $\mathbf{r}$ in Eq. (1.59) by $\mathbf{R} = \mathbf{r} - \mathbf{r}_s$.
Example:
Find out the solid angle of a spherical cap with respect to the origin, as shown in Fig. 9(c).
Sol'n:

$$\begin{aligned} \Omega &= \int \frac{\hat{\mathbf{r}} \cdot d\mathbf{s}}{r^2} && (1.67) \\ &= \int_0^\theta \sin\theta d\theta \int_0^{2\pi} d\phi && (1.68) \\ \text{or} \quad &= \int_{\cos\theta}^1 d\cos\theta \int_0^{2\pi} d\phi && (1.69) \\ &= 2\pi(1-\cos\theta). && (1.70) \end{aligned}$$

When the cap covers the whole sphere ($\theta = \pi$), $\Omega = 4\pi$, as it should be.
Application
There is a point charge $q(>0)$ at the origin in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$ ($E_0 > 0$). Find out the electric field lines of this system.
Sol'n:
It's easy to get the electric field of this system,

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2} + E_0\hat{\mathbf{z}}. \tag{1.71}$$

However, we are interested in field lines, not $\mathbf{E}$, which is the tangent of a field line.

To obtain the mathematical expression of field lines, let's adopt the following method. In Fig. 10(a) we see that the electric flux is conserved along the flow of field lines,

$$\Phi_E(S) = \Phi_E(S'), \tag{1.72}$$

where S and S' are flat disks perpendicular to the z-axis. If we can write Φ_E as a function of r, θ (no ϕ because of the rotation symmetry around the z-axis), then flux conservation should give us an equation of field lines.

Note that the flux through S is the same as the flux through the cap S_c in Fig. 10(a), thus with spherical coordinate,

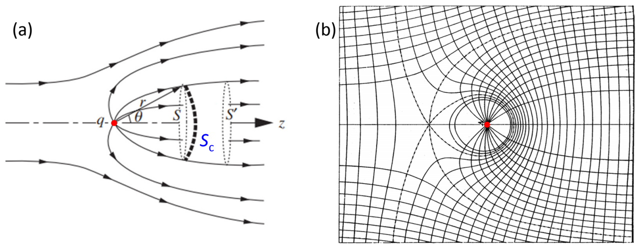

FIG. 10 (a) A point charge in a uniform electric field (Fig. from Zangwill). (b) The field lines and equipotential lines of the system in (a). Fig. from Maxwell, 1891.

$$\begin{aligned} \Phi_E(S) &= \int_{S_c} \mathbf{E}\cdot d\mathbf{s}, \quad d\mathbf{s} = r^2\hat{\mathbf{r}}d\Omega && (1.73) \\ &= \frac{q}{4\pi\varepsilon_0}\int_{S_c}\frac{\hat{\mathbf{r}}}{r^2}\cdot d\mathbf{s} + \int_{S_c}\mathbf{E}_0\cdot d\mathbf{s} && (1.74) \\ &= \frac{q}{4\pi\varepsilon_0}\Omega(S_c) + E_0\int_{\cos\theta}^1 \cos\theta r^2 d\cos\theta d\phi \\ &= \frac{q}{4\pi\varepsilon_0}2\pi(1-\cos\theta) + E_0\pi r^2\sin^2\theta && (1.75) \\ &= \text{constant } \alpha, && (1.76) \end{aligned}$$

in which $\Omega(S_c)$ is the solid angle of S_c. Note that the choice of S_c (instead of S) makes the second term harder to calculate. However, if we choose S, then the first term would be even harder to calculate.

Finally, one can write r in terms of θ, and different constants give different field lines (Fig. 10(b)),

$$r^2 = \frac{\alpha - \frac{q}{\varepsilon_0}\sin^2\frac{\theta}{2}}{E_0\pi\sin^2\theta}, \quad \theta \neq 0. \tag{1.77}$$

G. Electric potential energy

Electric potential energy is the potential energy of charges in an *external* electric potential. Suppose there are two sets of charge distribution $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$. They can be spatially separated or mixed (but remain different sets). The first set produces electric potential,

$$\phi_1(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho_1(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{1.78}$$

similarly for the second set. Then $\rho_2(\mathbf{r})$ in $\phi_1(\mathbf{r})$ has the electric potential energy,

$$\begin{aligned} V_E &= \int dv\rho_2(\mathbf{r})\phi_1(\mathbf{r}) && (1.79) \\ &= \frac{1}{4\pi\varepsilon_0}\int dvdv'\frac{\rho_2(\mathbf{r})\rho_1(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} && (1.80) \\ &= \int dv'\rho_1(\mathbf{r}')\phi_2(\mathbf{r}'). && (1.81) \end{aligned}$$

That is, the potential energy of $\rho_2(\mathbf{r})$ in $\phi_1(\mathbf{r})$ is the same as that of $\rho_1(\mathbf{r})$ in $\phi_2(\mathbf{r})$. This is called **Green's reciprocity relation**.
Application:
In a finite region without any charge, the average of potential $\phi(\mathbf{r})$ over a spherical surface S is equal to its value at the center of the sphere (Fig. 3-11). That is, if the radius of the fictitious sphere S is R (which does not need to be small), then

$$\langle\phi_1(\mathbf{r})\rangle_S \equiv \frac{1}{4\pi R^2}\int_S ds\phi(\mathbf{r}) = \phi(0). \tag{1.82}$$

This is called the **mean value theorem** of electrostatic potential.
Pf: The are more than one way to prove this theorem. Here we use a trick using Green's reciprocity relation. Suppose that the charge density that produces the potential is $\rho(\mathbf{r})$, which is *outside* S. Let

$$\rho_1(\mathbf{r}) = \rho(\mathbf{r}), \phi_1(\mathbf{r}) = \phi(\mathbf{r}). \tag{1.83}$$

In order to select the potential on the surface of S, choose

$$\rho_2(\mathbf{r}) = \delta(r - R). \tag{1.84}$$

It has a total charge

$$Q_2 = \int dv\delta(r - R) = 4\pi R^2. \tag{1.85}$$

The charge $\rho_2(\mathbf{r})$ produces a potential $\phi_2(\mathbf{r})$,

$$\phi_2(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{1.86}$$

$$= \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{r}, & r \geq R, \\ \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{R}, & r \leq R. \end{cases} \tag{1.87}$$

Be aware that $\rho_2(\mathbf{r})$ is simply a mathematical apparatus and does not really physically coexist with $\rho_1(\mathbf{r})$.
According to Green's reciprocity relation, one has

$$\int dv\delta(r - R)\phi_1(\mathbf{r}) = \int dv\rho_1(\mathbf{r})\phi_2(\mathbf{r}), \tag{1.88}$$

The integration is over the whole space. First, the LHS gives

$$LHS = \int r^2 dr d\Omega\delta(r - R)\phi_1(\mathbf{r}) \tag{1.89}$$

$$= \int_S R^2 d\Omega\phi_1(R),\ R^2 d\Omega = ds \tag{1.90}$$

$$= \int_S ds\phi_1(\mathbf{r}). \tag{1.91}$$

Since the charge density ρ is outside S, the integrand of the RHS is nonzero only when $r > R$,

$$RHS = \int_{r>R} dv\rho_1\frac{1}{4\pi\varepsilon_0}(\mathbf{r})\frac{Q_2}{r} \tag{1.92}$$

$$= \frac{Q_2}{4\pi\varepsilon_0}\int_{r>R} dv\frac{\rho_1(\mathbf{r})}{r} \tag{1.93}$$

$$= \underbrace{4\pi R^2}_{Q_2}\phi_1(0). \tag{1.94}$$

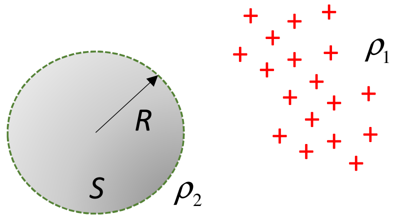

FIG. 11 A fictitious sphere outside the distribution ρ_1 of charges. The charges ρ_2 on the surface of the sphere are employed to prove the mean value theorem.

Equate the LHS with the RHS, we have

$$\langle\phi_1(\mathbf{r})\rangle_S = \phi_1(0), \quad \text{QED.} \tag{1.95}$$

Note that the mean value theorem implies Earnshaw's theorem: If there is a local min or max in a charge-free region, then the mean value theorem would no longer be true. Thus, in order for the later to be true, there cannot be local min/max in a charge-free region.

H. Electrostatic energy

The electrostatic energy of a charge distribution equals the total work required to assemble these charges, starting from an initial state with energy zero, when all of the charges are dispersed far away from each other. First, consider two point charges q_1, q_2 at $\mathbf{r}_1$, $\mathbf{r}_2$. The electrostatic energy is (ignoring the self-energy of point charges),

$$U_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{1.96}$$

which is the same as the potential energy of q_2 in the field produced by q_1, or vice versa.
If there are N charges $q_1, \cdots, q_N$ at $\mathbf{r}_1, \cdots, \mathbf{r}_N$, then the electrostatic energy is (again ignoring the self-energy),

$$U_E = \sum_{i<j} U_{ij} = \frac{1}{2}\sum_{i,j=1}^{N} U_{ij} \tag{1.97}$$

$$= \frac{1}{8\pi\varepsilon_0}\sum_{i,j=1}^{N}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{1.98}$$

$$= \frac{1}{2}\sum_{i=1}^{N} q_i\phi(\mathbf{r}_i), \tag{1.99}$$

where

$$\phi(\mathbf{r}_i) = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{N}\frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{1.100}$$

It is half (to avoid double counting) of the sum of the potential energy from each charge.

A continuous charge distribution can be divided into volume elements with charges $q_i = \rho(\mathbf{r}_i)dv_i$. Thus, just replace the q_i in Eq. (1.98) with $\rho(\mathbf{r}_i)dv_i$, and replace the summation with integral to get

$$U_E = \frac{1}{8\pi\varepsilon_0}\int dvdv' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \quad (1.101)$$
$$= \frac{1}{2}\int dv\rho(\mathbf{r})\phi(\mathbf{r}). \quad (1.102)$$

We can rewrite this expression as,

$$U_E = \frac{\varepsilon_0}{2}\int dv|\mathbf{E}|^2. \quad (1.103)$$

Pf: The charge density can be related to field using Gauss's law,

$$\rho(\mathbf{r}) = \varepsilon_0 \nabla\cdot\mathbf{E}. \quad (1.104)$$

With integration by parts, Eq. (1.102) becomes

$$U_E = \frac{\varepsilon_0}{2}\int dv\nabla\cdot\mathbf{E}\ \phi(\mathbf{r}) \quad (1.105)$$
$$= -\frac{\varepsilon_0}{2}\int dv\mathbf{E}\cdot\nabla\phi + \text{surface term} \quad (1.106)$$
$$= \frac{\varepsilon_0}{2}\int dv|\mathbf{E}|^2. \quad (1.107)$$

The surface term can be dropped since the surface (of the whole space) is at infinity. The integrand above is the **energy density** of electric field,

$$u_E = \frac{\varepsilon_0}{2}|\mathbf{E}|^2 \quad (1.108)$$

Note that the electrostatic energy in Eq. (1.101) is always positive but the one in Eq. (1.98) can be positive or negative. This is because the self-energy of point charge, which is positive and infinite, is not included in Eq. (1.98).

To illustrate this, consider two different charge distributions ρ_1 and ρ_2. The electrostatic energy of the whole system with $\rho(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho(\mathbf{r})$ is, according to Eq. (1.101),

$$U_E = U_1 + U_2 + \frac{1}{4\pi\varepsilon_0}\int dvdv' \frac{\rho_1(\mathbf{r})\rho_2(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}, \quad (1.109)$$

where $U_{1,2}$ are the "self-energies" of $\rho_{1,2}$. U_E is always positive, but the "interaction energy" (the last term above) can be either positive or negative.

Example:
Calculate the electrostatic energy of a uniformly charged ball with radius a and charge Q.

Sol'n:
Instead of using Eq. (1.107), let's calculate U_E with the work required to build this charged ball. Write the charge density of the ball as ρ_0. A ball with radius r has charge $Q(r) = \rho_0(4\pi r^3/3)$. The work required to add an additional layer with thickness dr is $dW = dQ\ \phi_s$, where ϕ_s is the potential at the surface,

$$dQ = \rho_0 4\pi r^2 dr, \quad (1.110)$$
$$\phi_s = \frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r} = \frac{\rho_0}{3\varepsilon_0}r^2. \quad (1.111)$$

Thus,

$$dW = dQ\ \phi_s = \frac{4\pi}{3\varepsilon_0}\rho_0^2 r^4 dr, \quad (1.112)$$

and

$$U_E = \int dW \quad (1.113)$$
$$= \frac{4\pi}{3\varepsilon_0}\rho_0^2\int_0^a r^4 dr,\ \ \rho_0 = \frac{Q}{4\pi a^3/3} \quad (1.114)$$
$$= \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 a}. \quad (1.115)$$

You may also calculate U_E using Eq. (1.107). This is left as an exercise.

Problem:
1. Starting from the electric potential for a finite, charged wire in Eq. (1.22), verify that (a) at large distance it reduces to Eq. (1.24); (b) at short distance, it reduces to (1.25).
2. Suppose a metallic, spherical shell with radius 1 m has total charge $Q = 10^{-3}$ C.
(a) Find out its surface charge density σ.
(b) Find out magnitude and direction of the pressure $\mathbf{f}$ (due to the electric field) on the wall of the spherical shell.
3. Two concentric, spherical metal *shells* have radii a and b ($b > a$). The inner shell and the outer shell have charges Q and $-Q$ respectively. Two shells are separated by vacuum.
(a) What is the electric field inside and outside the two shells?
(b) What is the total electrostatic energy of this system?

References

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