Lecture notes on classical electrodynamics

Ming-Che Chang

Department of Physics, National Taiwan Normal University, Taipei, Taiwan

(Dated: September 26, 2021)

Contents

I. MATHEMATICAL PRELIMINARIES

In this chapter, we collect some mathematics that is *essential* to the learning of electrodynamics.

A. Coordinate system

A coordinate system combines geometry with algebra. That is, we can use numbers to describe geometrical objects. Here we introduce three of the most popular coordinate systems.

1. Cartesian coordinate
The word Cartesian comes from the latinized name of Descartes - Cartesius. The three coordinate axes are perpendicular to each other. A point has coordinates (x, y, z), and unit basis vectors are $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$.

A **position vector** is,

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{1.1}$$

A vector at point $\mathbf{r}$ is

$$\mathbf{V}(\mathbf{r}) = V_x(\mathbf{r})\hat{\mathbf{x}} + V_y(\mathbf{r})\hat{\mathbf{y}} + V_z(\mathbf{r})\hat{\mathbf{z}}. \tag{1.2}$$

The distribution of vectors $\mathbf{V}(\mathbf{r})$ in space is a **vector field**.

2. Cylindrical coordinate
As shown in Fig. 2(a), a point in cylindrical coordinate has coordinates (ρ, ϕ, z). The unit basis vectors are $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$, which are along the direction of increase of ρ, ϕ, z and are perpendicular to each other. The connection between Cartesian and cylindrical coordinates are,

$$x = \rho\cos\phi, \tag{1.3}$$
$$y = \rho\sin\phi, \tag{1.4}$$
$$z = z. \tag{1.5}$$

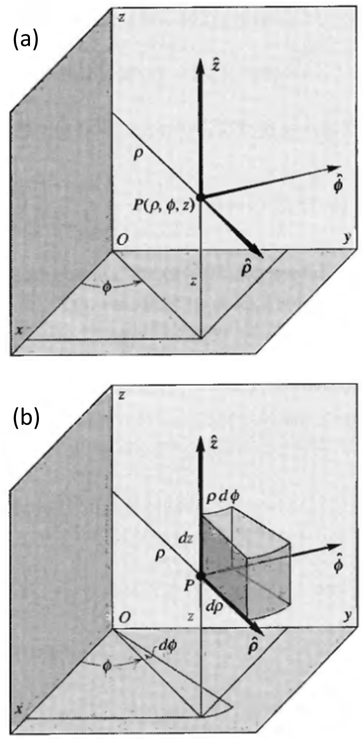

FIG. 1 Cylindrical coordinate (a) and its volume element (b). Figs. from Lorrain and Corson, 1970.

A position vector is

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}. \tag{1.6}$$

The coordinates ρ, z account for two degrees of freedom. The third one is hidden in the angle ϕ of $\hat{\boldsymbol{\rho}}$. A vector at point $\mathbf{r}$ can be expanded as,

$$\mathbf{V}(\mathbf{r}) = V_\rho(\mathbf{r})\hat{\boldsymbol{\rho}} + V_\phi(\mathbf{r})\hat{\boldsymbol{\phi}} + V_z(\mathbf{r})\hat{\mathbf{z}}. \tag{1.7}$$

3. Spherical coordinate

As shown in Fig. 1(c), a point in spherical coordinate has coordinates (r, θ, ϕ). These are standard notations used by most people, so you need to keep them in mind, since a figure is not always drawn to remind you of their meaning. The unit basis vectors are $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$, which are along the direction of increase of r, θ, ϕ and are perpendicular to each other. The connection between Cartesian

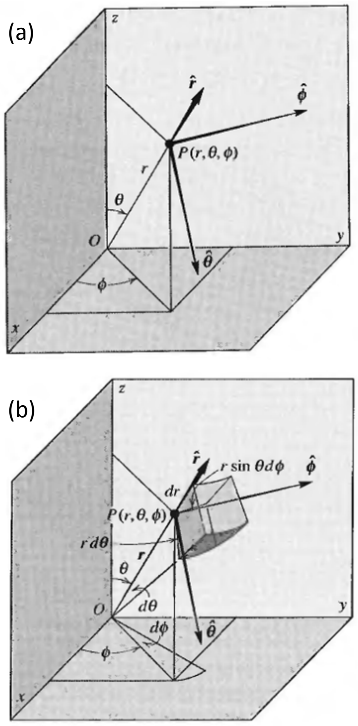

FIG. 2 Spherical coordinate (a) and its volume element (b). Figs. from Lorrain and Corson, 1970.

and spherical coordinates are,

$$x = r\sin\theta\cos\phi, \quad (1.8)$$
$$y = r\sin\theta\sin\phi, \quad (1.9)$$
$$z = r\cos\theta. \quad (1.10)$$

A position vector is simply

$$\mathbf{r} = r\hat{\mathbf{r}}. \quad (1.11)$$

The coordinate r accounts for one degree of freedom. The other two are hidden in the angles θ, ϕ of $\hat{\mathbf{r}}$. A vector at point $\mathbf{r}$ can be expanded as,

$$\mathbf{V}(\mathbf{r}) = V_r(\mathbf{r})\hat{\mathbf{r}} + V_\theta(\mathbf{r})\hat{\boldsymbol{\theta}} + V_\phi(\mathbf{r})\hat{\boldsymbol{\phi}}. \quad (1.12)$$

Note that the volume elements in Cartesian, cylindrical, and spherical coordinates are (Fig. 1(b) and Fig. 2(b))

$$dv = dxdydz, \quad (1.13)$$
$$dv = \rho d\rho d\phi dz, \quad (1.14)$$
$$dv = r^2\sin\theta dr d\theta d\phi, \quad (1.15)$$

The major difference between Newton's dynamics and Maxwell's dynamics is that in the former we simply deal with particle trajectory $\mathbf{r}(t)$, while the latter we need to deal with field distribution $\mathbf{V}(\mathbf{r}, t)$. This makes electrodynamic much harder to learn.

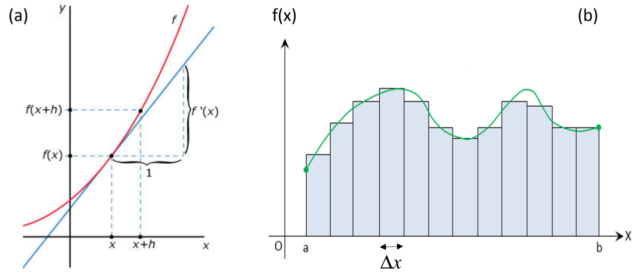

FIG. 3 (a) A derivative. (b) An integration.

B. Basics of calculus

Recall that the derivative of $f(x)$ at x is defined as (see Fig. 3(a)),

$$\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}. \quad (1.16)$$

When $h = \Delta x$ is small but finite, one has

$$\frac{df(x)}{dx} \simeq \frac{f(x+\Delta x) - f(x)}{\Delta x}. \quad (1.17)$$

Thus,

$$f(x+\Delta x) \simeq f(x) + \frac{df(x)}{dx}\Delta x. \quad (1.18)$$

For a function $f(\mathbf{r})$ in three dimensions,

$$f(\mathbf{r}+\Delta\mathbf{r}) \simeq f(\mathbf{r}) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z. \quad (1.19)$$

We'll often write Δx as dx (or $\Delta\mathbf{r}$ as $d\mathbf{r}$), without distinguishing between finite and infinitesimal, when the limit $\Delta x \to dx$ (or $\Delta\mathbf{r} \to d\mathbf{r}$) needs to be taken at the end of a derivation.

The integral of $f(x)$ is the area between the curve $f(x)$ and the x-axis, which can be approximated as a sum of the areas of rectangles (Fig. 3(b))

$$\int_a^b dx f(x) \simeq \sum_i \Delta x f(x_i), \quad (1.20)$$

where x_i can be any point (e.g., the middle one) inside an interval Δx. The equation above becomes an equality when the division becomes infinitesimal, $\Delta x \to dx$. It follows from the equation above that,

$$\sum_i f(x_i) \simeq \frac{1}{\Delta x}\int_a^b dx f(x). \quad (1.21)$$

That is, if $f(x)$ is smooth, then you can evaluate its summation with the help of integration.

In three dimensions, the integral of $f(\mathbf{r})$ over a region V is given as,

$$\int_V dv f(\mathbf{r}) \simeq \sum_i \Delta v f(\mathbf{r}_i), \quad (1.22)$$

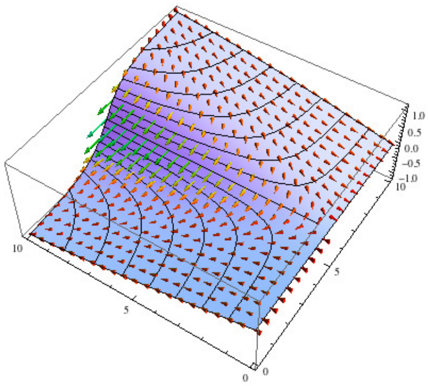

FIG. 4 The gradient vectors $-\nabla f$ of a function $f(x,y)$ in two dimension. Fig. from the web.

where the region V is divided into many small boxes, and dv is a volume element (the volume of a box) around $\mathbf{r}_i$. The equation above approaches an equality when the division gets finer and finer, $\Delta v \to 0$.

Finally,

$$\int^x dx' \frac{df}{dx'} = f(x) + c, \tag{1.23}$$

where c is a constant. Also,

$$\frac{d}{dx}\int^x dx' f(x') = f(x). \tag{1.24}$$

That is, integration is the opposite of differentiation, and vice versa. This is called the **fundamental theorem of calculus**.

C. Differentiation of field

A *scalar field* $f(\mathbf{r})$, or $f(x,y,z)$, describes, e.g., the distribution of temperature or charge density in space. A *vector field* $\mathbf{V}(\mathbf{r})$, or $\mathbf{V}(x,y,z)$, describes, e.g., the distribution of fluid velocity or electric field in space. We review three major differential operations of fields: gradient, divergence, and curl.

1. **Gradient**

The gradient of a scalar function $f(\mathbf{r})$ is defined as,

$$\nabla f(\mathbf{r}) \left(\text{or } \frac{\partial f}{\partial \mathbf{r}}\right) = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \tag{1.25}$$

in which ∇ is called *del*.

The **total derivative** of $f(\mathbf{r})$ (see Eq. (1.19)),

$$\begin{aligned} df(\mathbf{r}) &\equiv f(\mathbf{r}+d\mathbf{r}) - f(\mathbf{r}) && (1.26)\\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz && (1.27)\\ &= \nabla f \cdot d\mathbf{r}. && (1.28)\end{aligned}$$

That is,

$$\nabla f \cdot d\mathbf{r} = df, \text{the change of } f \text{ along } d\mathbf{r}. \tag{1.29}$$

Since $df = |\nabla f||d\mathbf{r}|\cos\theta$ (θ is the angle between ∇f and $d\mathbf{r}$), if we fix $|d\mathbf{r}|$ and swivel the vector $d\mathbf{r}$ around, then df is maximum when $d\mathbf{r} \parallel \nabla f$. Therefore, *the direction of* ∇f = *The direction of maximum increase of* $f(\mathbf{r})$ (i.e., the steepest *ascent*). Conversely, $-\nabla f$ points to the direction of steepest *descent* (Fig. 4). For example, given a temperature distribution $T(\mathbf{r})$, the heat current $\mathbf{J}_T(\mathbf{r})$ flows along the steepest descent of the temperature,

$$\mathbf{J}_T(\mathbf{r}) = -\kappa \nabla T(\mathbf{r}), \tag{1.30}$$

where κ is the **thermal conductivity**. This is **Fourier's law** of heat conduction.

Similarly, given an electric potential $\phi(\mathbf{r})$, the current are flowing along the steepest descent of the potential,

$$\mathbf{J}(\mathbf{r}) = -\sigma\nabla\phi(\mathbf{r}) = \sigma\mathbf{E}, \tag{1.31}$$

where σ is the electric conductivity, and $\mathbf{E} = -\nabla\phi$ is the electric field. This is the **Ohm's law**.

On the other hand, when $d\mathbf{r} \perp \nabla f(\mathbf{r})$, then $df = 0$. Thus $f(\mathbf{r})$ is not changed (to the first order) along the plane perpendicular to $\nabla f(\mathbf{r})$.

For reference, in cylindrical and spherical coordinates,

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, && (1.32)\\ \nabla f &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}. && (1.33)\end{aligned}$$

2. **Divergence**

The divergence of a vector field $\mathbf{V}(\mathbf{r})$ is defined as,

$$\nabla \cdot \mathbf{V}(\mathbf{r}) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}. \tag{1.34}$$

Given a volume element $dv = dxdydz$, which is a small box around point $P = (x,y,z)$ (Fig. 5(a)), we have

$$\begin{aligned} \nabla \cdot \mathbf{V}(\mathbf{r})dv &= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dxdydz \\ &\simeq \Delta V_x dydz + \Delta V_y dzdx + \Delta V_z dxdy \\ &= (V_{x,+} - V_{x,-})dydz + (V_{y,+} - V_{y,-})dzdx \\ &+ (V_{z,+} - V_{z,-})dxdy, && (1.35)\end{aligned}$$

where $V_{x,\pm} \equiv V_x(x \pm dx/2, y, z)$, and similarly for $V_{y,\pm}$ and $V_{z,\pm}$.

The term $V_{x,+}dydz$ is the flux passing through the area $dydz$ at $x + dx/2$; $V_{x,-}dydz$ is the flux passing through the area $dydz$ at $x - dx/2$. Similarly for the other terms. Thus, $\nabla \cdot \mathbf{V}dv$ is the **flux** *out of* the box dv (Fig. 5(b)),

$$\nabla \cdot \mathbf{V}dv = \int_{\text{box}} d\mathbf{s} \cdot \mathbf{V}(\mathbf{r}), \quad \text{box} \to 0, \tag{1.36}$$

where $d\mathbf{s} = ds\hat{\mathbf{n}}$, $\hat{\mathbf{n}}$ is the unit normal vector of the box (pointing *outward*).

For reference, in cylindrical and spherical coordinates,

$$\nabla \cdot \mathbf{V} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}, \tag{1.37}$$

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}.$$

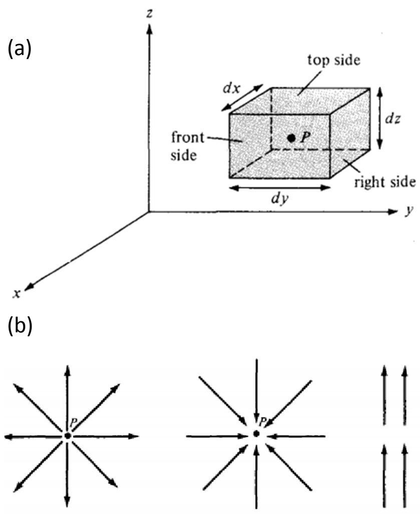

FIG. 5 (a) A box as a volume element dv near point P. (b) From left to right, vector fields with positive, negative, and zero divergence at point P.

3. **Curl**

The curl of a vector field $\mathbf{V}(\mathbf{r})$ is defined as,

$$\nabla \times \mathbf{V}(\mathbf{r}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}. \tag{1.38}$$

Given a surface element $d\mathbf{s} = dxdy\hat{\mathbf{z}}$, which is a small rectangle on the x-y plane around point $P = (x, y, 0)$ (Fig. 6(a)), then

$$\begin{aligned} \nabla \times \mathbf{V}(\mathbf{r}) \cdot d\mathbf{s} &= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dxdy \quad (1.39) \\ &\simeq (V_{y+} - V_{y-})dy - (V_{x+} - V_{x-})dx, \end{aligned}$$

where

$$\begin{aligned} V_{x\pm} &\equiv V_x(x, y \pm dy/2, z), \\ V_{y\pm} &\equiv V_y(x \pm dx/2, y, z). \end{aligned}$$

Thus, $\nabla \times \mathbf{V} \cdot d\mathbf{s}$ is the *right-hand* **circulation** around the rectangle $d\mathbf{s}$ (Fig. 6(b)),

$$\begin{aligned} & \nabla \times \mathbf{V} \cdot d\mathbf{s} \quad (1.40) \\ \simeq\ & V_{x-}dx + V_{y+}dy - V_{x+}dx - V_{y-}dy \\ \simeq\ & \int_{\rightarrow} dxV_{x-} + \int_{\uparrow} dyV_{y+} + \int_{\leftarrow} dxV_{x+} + \int_{\downarrow} dyV_{y-} \\ =\ & \oint_{\Box} d\mathbf{r} \cdot \mathbf{V}(\mathbf{r}), \ \Box \to 0. \end{aligned}$$

For reference, in cylindrical and spherical coordinates,

$$\nabla \times \mathbf{V} = \frac{1}{\rho} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\phi & V_z \end{vmatrix}, \tag{1.41}$$

$$\nabla \times \mathbf{V} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & rV_\theta & r\sin\theta V_\phi \end{vmatrix}. \tag{1.42}$$

Note that some of the surface elements in Cartesian, cylindrical, and spherical coordinates are,

$$\begin{aligned} d\mathbf{s} &= dxdy\ \hat{\mathbf{z}}, & (1.43) \\ d\mathbf{s} &= \rho d\phi dz\ \hat{\boldsymbol{\rho}}, & (1.44) \\ d\mathbf{s} &= r^2 \sin\theta d\theta d\phi\ \hat{\mathbf{r}}. & (1.45) \end{aligned}$$

They lie on the x-y plane, the surface of a cylinder with radius ρ, and the surface of a sphere with radius r respectively.

4. Combined operation

It is very useful to know that *a gradient has no curl*, and *a curl has no divergence*:

$$\begin{aligned} \nabla \times \nabla f(\mathbf{r}) &= 0, & (1.46) \\ \nabla \cdot \nabla \times \mathbf{V}(\mathbf{r}) &= 0. & (1.47) \end{aligned}$$

These can be easily verified in Cartesian coordinate.

It's important to keep in mind that, conversely,

$$\begin{aligned} \text{if } \nabla \times \mathbf{V} &= 0, \text{ then } \mathbf{V} = \nabla f, & (1.48) \\ \text{if } \nabla \cdot \mathbf{V} &= 0, \text{ then } \mathbf{V} = \nabla \times \mathbf{W}. & (1.49) \end{aligned}$$

That is, *if a vector field is curless, then it can be written as a gradient. If a vector field is divergenceless, then it can be written as a curl.*

Finally, $\nabla^2 \equiv \nabla \cdot \nabla$ is called **Laplace operator**, or **Laplacian**. In Cartesian, cylindrical, and spherical coordinates, they are

$$\begin{aligned} \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, & (1.50) \\ \nabla^2 f &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}, & (1.51) \\ \nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) \\ &+ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. & (1.52) \end{aligned}$$

D. Integration of field

1. **Gradient theorem**

The integral of a gradient ∇f along a line C equals the difference of function values at end points,

$$\int_C d\mathbf{r} \cdot \nabla f = \int_a^b df = f(b) - f(a), \tag{1.53}$$

where a, b are the end points of a curve C. This is a generalization of Eq. (1.23) to higher dimension.

2. **Divergence theorem**

The integral of a divergence $\nabla \cdot \mathbf{V}$ over a volume V can be written as a surface integral of flux,

$$\int_V dv \nabla \cdot \mathbf{V}(\mathbf{r}) = \int_S d\mathbf{s} \cdot \mathbf{V}(\mathbf{r}), \tag{1.54}$$

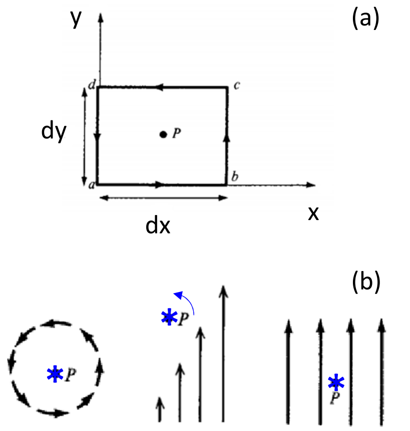

FIG. 6 (a) A rectangle as a surface element $d\mathbf{s} = da\hat{\mathbf{z}}$ near point P. (b) Vector fields with (left and middle) and without (right) curl at point P. If the vector fields are flows of water, then a paddle wheel at P would rotate when the curl of the field at P is not zero, and vice versa.

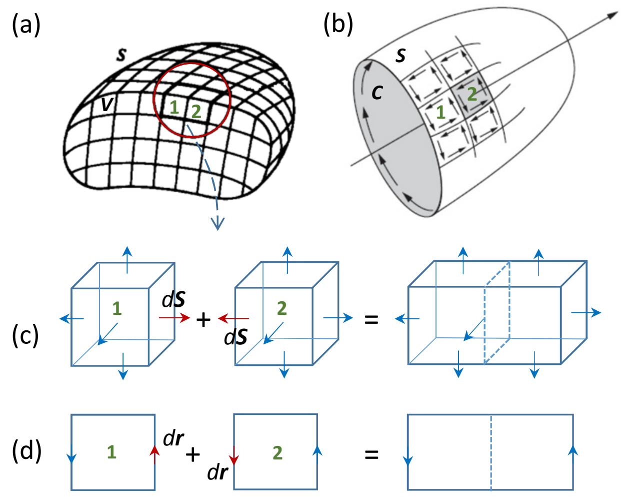

FIG. 7 (a) A finite volume V with surface S can be divided into many small volume elements dv. (b) A finite surface S with boundary C can be divided into many surface elements $d\mathbf{s}$. (c) At the interface between adjacent boxes, the normal vectors (in red) from these two boxes are opposite. (d) At the boundary between adjacent rectangles, the circulations (in red) from these two rectangles are opposite.

where S is the surface of V, and $d\mathbf{s}$ points *out of* volume V. This can be understood as follows: First, divide the volume V into boxes (Fig. 7(a)). Then (see Eq. (1.22))

$$\int_V dv \nabla \cdot \mathbf{V}(\mathbf{r}) \simeq \sum_i dv \nabla \cdot \mathbf{V}(\mathbf{r}_i). \tag{1.55}$$

This becomes an equality when $dv \to 0$. For each box, Eq. (1.36) applies, so that

$$\sum_i dv \nabla \cdot \mathbf{V}(\mathbf{r}_i) = \sum_i \int_{S_i} d\mathbf{s} \cdot \mathbf{V}(\mathbf{r}) \tag{1.56}$$

$$= \int_{\sum_i S_i} d\mathbf{s} \cdot \mathbf{V}(\mathbf{r}), \tag{1.57}$$

where S_i is the surface of box-i (with normal vectors pointing *outward*). But since the sum of the surfaces of two boxes equals their outer surface (Fig. 7(d)), so eventually $\sum_i S_i = S$, and Eq. (1.54) follows. That is, the divergence theorem is the macroscopic version of Eq. (1.36).

3. **Curl theorem** (aka **Stokes theorem**)

The integral of a curl $\nabla \times \mathbf{V}$ over a surface S can be written as a line integral of circulation,

$$\int_S d\mathbf{s} \cdot \nabla \times \mathbf{V}(\mathbf{r}) = \oint_C d\mathbf{r} \cdot \mathbf{V}(\mathbf{r}), \tag{1.58}$$

where C is the boundary of S, and the orientation of C is determined by the direction of $d\mathbf{s}$ (see the Note below). This can be understood as follows: First, divide the surface S into rectangles (Fig. 7(b)). Then

$$\int_S d\mathbf{s} \cdot \nabla \times \mathbf{V}(\mathbf{r}) \simeq \sum_i d\mathbf{s} \cdot \nabla \times \mathbf{V}(\mathbf{r}_i). \tag{1.59}$$

This becomes an equality when $d\mathbf{s} \to 0$. For each rectangle, Eq. (1.36) applies, so that

$$\sum_i d\mathbf{s} \nabla \times \mathbf{V}(\mathbf{r}_i) = \sum_i \int_{C_i} d\mathbf{r} \cdot \mathbf{V}(\mathbf{r}) \tag{1.60}$$

$$= \int_{\sum_i C_i} d\mathbf{r} \cdot \mathbf{V}(\mathbf{r}), \tag{1.61}$$

where C_i is the boundary of rectangle-i (with *right-hand* circulation). But since the sum of the boundaries of two rectangles equals their outer boundary (Fig. 7(c)), so eventually $\sum_i C_i = C$, and Eq. (1.58) follows. That is, the curl theorem is the macroscopic version of Eq. (1.40).

Note: For an *open* surface S with boundary C, there are two possible choices of $\hat{\mathbf{n}}$'s: it either points up or points down. *Once* $\hat{\mathbf{n}}$ *is chosen, the direction of* C *is determined by the right-hand rule*: $\hat{\mathbf{n}}$ is along the thumb, and the direction of C is along four curved fingers.

E. Some useful symbols and identities

1. First, two symbols ($i, j, k = 1, 2, 3$, or $x.y.z$):
Kronecker delta symbol:

$$\delta_{ij} \equiv \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{1.62}$$

Levi-Civita symbol:

$$\epsilon_{ijk} \equiv \begin{cases} 0 \text{ if any two subscripts are the same} \\ +1 \text{ if } (i,j,k) = (1,2,3), (2,3,1), (3,1,2) \\ -1 \text{ if } (i,j,k) = (2,1,3), (1,3,2), (3,2,1) \end{cases} \tag{1.63}$$

It follows that $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$.

For example, if $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, then when written in components, one has

$$c_i = \epsilon_{ijk} a_j b_k. \tag{1.64}$$

We have used **Einstein's summation convention**: *repeated subscripts are automatically summed.* Also, when written in components,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk} a_i b_j c_k. \tag{1.65}$$

It's not difficult to see that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \tag{1.66}$$

It is helpful to know that

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}. \tag{1.67}$$

Pf: First, if any two numbers in the triplet (i, j, k) or the triplet (l, m, n) are the same, then the left-hand side (LHS) is zero (see Eq. (1.63)). The right-hand side (RHS) is also zero since two rows or two columns in the determinant are the same. So the equality is valid.

Next, consider the cases when the numbers in a triplet are different. If $(i, j, k) = (1, 2, 3)$ and $(l, m, n) = (1, 2, 3)$, then it's obvious that the LHS equals the RHS. Now if you exchange any two numbers in the first triplet or the second triplet, then the LHS changes sign (see Eq. (1.63)). The RHS also changes sign since two rows or two columns in the determinant are exchanged. So the equality remains valid. It's not difficult to see that this applies to other permutations of the triplets. QED.

A special case:

$$\epsilon_{ijk}\epsilon_{imn} = \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix}. \tag{1.68}$$

Note that the subscript i is repeated and needs be summed. It is a *dummy index* that would not appear in the result. The proof of this equation is left as an exercise.

2. A frequently used identity is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{1.69}$$

This is called the **BAC-CAB rule**:
Pf: When written in components,

$$\begin{aligned} [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i &= \epsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k & (1.70)\\ &= \underbrace{\epsilon_{ijk}\epsilon_{mnk}}_{\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}} \ a_j b_m c_n & (1.71)\\ &= a_j b_i c_j - a_j b_j c_i & (1.72)\\ &= b_i \mathbf{a} \cdot \mathbf{c} - c_i \mathbf{a} \cdot \mathbf{b}. & (1.73) \end{aligned}$$

QED.
Note that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

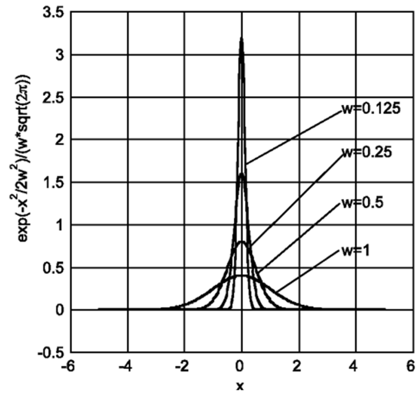

FIG. 8 A Dirac delta function can be considered as a Gaussian function with zero width and infinite height.

3. Gradient

Assume that function $f(r)$ depends only on $r = |\mathbf{r}|$, then

$$\nabla f(r) = \frac{df(r)}{dr}\hat{\mathbf{r}}, \text{ or } f'(r)\hat{\mathbf{r}}. \tag{1.74}$$

Furthermore, let $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$, $R = |\mathbf{R}|$, then

$$\begin{aligned} \nabla f(R), \text{ or } \left.\frac{\partial f(R)}{\partial \mathbf{r}}\right|_{\mathbf{r}' \text{ fixed}} &= f'(R)\hat{\mathbf{R}}, & (1.75)\\ \nabla' f(R), \text{ or } \left.\frac{\partial f(R)}{\partial \mathbf{r}'}\right|_{\mathbf{r} \text{ fixed}} &= -f'(R)\hat{\mathbf{R}}, & (1.76) \end{aligned}$$

in which $f'(R) = df(R)/dR$.

For example, $f(R) = 1/|\mathbf{r} - \mathbf{r}'|$, then

$$\begin{aligned} \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}, & (1.77)\\ \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= +\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. & (1.78) \end{aligned}$$

In electrodynamics, $\mathbf{r}$ and $\mathbf{r}'$ often refer to the distributions of *field* and *source* respectively, and R is the distance between source and field (observation point). Sometimes we need to take a derivative with respect to $\mathbf{r}$, sometimes $\mathbf{r}'$, and the results differ by a sign, $\nabla' f(R) = -\nabla f(R)$.

F. Dirac delta function

Dirac delta function looks like a spike,

$$\delta(x - x') = \begin{cases} 0 & \text{if } x \neq x' \\ +\infty & \text{if } x = x' \end{cases} \tag{1.79}$$

It is an even function, $\delta(-x) = \delta(x)$. You may think of it as a very sharp **Gaussian distribution** (Fig. 8),

$$\delta(x - x') = \lim_{w \to 0} \frac{1}{\sqrt{2\pi} w} e^{-(x-x')^2/2w^2}. \tag{1.80}$$

In addition, the Dirac delta function has to satisfy

$$\int_{-\infty}^{\infty} dx\ \delta(x-x') = 1, \tag{1.81}$$

$$\int_{-\infty}^{\infty} dx f(x)\delta(x-x') = f(x'). \tag{1.82}$$

It's almost always a good news to have the delta function inside an integral, since the integration then becomes trivial.

If c is a nonzero constant, then

$$\delta[c(x-x')] = \frac{1}{|c|}\delta(x-x'). \tag{1.83}$$

The "$|\ |$" is required since the delta function is always positive. If a function $f(x)$ has roots at $x = x_i$, then

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df(x)}{dx_i}\right|}. \tag{1.84}$$

For example,

$$\delta(x^2-a^2) = \frac{1}{2|a|}\left[\delta(x-a)+\delta(x+a)\right]. \tag{1.85}$$

The delta function is the **Fourier transformation** of "1",

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ e^{ikx}, \tag{1.86}$$

$$\text{or} \quad \int_{-\infty}^{\infty} dk\ e^{ik(x-x')} = 2\pi\delta(x-x'). \tag{1.87}$$

The delta function can be generalized to higher dimensions. In three dim,

$$\delta(\mathbf{r}-\mathbf{r}') \equiv \delta(x-x')\delta(y-y')\delta(z-z'). \tag{1.88}$$

It is zero everywhere in space, except being infinite at a single point $\mathbf{r}'$. Also (*all* means *all space*),

$$\int_{all} dv\ \delta(\mathbf{r}-\mathbf{r}') = 1, \tag{1.89}$$

$$\int_{all} dv f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') = f(\mathbf{r}'). \tag{1.90}$$

The 3-dim generalization of Eqs. (1.86) and (1.87) are

$$\delta(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{1.91}$$

$$\text{or} \quad \int d^3k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} = (2\pi)^3\delta(\mathbf{r}-\mathbf{r}'). \tag{1.92}$$

This is the **orthogonal relation for plane waves** $e^{i\mathbf{k}\cdot\mathbf{r}}$.

The Dirac delta function is ideal for describing a *point charge*. If there is a point charge q at location $\mathbf{r}'$, then its *charge density* can be described as,

$$\rho(\mathbf{r}) = q\delta(\mathbf{r}-\mathbf{r}'). \tag{1.93}$$

It is zero everywhere in space, except being infinite at a single point $\mathbf{r}'$. After integration, we get the total charge,

$$\int_{all} dv\rho(\mathbf{r}) = q\int_{all} dv\delta(\mathbf{r}-\mathbf{r}') = q, \tag{1.94}$$

as it should be.

Finally, we show that

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}'). \tag{1.95}$$

Pf: First, since

$$\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{1.96}$$

it follows that

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\nabla\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = 0. \tag{1.97}$$

However, this is valid only if $\mathbf{r}' \neq \mathbf{r}$. When $\mathbf{r}' = \mathbf{r}$, the function diverges and its derivative cannot be taken. However, if we integrate the $\nabla^2(1/R)$ over a tiny sphere V centered at the point $\mathbf{r}'$, then with the divergence theorem, one has

$$\int_V dv\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \int_S d\mathbf{s}\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{1.98}$$

$$= -\int_{S_0} d\mathbf{s}\cdot\frac{\mathbf{r}}{|\mathbf{r}|^3},\ d\mathbf{s} = r^2\sin\theta d\theta d\phi\ \hat{\mathbf{r}}$$

$$= -4\pi, \tag{1.99}$$

where the center of S_0 is at 0. We get a finite result -4π no matter how small V is, as long as it encloses $\mathbf{r}'$. This shows that $\nabla^2\frac{1}{R}$ is a delta function with strength -4π, thus Eq. (1.95) follows. QED.

This mathematical identity is consistent with the fact that, if there is a point charge q at $\mathbf{r}'$, then its Coulomb potential is $\phi(\mathbf{r}) = q/4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|$, its charge density is given as Eq. (1.93). From the **Poission equation** in electrostatics,

$$\nabla^2\phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}, \tag{1.100}$$

we can also reach Eq. (1.95).

G. Series expansion

Series expansions are really useful for approximations. Here we mention two of them:

1. **Binomial expansion**

If $|x| < 1$, and α is a real number, then

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3\cdots. \tag{1.101}$$

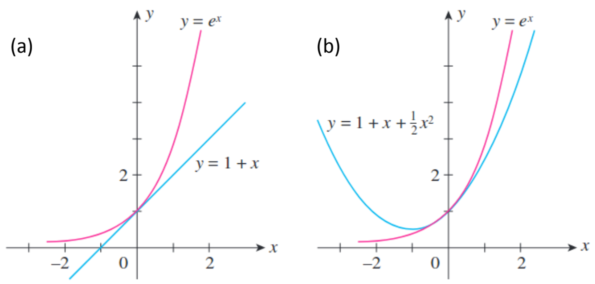

FIG. 9 Use Taylor expansion to approximate e^x to first order (a) and second order (b).

For example, when $|x| \ll 1$,

$$\frac{1}{\sqrt{2-(1+x)^2}} \simeq 1 + x + 2x^2 + O(x^3). \tag{1.102}$$

2. **Taylor expansion**
For small a, we have

$$f(x+a) = f(x) + a\frac{df}{dx} + \frac{a^2}{2!}\frac{d^2 f}{dx^2} + \cdots . \tag{1.103}$$

An alternative form is,

$$f(a+x) = f(a) + x\left.\frac{df}{dx}\right|_{x=a} + \frac{x^2}{2!}\left.\frac{d^2 f}{dx^2}\right|_{x=a} + \cdots , \tag{1.104}$$

in which x is small.

For example, expand $f(x) = e^x$ with respect to $x = 0$, one has (see Fig. 9)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots . \tag{1.105}$$

Eq. (1.103) is sometimes written as,

$$f(x+a) = e^{a\frac{d}{dx}} f(x), \tag{1.106}$$

in which $e^{a\frac{d}{dx}}$ is expanded as in Eq. (1.105).

You may also check that the binomial expansion is a special case of the Taylor expansion, if we expand $f(1+x) = (1+x)^\alpha$ with respect to $x = 0$.

In three dimension, we have

$$\begin{aligned} f(\mathbf{r}+\mathbf{a}) &= e^{a_x\frac{\partial}{\partial x}} e^{a_y\frac{\partial}{\partial y}} e^{a_z\frac{\partial}{\partial z}} f(\mathbf{r}) & (1.107)\\ &= e^{\mathbf{a}\cdot\frac{\partial}{\partial \mathbf{r}}} f(\mathbf{r}) & (1.108)\\ &= f(\mathbf{r}) + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{r}} + \frac{1}{2!}\left(\mathbf{a}\cdot\frac{\partial}{\partial \mathbf{r}}\right)^2 f + \cdots . \end{aligned}$$

More mathematical tools can be found in the first chapter of Zangwill, 2013. In addition to the mathematics reviewed here, we will pick up and learn some more along the way when they are needed.

Problems:

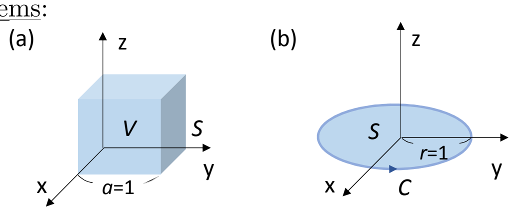

FIG. 10 (a) The cubic box for Prob. 2(a). (b) The circle for Prob. 2(b).

1. Draw the following vector fields,

$$\begin{aligned} \mathbf{V}_1(\mathbf{r}) &= (x, y, 0), & (1.109)\\ \mathbf{V}_2(\mathbf{r}) &= (0, x, 0), & (1.110)\\ \mathbf{V}_3(\mathbf{r}) &= (-y, x, 0). & (1.111) \end{aligned}$$

Calculate their divergence and curl. Which one has non-zero divergence, and which one has non-zero curl?

2. (a) Calculate the *flux* of the vector field $\mathbf{V}(\mathbf{r}) = (0, x, 0)$ out of a cubic box with side length 1 in the first octant (See Fig. 10(a)).
(b) Calculate the *circulation* of the vector field $\mathbf{V}(\mathbf{r}) = (-y, x, 0)$ around a circle with radius 1 lying on the $x-y$ plane centered at the origin (See Fig. 10(b)).

3. Evaluate the following expressions,

$$\begin{aligned} &(a) \quad \delta_{ii} & (1.112)\\ &(b) \quad \delta_{ij}\epsilon_{ijk} & (1.113)\\ &(c) \quad \epsilon_{ijk}\epsilon_{imn} & (1.114)\\ &(d) \quad \epsilon_{ijk}\epsilon_{\ell jk} & (1.115) \end{aligned}$$

A repeated index is summed, according to the Einstein summation convention.

4. Prove the following identities,

$$\begin{aligned} &(a) \quad \nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\nabla\times\mathbf{a} - \mathbf{a}\cdot\nabla\times\mathbf{b} & (1.116)\\ &(b) \quad \nabla\times\nabla\times\mathbf{a} = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a} & (1.117) \end{aligned}$$

Hint: Write $\nabla\cdot(\mathbf{a}\times\mathbf{b})$ in components $\epsilon_{ijk}\frac{\partial}{\partial x_i}(a_j b_k)$ and differentiate it, similarly for $\nabla\times\nabla\times\mathbf{a}$.

5. With the help of the divergence theorem and the Stokes theorem, prove that

$$\begin{aligned} &(a) \quad \int_V dv \nabla\times\mathbf{v}(\mathbf{r}) = \int_S d\mathbf{s}\times\mathbf{v}(\mathbf{r}) & (1.118)\\ &(b) \quad \int_S d\mathbf{s}\times\nabla f(\mathbf{r}) = \oint_C d\mathbf{r} f(\mathbf{r}) & (1.119) \end{aligned}$$

Hint: Let the vector fields in the divergence theorem and the Stokes theorem be $\mathbf{V}(\mathbf{r}) = \mathbf{v}(\mathbf{r})\times\mathbf{c}$ or $\mathbf{V}(\mathbf{r}) = \mathbf{c}f(\mathbf{r})$, $\mathbf{c}$ is a constant vector.

References

Lorrain, P., and D. Corson, 1970, *Electromagnetic fields and waves* (W. H. Freeman).

Zangwill, A., 2013, *Modern electrodynamics* (Cambridge Univ. Press, Cambridge).