Lecture notes on classical electrodynamics

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Contents

I. THE MAXWELL EQUATIONS

In this chapter, we outline the fundamental equations in electrodynamics.

A. Charge and current

1. Charge density

Consider a distribution of charge inside a volume V. If in a volume element dv near point $\mathbf{r}$, there is charge dQ, then the **charge density** at this location is

$$\rho(\mathbf{r}) \equiv \frac{dQ}{dv}. \tag{1.1}$$

By the integration of $\rho(\mathbf{r})$, we can have the total charge Q inside a volume V,

$$Q = \int_V dv \rho(\mathbf{r}). \tag{1.2}$$

As we have mentioned in Chap 1, for a point charge q at $\mathbf{r}_1$, its charge density is,

$$\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}_1). \tag{1.3}$$

If there are point charges $q_1, q_1, \cdots, q_N$ at locations $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$, then the charge density of this system is,

$$\rho(\mathbf{r}) = \sum_{i=1}^{N} q_i \delta(\mathbf{r} - \mathbf{r}_i). \tag{1.4}$$

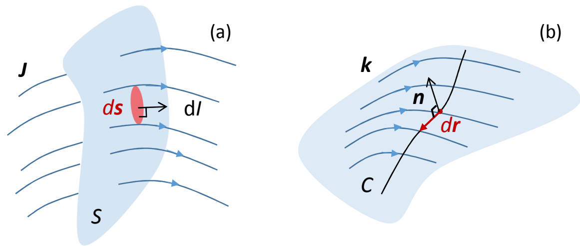

FIG. 1 (a) Current density $\mathbf{J}$ is the current flowing through a unit area ds. (b) Surface current density $\mathbf{K}$ is the surface current flowing pass a unit length dr.

The total charge inside a volume V that encloses these charges is,

$$Q = \int_V dv \rho(\mathbf{r}) = \sum_{i=1}^{N} q_i \int_V dv \delta(\mathbf{r} - \mathbf{r}_i) \tag{1.5}$$

$$= \sum_{i=1}^{N} q_i. \tag{1.6}$$

Given a distribution of charges on a surface S. If on a surface element ds near point $\mathbf{r}$, there is charge dQ, then the **surface charge density** at this location is

$$\sigma(\mathbf{r}) \equiv \frac{dQ}{ds}. \tag{1.7}$$

By the integration of $\sigma(\mathbf{r})$, we can have the total charge Q on a surface S,

$$Q = \int_S ds\ \sigma(\mathbf{r}). \tag{1.8}$$

2. Current density

Electric current passing through a surface S is defined as the amount of charge passing through S per unit time. Current density is the current per unit area. Its dimension is [current]/[area], the dimension of current divided by the dimension of area. If there is current dI passing through a surface element $d\mathbf{s} = ds\hat{\mathbf{n}}$, then (Fig. 1(a))

$$dI = \mathbf{J}(\mathbf{r}) \cdot d\mathbf{s} = J_\parallel(\mathbf{r}) ds, \tag{1.9}$$

where $\mathbf{J}(\mathbf{r})$ is the **current density** along the direction of charge motion, and $J_\parallel = \mathbf{J} \cdot \hat{\mathbf{n}}$ is its component along the surface normal $\hat{\mathbf{n}}$.

After integration, we can find out the total current passing through surface S,

$$I = \int_S d\mathbf{s} \cdot \mathbf{J}(\mathbf{r}). \tag{1.10}$$

If a small packet of charge dQ is moving with velocity $\mathbf{v}$, then within a time dt, the charges passing through $d\mathbf{s}$ have spanned a volume $dv = (\mathbf{v}dt) \cdot d\mathbf{s}$. Inside this volume,

$$dQ = \rho dv = \rho(\mathbf{v}dt) \cdot d\mathbf{s}, \tag{1.11}$$

which delivers a current,

$$dI = \frac{dQ}{dt} = \rho \mathbf{v} \cdot d\mathbf{s}. \tag{1.12}$$

Compared with Eq. (1.9), one has

$$\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}(\mathbf{r}). \tag{1.13}$$

For point charges, with Eq. (1.4), one has

$$\mathbf{J}(\mathbf{r}) = \sum_{i=1}^{N} q_i \mathbf{v}_i \delta(\mathbf{r} - \mathbf{r}_i), \tag{1.14}$$

where $\mathbf{v}_i$ is the velocity of charge i.

Next, consider the current flowing on a surface. The surface has normal vector $\hat{\mathbf{n}}$, and there is a line element $d\mathbf{r} \perp \hat{\mathbf{n}}$ on the surface (see Fig. 1(b)). The vector $\hat{\mathbf{n}} \times d\mathbf{r}$ is tangent to the surface and perpendicular to $d\mathbf{r}$. The current dI passes through $d\mathbf{r}$ is,

$$dI = \mathbf{K}(\mathbf{r}) \cdot (\hat{\mathbf{n}} \times d\mathbf{r}), \tag{1.15}$$

where $\mathbf{K}(\mathbf{r})$ is the **surface current density** along the direction of charge motion. Its dimension is [current]/[length].

After integration, we can find out the total current passing through a curve C on the surface,

$$I = \int_C \mathbf{K}(\mathbf{r}) \cdot \hat{\mathbf{n}} \times d\mathbf{r} = \int_C \mathbf{K}(\mathbf{r}) \times \hat{\mathbf{n}} \cdot d\mathbf{r} \tag{1.16}$$

3. Conservation of charge

Suppose the charge Q inside a volume V is leaking through its surface S to the outside (Fig. 2). The leaking current through S is,

$$I = -\frac{dQ}{dt}. \tag{1.17}$$

With (1.10), we have

$$I = \int_S d\mathbf{s} \cdot \mathbf{J} = \int_V dv \nabla \cdot \mathbf{J}, \tag{1.18}$$

and from Eqs. (1.2),

$$\frac{dQ}{dt} = \frac{d}{dt}\int_V dv \rho(\mathbf{r},t) = \int_V dv \frac{\partial \rho(\mathbf{r},t)}{\partial t}, \tag{1.19}$$

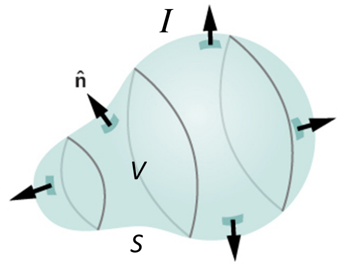

FIG. 2 Charges I flowing out of the surface S of volume V.

in which the region V of integration is fixed. Hence,

$$\int_V dv \nabla \cdot \mathbf{J} = -\int_V dv \frac{\partial \rho(\mathbf{r},t)}{\partial t} \tag{1.20}$$

$$\text{or} \quad \int_V dv \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right) = 0. \tag{1.21}$$

Since the charge should be conserved for any dv in any location, so we can choose V to be one of the dv, then

$$\int_V dv \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right) \simeq dv \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right) \tag{1.22}$$

$$\rightarrow \nabla \cdot \mathbf{J}(\mathbf{r},t) + \frac{\partial \rho(\mathbf{r},t)}{\partial t} = 0, \text{ at any } \mathbf{r}. \tag{1.23}$$

This is **equation of continuity**, which is valid *if and only if* charge is conserved.

B. Maxwell equations in vacuum

1. Electrostatics

According to **Coulomb's law**, the electric force between two charges q, q_1 at positions $\mathbf{r}, \mathbf{r}_1$ is,

$$\mathbf{F} = \frac{qq_1}{4\pi\varepsilon_0}\frac{\mathbf{r} - \mathbf{r}_1}{|\mathbf{r} - \mathbf{r}_1|^3}, \tag{1.24}$$

where the **electric permittivity** of free space $\varepsilon_0 = 8.8542 \times 10^{-12}$ C^2/Nm^2.

If there are N charges $q_1, q_2, \cdots, q_N$ at positions $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$, then a test charge charge q at $\mathbf{r}$ feels a force,

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N} qq_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}. \tag{1.25}$$

The electric field $\mathbf{E}$ from these N charges is given as,

$$\mathbf{E}(\mathbf{r}) \equiv \frac{\mathbf{F}}{q} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N} q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}. \tag{1.26}$$

A continuous charge distribution can be divided into small packets with charges $\rho(\mathbf{r}')dv'$ (Fig. 3). Identify q_i

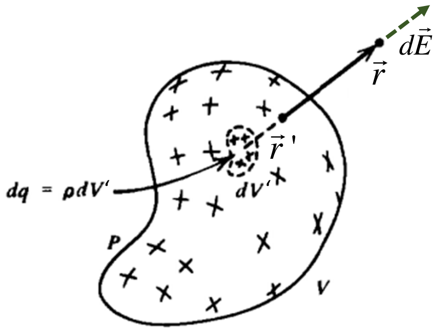

FIG. 3 The electric field $\mathbf{E}(\mathbf{r})$ at point $\mathbf{r}$ is the sum of the electric fields $d\mathbf{E}$ produced by charges $\rho(\mathbf{r}')dv'$ in volume elements.

with $\rho(\mathbf{r}')dv'$ and replace the summation with an integral, one then has

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \quad (1.27)$$

This form is valid for all kinds of charge distribution, continuous or discrete. You may check that with Eq. (1.4), Eq. (1.27) reduces to Eq. (1.26).

We can rewrite

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \quad (1.28)$$

so that

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -\frac{1}{4\pi\varepsilon_0}\int dv'\rho(\mathbf{r}')\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}, && (1.29)\\ &= -\nabla\phi(\mathbf{r}), && (1.30)\end{aligned}$$

with **electric potential**,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \quad (1.31)$$

Note that the order of $\int dv'$ and ∇ can be exchanged, since $\mathbf{r}'$ and $\mathbf{r}$ are independent variables.

Also, remember that

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}'). \quad (1.32)$$

Thus,

$$\begin{aligned}\nabla\cdot\mathbf{E}(\mathbf{r}) &= -\frac{1}{4\pi\varepsilon_0}\int dv'\rho(\mathbf{r}')\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} && (1.33)\\ &= \frac{1}{\varepsilon_0}\int dv'\rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') && (1.34)\\ &= \frac{\rho(\mathbf{r})}{\varepsilon_0}. && (1.35)\end{aligned}$$

This is **Gauss's law**. When written in electric potential, we have

$$\nabla^2\phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}. \quad (1.36)$$

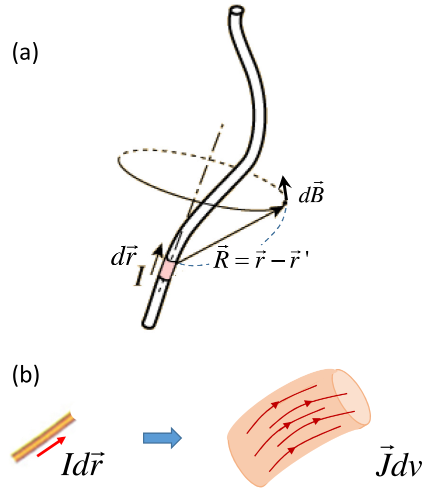

FIG. 4 (a) The magnetic field $d\mathbf{B}$ produced by a segment $d\mathbf{r}$ of a current-carrying wire. (b) A thin wire is replaced by a body (or a region) with volume element dv.

This is the **Poisson equation** that has been mentioned in Chap 1.

Furthermore, since the curl of divergence is zero, so

$$\nabla\times\mathbf{E}(\mathbf{r}) = -\nabla\times\nabla\phi = 0. \quad (1.37)$$

In short, the fundamental equations of electrostatics are

$$\begin{aligned}\nabla\cdot\mathbf{E}(\mathbf{r}) &= \frac{\rho(\mathbf{r})}{\varepsilon_0}, && (1.38)\\ \nabla\times\mathbf{E}(\mathbf{r}) &= 0. && (1.39)\end{aligned}$$

If we integrate Eq. (1.38) over a region V enclosed by surface S, then

$$\int_v dv\nabla\cdot\mathbf{E}(\mathbf{r}) = \frac{1}{\varepsilon_0}\int_V dv\rho(\mathbf{r}), \quad (1.40)$$

$$\text{or} \quad \int_S d\mathbf{s}\cdot\mathbf{E}(\mathbf{r}) = \frac{Q}{\varepsilon_0}, \quad (1.41)$$

where Q is the total amount of charge inside V. This is the integral form of the Gauss's law.

If we integrate Eq. (1.39) over a surface S with boundary C, then

$$\begin{aligned}\int_S d\mathbf{s}\cdot\nabla\times\mathbf{E}(\mathbf{r}) &= 0, && (1.42)\\ \text{or} \quad \oint_C d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) &= 0. && (1.43)\end{aligned}$$

2. Magnetostatics

According to **Biot-Savart law**, the magnetic field produced by a short segment $d\mathbf{r}'$ of a thin wire carrying current I is (see Fig. 4(a)),

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}Id\mathbf{r}'\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \quad (1.44)$$

where the **magnetic permeability** in vacuum $\mu_0 = 4\pi \times 10^{-7}$ N/A^2. For a closed loop C of thin wire,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint_C I d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{1.45}$$

Given a general current distribution, just (see Fig. 4(b))

$$\text{replace} \;\; I d\mathbf{r}' \;\; \text{with} \;\; \mathbf{J}(\mathbf{r}') dv', \tag{1.46}$$

so that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V dv' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{1.47}$$

This is the most general form of the Biot-Savart law that applies to all kinds of current distribution.

Again we can rewrite

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.48}$$

With the identity,

$$\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f \nabla \times \mathbf{v}, \tag{1.49}$$

we have

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= -\frac{\mu_0}{4\pi} \int dv' \mathbf{J}(\mathbf{r}') \times \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} && (1.50) \\ &= \nabla \times \mathbf{A}(\mathbf{r}), && (1.51) \end{aligned}$$

with the **vector potential**,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dv' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.52}$$

For a thin wire, it reduces to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \oint_C d\mathbf{r}' \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{1.53}$$

Since the divergence of curl is zero, so

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \nabla \cdot \nabla \times \mathbf{A}(\mathbf{r}) = 0. \tag{1.54}$$

This is **Gauss's law in magnetism**. Also, if we take the curl of $\mathbf{B}$, then

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}). \tag{1.55}$$

This is **Ampère's law**.
Pf: First, we can show that for the steady case $\nabla \cdot \mathbf{J} = 0$, one has $\nabla \cdot \mathbf{A} = 0$. This is because

$$\begin{aligned} \nabla \cdot \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int dv' \mathbf{J}(\mathbf{r}') \cdot \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} && (1.56) \\ &= -\frac{\mu_0}{4\pi} \int dv' \mathbf{J}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} && (1.57) \\ &= \frac{\mu_0}{4\pi} \int dv' \underbrace{\nabla' \cdot \mathbf{J}(\mathbf{r}')}_{=0} \frac{1}{|\mathbf{r}-\mathbf{r}'|} && (1.58) \\ &= 0, && (1.59) \end{aligned}$$

where we have used the identity,

$$\nabla \cdot (f\mathbf{v}) = \nabla f \cdot \mathbf{v} + f \nabla \cdot \mathbf{v}. \tag{1.60}$$

Also, a surface term (for the surface at infinity) has been dropped.

Second,

$$\begin{aligned} \nabla \times \mathbf{B}(\mathbf{r}) &= \nabla \times (\nabla \times \mathbf{A}) && (1.61) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} && (1.62) \\ &= -\nabla^2 \mathbf{A}(\mathbf{r}) \quad \because \; \nabla \cdot \mathbf{A} = 0 && (1.63) \\ &= -\frac{\mu_0}{4\pi} \int dv' \mathbf{J}(\mathbf{r}') \underbrace{\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|}}_{=-4\pi\delta(\mathbf{r}-\mathbf{r}')} && (1.64) \\ &= \mu_0 \mathbf{J}(\mathbf{r}). \quad QED && (1.65) \end{aligned}$$

When written in potential, we have

$$\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r}). \tag{1.66}$$

This is the **vector Poission equation** in magnetostatics.

In short, the fundamental equations of magnetostatics are

$$\begin{aligned} \nabla \cdot \mathbf{B}(\mathbf{r}) &= 0, && (1.67) \\ \nabla \times \mathbf{B}(\mathbf{r}) &= \mu_0 \mathbf{J}(\mathbf{r}). && (1.68) \end{aligned}$$

If we integrate Eq. (1.67) over a region V enclosed by surface S, then

$$\int_v dv \nabla \cdot \mathbf{B}(\mathbf{r}) = \int_S d\mathbf{s} \cdot \mathbf{B}(\mathbf{r}) = 0, \tag{1.69}$$

This shows that the magnetic flux through a closed surface is always zero. The existence of a magnetic monopole would contradict this result, but no magnetic monopole has been found so far.

If we integrate Eq. (1.68) over a surface S with boundary C, then

$$\begin{aligned} \int_S d\mathbf{s} \cdot \nabla \times \mathbf{B}(\mathbf{r}) &= \mu_0 \int_S d\mathbf{s} \cdot \mathbf{J}(\mathbf{r}), && (1.70) \\ \text{or} \quad \oint_C d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) &= \mu_0 I, && (1.71) \end{aligned}$$

where I is the total current flowing through S. This is the integral form of the Ampère's law.

3. Dynamic electromagnetic field

Eqs. (1.38), (1.39), (1.67), and (1.68) are the Maxwell equations for static electromagnetic field. For dynamics fields, we need to add two new terms,

$$\begin{aligned} \nabla \cdot \mathbf{E}(\mathbf{r},t) &= \frac{\rho(\mathbf{r},t)}{\varepsilon_0}, && (1.72) \\ \nabla \cdot \mathbf{B}(\mathbf{r},t) &= 0, && (1.73) \\ \nabla \times \mathbf{E}(\mathbf{r},t) &= -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t), && (1.74) \\ \nabla \times \mathbf{B}(\mathbf{r},t) &= \mu_0 \mathbf{J}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t). && (1.75) \end{aligned}$$

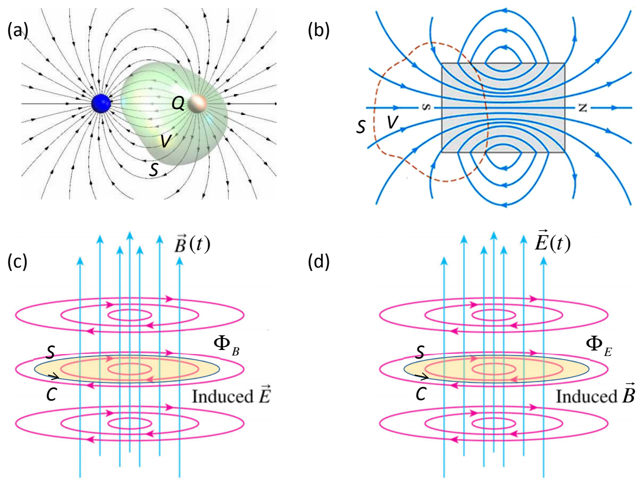

FIG. 5 Illustration of the Maxwell's equations: (a) Gauss's law. (b) Gauss's law in magnetism. (c) Faraday's law. (d) Ampère-Maxwell's law (with $I = 0$). Figs. from the web.

The charge density and the current density are related by the equation of continuity,

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0. \tag{1.76}$$

The first change is that in Eq. (1.74), the right hand side (RHS) is no longer zero. This is **Faraday's law**:*a time-changing magnetic field produces an electric field.*

The second change is that there is an extra term on the RHS of Eq. (1.75). This is the famous **displacement current** added by Maxwell: *a time-changing electric field produces a magnetic field.* This modified equation is called **Ampère-Maxwell's law**.

When the fields are static, the Maxwell's equations decouple into two sets of equations: two for electric field, and two for magnetic field. Thus, electrostatics and magnetostatics are independent of each other.

Integrating a divergence (e.g., $\nabla \cdot \mathbf{E}$) over a volume V or a curl (e.g., $\nabla \times \mathbf{E}$) over a surface, and using the divergence theorem or the Stokes theorem, we have the *integral form* of the Maxwell equations (Fig. 5):

$$\int_S d\mathbf{s} \cdot \mathbf{E}(\mathbf{r}, t) = \frac{Q}{\varepsilon_0}, \tag{1.77}$$

$$\int_S d\mathbf{s} \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{1.78}$$

$$\oint_C d\mathbf{r} \cdot \mathbf{E}(\mathbf{r}, t) = -\frac{d\Phi_B}{dt}, \tag{1.79}$$

$$\oint_C d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}, t) = \mu_0 I + \frac{1}{c^2}\frac{d\Phi_E}{dt}, \tag{1.80}$$

in which

$$\Phi_B \equiv \int_S d\mathbf{s} \cdot \mathbf{B}, \tag{1.81}$$

$$\Phi_E \equiv \int_S d\mathbf{s} \cdot \mathbf{E}. \tag{1.82}$$

They are the **magnetic flux** and the **electric flux** passing through surface S. Eq. (1.79) (Eq. (1.80)) tells us that a changing magnetic (electric) flux through surface S would induce electric (magnetic) circulation around the boundary C of S.

Note: The first order derivatives of a vector $\mathbf{V}(\mathbf{r})$ have 9 components, $\partial V_i/\partial x_j$ $(i, j = 1, 2, 3)$. The Maxwell equations are written in terms of divergence and curl of $\mathbf{E}$ (or $\mathbf{B}$), which does not exhaust the possibilities just mentioned. This is all right since according to the **Helmholtz theorem**, a vector field $\mathbf{V}(\mathbf{r})$ that vanishes at infinity is completely determined by giving its divergence and curl everywhere in space.

C. Some history

In 1873, James C. Maxwell published " *Treatise on electricity and magnetism*" (Maxwell, 1891), in which he constructed a mathematical framework to describe the phenomena of electromagnetism. It has all the essence included but it's hard to find "Maxwell equations" in the *Treatise*, since they are written as 20 equations in 20 variables scattered through the monograph. Some of the equations describe things like $\mathbf{D} = \varepsilon\mathbf{E}$, or $\mathbf{B} = \nabla \times \mathbf{A}$. It's a pity that Maxwell died six years later at the age of 48, and was unable to pursue this subject further.

The four Maxwell equations we are familiar with nowadays are mainly the works of Oliver Heaviside and, independently, Heinrich R. Hertz (Fig. 6). It's interesting to know that when the *Treatise* was just published, Heaviside (then 24 years old) flipped through it in library and immediately saw the "prodigious possibilities in its power". He then "determined to master the book". (Mahon, 2017) Remember that at that time Maxwell is still not "the Maxwell" and not many people trust his obscure, sometimes unintelligible theory of electromagnetism.

Heaviside has no college education, and has forgotten most of the algebra and trigonometry learned in school. Thus, he quit his job that has a decent pay, stayed at home with his far-from-rich parents and started studying the *Treatise*. He remained "self-employed" ever since and never to get a job again. Heaviside has to learn all of the difficult mathematics of divergence, curl, and related theorems on his own, without friendly textbooks to ease the job. In his later years, Heaviside recalls that "It took me several years before I could understand as much as I possibly could. Then I set Maxwell aside and followed my own course."

The effort and sacrifice pay off. With his own formulation of Maxwell equations, Heaviside discovered things like **electric inductance**, contraction of the electric field of a moving charge (**Heaviside ellipsoid**), and magnetic-like field of gravity (**gravito-magnetism**).

In 1888, to the surprise of everybody, Hertz generated and detected electromagnetic wave in free space. This is the strongest boost to the status of Maxwell's electro-

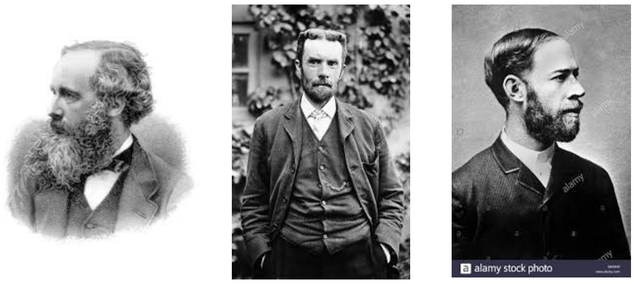

FIG. 6 From left to right, Maxwell, Heaviside, and Hertz.

magnetic theory since at that time there was no other theory of electromagnetism that predicted the existence of EM wave. Afterwards, optics becomes a branch of electromagnetism.

More progress followed, such as the discovery of **electron** (the source of electric field) by J. J. Thomson in 1897, the theory of **thermal radiation** (randomized EM field) by Ludwig E. Boltzmann and others. The latter pursuit eventually leads to Max Planck's important discovery of **energy quantum** at 1900.

Furthermore, in an attempt to resolve a paradox regarding motional electromotive force, Einstein discovered the theory of **special relativity** in 1905. As a result, Newton's theory of mechanics needs to be revised. Nevertheless, *Maxwell's theory remains intact*, since it is based on experimental observations that have already included relativistic effects.

Problem:

1. The electric potential of an atom is given by

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}, \tag{1.83}$$

where $q(>0), \alpha$ are constants.
(a) Find out the electron charge density $\rho(\mathbf{r})$ outside the nucleus.
(b) Find out the total charge of this charge distribution.
Hint: Poission equation.

2. (a) Show that Eqs. (1.72) and (1.75) are consistent with the equation of continuity in Eq. (1.76).
Hint: Take the time derivative of ρ on the right-hand side of Eq. (1.72), and the divergence of $\mathbf{J}$ on the right-hand side of Eq. (1.75).
(b) Suppose there are magnetic monopoles, such that

$$\nabla \cdot \mathbf{B} = \mu_o \rho_m, \tag{1.84}$$

where ρ_m is the magnetic charge density Similar to electric charges, the equation of continuity of magnetic charges is,

$$\nabla \cdot \mathbf{J}_m + \frac{\partial \rho_m}{\partial t} = 0, \tag{1.85}$$

where $\mathbf{J}_m$ is the magnetic current density. What type of term should be added to the right-hand-side of Eq. (1.74), so that new Maxwell equations can be consistent with the equation of continuity above?

References

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