Lecture notes on classical electrodynamics

Ming-Che Chang

Department of Physics,
National Taiwan Normal University, Taipei,
Taiwan

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CONTENTS

I. MATHEMATICAL PRELIMINARIES

In this chapter, we collect some mathematics that is *essential* to the learning of electrodynamics.

A. Coordinate system

A coordinate system combines geometry with algebra. That is, we can use numbers to describe geometrical objects. Here we introduce three of the most popular coordinate systems.

1. Cartesian coordinate
The word Cartesian comes from the latinized name of Descartes - Cartesius. The three coordinate axes are perpendicular to each other. A point has coordinates (x, y, z), and unit basis vectors are $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$.

A **position vector** is,

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{1.1}$$

A vector at point $\mathbf{r}$ is

$$\mathbf{V}(\mathbf{r}) = V_x(\mathbf{r})\hat{\mathbf{x}} + V_y(\mathbf{r})\hat{\mathbf{y}} + V_z(\mathbf{r})\hat{\mathbf{z}}. \tag{1.2}$$

The distribution of vectors $\mathbf{V}(\mathbf{r})$ in space is a **vector field**.

2. Cylindrical coordinate
As shown in Fig. 2(a), a point in cylindrical coordinate has coordinates (ρ, ϕ, z). The unit basis vectors are

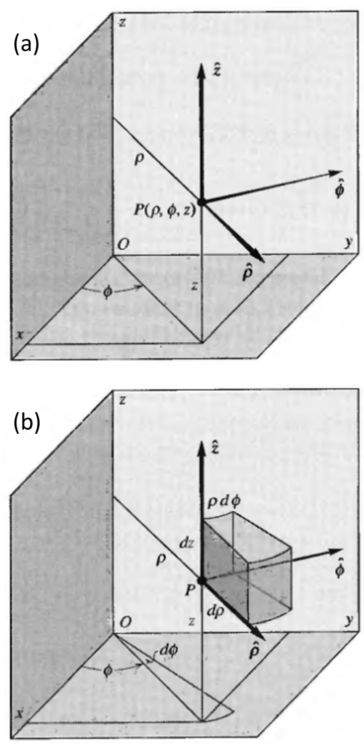

FIG. 1 Cylindrical coordinate (a) and its volume element (b). Figs. from Lorrain and Corson, 1970.

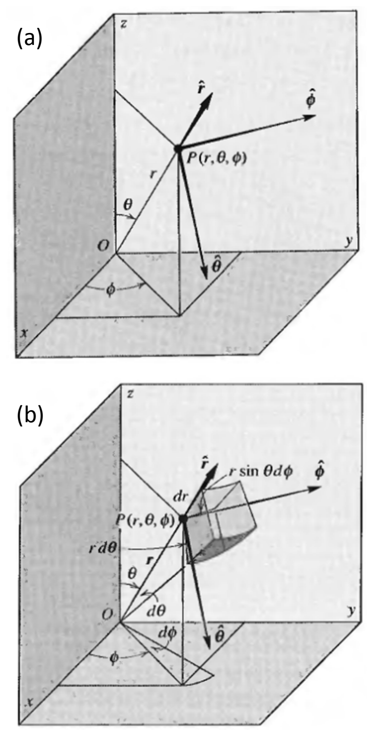

FIG. 2 Spherical coordinate (a) and its volume element (b). Figs. from Lorrain and Corson, 1970.

$\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$, which are along the direction of increase of ρ, ϕ, z and are perpendicular to each other. The connection between Cartesian and cylindrical coordinates are,

$$x = \rho \cos\phi, \tag{1.3}$$
$$y = \rho \sin\phi, \tag{1.4}$$
$$z = z. \tag{1.5}$$

A position vector is

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}. \tag{1.6}$$

The coordinates ρ, z account for two degrees of freedom. The third one is hidden in the angle ϕ of $\hat{\boldsymbol{\rho}}$. A vector at point $\mathbf{r}$ can be expanded as,

$$\mathbf{V}(\mathbf{r}) = V_\rho(\mathbf{r})\hat{\boldsymbol{\rho}} + V_\phi(\mathbf{r})\hat{\boldsymbol{\phi}} + V_z(\mathbf{r})\hat{\mathbf{z}}. \tag{1.7}$$

3. Spherical coordinate

As shown in Fig. 1(c), a point in spherical coordinate has coordinates (r, θ, ϕ). These are standard notations used by most people, so you need to keep them in mind, since a figure is not always drawn to remind you of their meaning. The unit basis vectors are $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$, which are along the direction of increase of r, θ, ϕ and are perpendicular to each other. The connection between Cartesian and spherical coordinates are,

$$x = r\sin\theta\cos\phi, \tag{1.8}$$
$$y = r\sin\theta\sin\phi, \tag{1.9}$$
$$z = r\cos\theta. \tag{1.10}$$

A position vector is simply

$$\mathbf{r} = r\hat{\mathbf{r}}. \tag{1.11}$$

The coordinate r accounts for one degree of freedom. The other two are hidden in the angles θ, ϕ of $\hat{\mathbf{r}}$. A vector at point $\mathbf{r}$ can be expanded as,

$$\mathbf{V}(\mathbf{r}) = V_r(\mathbf{r})\hat{\mathbf{r}} + V_\theta(\mathbf{r})\hat{\boldsymbol{\theta}} + V_\phi(\mathbf{r})\hat{\boldsymbol{\phi}}. \tag{1.12}$$

Note that the volume elements in Cartesian, cylindrical, and spherical coordinates are (Fig. 1(b) and Fig. 2(b))

$$dv = dxdydz, \tag{1.13}$$
$$dv = \rho d\rho d\phi dz, \tag{1.14}$$
$$dv = r^2\sin\theta dr d\theta d\phi, \tag{1.15}$$

The major difference between Newton's dynamics and Maxwell's dynamics is that in the former we simply deal with particle trajectory $\mathbf{r}(t)$, while the latter we need to deal with field distribution $\mathbf{V}(\mathbf{r}, t)$. This makes electrodynamic much harder to learn.

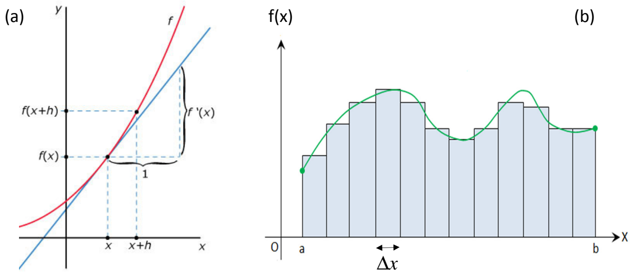

FIG. 3 (a) A derivative. (b) An integration.

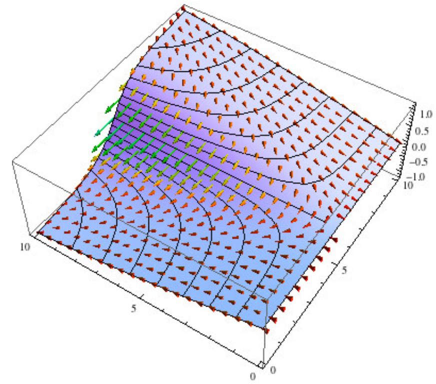

FIG. 4 The gradient vectors $-\nabla f$ of a function $f(x,y)$ in two dimension. Fig. from the web.

B. Basics of calculus

Recall that the derivative of $f(x)$ at x is defined as (see Fig. 3(a)),

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}. \tag{1.16}$$

When $h = \Delta x$ is small but finite, one has

$$\frac{df(x)}{dx} \simeq \frac{f(x+\Delta x)-f(x)}{\Delta x}. \tag{1.17}$$

Thus,

$$f(x+\Delta x) \simeq f(x) + \frac{df(x)}{dx}\Delta x. \tag{1.18}$$

For a function $f(\mathbf{r})$ in three dimensions,

$$f(\mathbf{r}+\Delta\mathbf{r}) \simeq f(\mathbf{r}) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z. \tag{1.19}$$

We'll often write Δx as dx (or $\Delta\mathbf{r}$ as $d\mathbf{r}$), without distinguishing between finite and infinitesimal, when the limit $\Delta x \to dx$ (or $\Delta\mathbf{r} \to d\mathbf{r}$) needs to be taken at the end of a derivation.

The integral of $f(x)$ is the area between the curve $f(x)$ and the x-axis, which can be approximated as a sum of the areas of rectangles (Fig. 3(b))

$$\int_a^b dx f(x) \simeq \sum_i \Delta x f(x_i), \tag{1.20}$$

where x_i can be any point (e.g., the middle one) inside an interval Δx. The equation above becomes an equality when the division becomes infinitesimal, $\Delta x \to dx$. It follows from the equation above that,

$$\sum_i f(x_i) \simeq \frac{1}{\Delta x}\int_a^b dx f(x). \tag{1.21}$$

That is, if $f(x)$ is smooth, then you can evaluate its summation with the help of integration.

In three dimensions, the integral of $f(\mathbf{r})$ over a region V is given as,

$$\int_V dv f(\mathbf{r}) \simeq \sum_i \Delta v f(\mathbf{r}_i), \tag{1.22}$$

where the region V is divided into many small boxes, and dv is a volume element (the volume of a box) around $\mathbf{r}_i$. The equation above approaches an equality when the division gets finer and finer, $\Delta v \to 0$.

Finally,

$$\int^x dx' \frac{df}{dx'} = f(x) + c, \tag{1.23}$$

where c is a constant. Also,

$$\frac{d}{dx}\int^x dx' f(x') = f(x). \tag{1.24}$$

That is, integration is the opposite of differentiation, and vice versa. This is called the **fundamental theorem of calculus**.

C. Differentiation of field

A *scalar field* $f(\mathbf{r})$, or $f(x,y,z)$, describes, e.g., the distribution of temperature or charge density in space. A *vector field* $\mathbf{V}(\mathbf{r})$, or $\mathbf{V}(x,y,z)$, describes, e.g., the distribution of fluid velocity or electric field in space. We review three major differential operations of fields: gradient, divergence, and curl.

1. **Gradient**

The gradient of a scalar function $f(\mathbf{r})$ is defined as,

$$\nabla f(\mathbf{r}) \left(\text{or } \frac{\partial f}{\partial \mathbf{r}}\right) = \frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, \tag{1.25}$$

in which ∇ is called *del*.

The **total derivative** of $f(\mathbf{r})$ (see Eq. (1.19)),

$$\begin{aligned} df(\mathbf{r}) &\equiv f(\mathbf{r}+d\mathbf{r}) - f(\mathbf{r}) && (1.26)\\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz && (1.27)\\ &= \nabla f \cdot d\mathbf{r}. && (1.28)\end{aligned}$$

That is,

$$\nabla f \cdot d\mathbf{r} = df, \text{the change of } f \text{ along } d\mathbf{r}. \quad (1.29)$$

Since $df = |\nabla f||d\mathbf{r}|\cos\theta$ (θ is the angle between ∇f and $d\mathbf{r}$), if we fix $|d\mathbf{r}|$ and swivel the vector $d\mathbf{r}$ around, then df is maximum when $d\mathbf{r} \parallel \nabla f$. Therefore, *the direction of* ∇f = *The direction of maximum increase of* $f(\mathbf{r})$ (i.e., the steepest *ascent*). Conversely, $-\nabla f$ points to the direction of steepest *descent* (Fig. 4). For example, given a temperature distribution $T(\mathbf{r})$, the heat current $\mathbf{J}_T(\mathbf{r})$ flows along the steepest descent of the temperature,

$$\mathbf{J}_T(\mathbf{r}) = -\kappa\nabla T(\mathbf{r}), \quad (1.30)$$

where κ is the **thermal conductivity**. This is **Fourier's law** of heat conduction.

Similarly, given an electric potential $\phi(\mathbf{r})$, the current are flowing along the steepest descent of the potential,

$$\mathbf{J}(\mathbf{r}) = -\sigma\nabla\phi(\mathbf{r}) = \sigma\mathbf{E}, \quad (1.31)$$

where σ is the electric conductivity, and $\mathbf{E} = -\nabla\phi$ is the electric field. This is the **Ohm's law**.

On the other hand, when $d\mathbf{r} \perp \nabla f(\mathbf{r})$, then $df = 0$. Thus $f(\mathbf{r})$ is not changed (to the first order) along the plane perpendicular to $\nabla f(\mathbf{r})$.

For reference, in cylindrical and spherical coordinates,

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}, & (1.32)\\ \nabla f &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}. & (1.33) \end{aligned}$$

2. **Divergence**

The divergence of a vector field $\mathbf{V}(\mathbf{r})$ is defined as,

$$\nabla \cdot \mathbf{V}(\mathbf{r}) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}. \quad (1.34)$$

Given a volume element $dv = dxdydz$, which is a small box around point $P = (x, y, z)$ (Fig. 5(a)), we have

$$\begin{aligned} \nabla \cdot \mathbf{V}(\mathbf{r})dv &= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dxdydz \\ &\simeq \Delta V_x dydz + \Delta V_y dzdx + \Delta V_z dxdy \\ &= (V_{x,+} - V_{x,-})dydz + (V_{y,+} - V_{y,-})dzdx \\ &+ (V_{z,+} - V_{z,-})dxdy, \quad (1.35) \end{aligned}$$

where $V_{x,\pm} \equiv V_x(x \pm dx/2, y, z)$, and similarly for $V_{y,\pm}$ and $V_{z,\pm}$.

The term $V_{x,+}dydz$ is the flux passing through the area $dydz$ at $x + dx/2$; $V_{x,-}dydz$ is the flux passing through the area $dydz$ at $x - dx/2$. Similarly for the other terms. Thus, $\nabla \cdot \mathbf{V}dv$ is the **flux** *out of* the box dv (Fig. 5(b)),

$$\nabla \cdot \mathbf{V}dv = \int_{\text{box}} d\mathbf{s} \cdot \mathbf{V}(\mathbf{r}), \quad \text{box} \to 0, \quad (1.36)$$

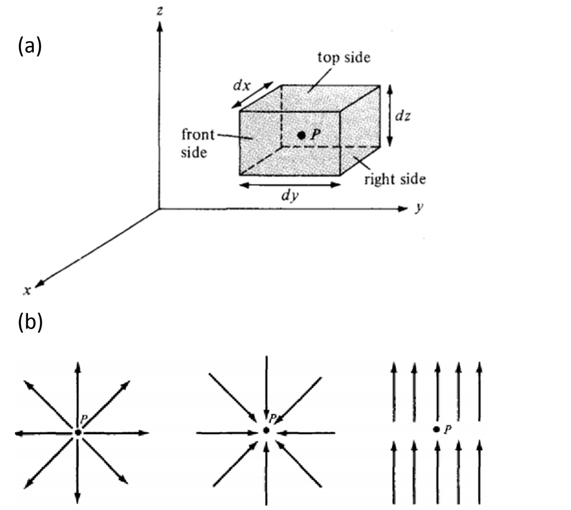

FIG. 5 (a) A box as a volume element dv near point P. (b) From left to right, vector fields with positive, negative, and zero divergence at point P.

where $d\mathbf{s} = ds\hat{\mathbf{n}}$, $\hat{\mathbf{n}}$ is the unit normal vector of the box (pointing *outward*).

For reference, in cylindrical and spherical coordinates,

$$\begin{aligned} \nabla \cdot \mathbf{V} &= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}, & (1.37)\\ \nabla \cdot \mathbf{V} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta V_\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}. \end{aligned}$$

3. **Curl**

The curl of a vector field $\mathbf{V}(\mathbf{r})$ is defined as,

$$\nabla \times \mathbf{V}(\mathbf{r}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}. \quad (1.38)$$

Given a surface element $d\mathbf{s} = dxdy\hat{\mathbf{z}}$, which is a small rectangle on the x-y plane around point $P = (x, y, 0)$ (Fig. 6(a)), then

$$\begin{aligned} \nabla \times \mathbf{V}(\mathbf{r}) \cdot d\mathbf{s} &= \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dxdy \quad (1.39)\\ &\simeq (V_{y+} - V_{y-})dy - (V_{x+} - V_{x-})dx, \end{aligned}$$

where

$$\begin{aligned} V_{x\pm} &\equiv V_x(x, y \pm dy/2, z), \\ V_{y\pm} &\equiv V_y(x \pm dx/2, y, z). \end{aligned}$$

Thus, $\nabla \times \mathbf{V} \cdot d\mathbf{s}$ is the *right-hand* **circulation** around the rectangle $d\mathbf{s}$ (Fig. 6(b)),

$$\begin{aligned} &\nabla \times \mathbf{V} \cdot d\mathbf{s} \quad (1.40)\\ &\simeq V_{x-}dx + V_{y+}dy - V_{x+}dx - V_{y-}dy \\ &\simeq \int_{\rightarrow} dxV_{x-} + \int_{\uparrow} dyV_{y+} + \int_{\leftarrow} dxV_{x+} + \int_{\downarrow} dyV_{y-} \\ &= \oint_{\square} d\mathbf{r} \cdot \mathbf{V}(\mathbf{r}), \ \square \to 0. \end{aligned}$$

For reference, in cylindrical and spherical coordinates,

$$\nabla \times \mathbf{V} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\phi & V_z \end{vmatrix}, \tag{1.41}$$

$$\nabla \times \mathbf{V} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & rV_\theta & r\sin\theta V_\phi \end{vmatrix}. \tag{1.42}$$

Note that some of the surface elements in Cartesian, cylindrical, and spherical coordinates are,

$$d\mathbf{s} = dxdy\ \hat{\mathbf{z}}, \tag{1.43}$$
$$d\mathbf{s} = \rho d\phi dz\ \hat{\boldsymbol{\rho}}, \tag{1.44}$$
$$d\mathbf{s} = r^2\sin\theta d\theta d\phi\ \hat{\mathbf{r}}. \tag{1.45}$$

They lie on the x-y plane, the surface of a cylinder with radius ρ, and the surface of a sphere with radius r respectively.

4. Combined operation

It is very useful to know that *a gradient has no curl*, and *a curl has no divergence*:

$$\nabla \times \nabla f(\mathbf{r}) = 0, \tag{1.46}$$
$$\nabla \cdot \nabla \times \mathbf{V}(\mathbf{r}) = 0. \tag{1.47}$$

These can be easily verified in Cartesian coordinate.

It's important to keep in mind that, conversely,

$$\text{if } \nabla \times \mathbf{V} = 0, \text{ then } \mathbf{V} = \nabla f, \tag{1.48}$$
$$\text{if } \nabla \cdot \mathbf{V} = 0, \text{ then } \mathbf{V} = \nabla \times \mathbf{W}. \tag{1.49}$$

That is, *if a vector field is curless, then it can be written as a gradient. If a vector field is divergenceless, then it can be written as a curl.*

Finally, $\nabla^2 \equiv \nabla \cdot \nabla$ is called **Laplace operator**, or **Laplacian**. In Cartesian, cylindrical, and spherical coordinates, they are

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \tag{1.50}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}, \tag{1.51}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{1.52}$$

D. Integration of field

1. **Gradient theorem**

The integral of a gradient ∇f along a line C equals the difference of function values at end points,

$$\int_C d\mathbf{r} \cdot \nabla f = \int_a^b df = f(b) - f(a), \tag{1.53}$$

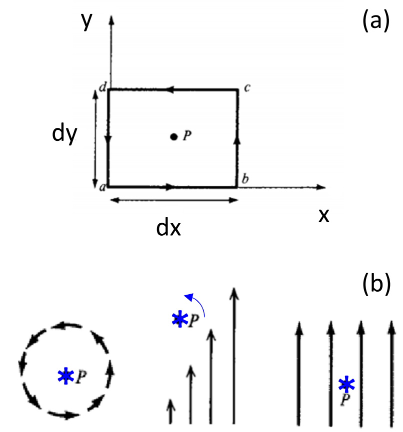

FIG. 6 (a) A rectangle as a surface element $d\mathbf{s} = da\hat{\mathbf{z}}$ near point P. (b) Vector fields with (left and middle) and without (right) curl at point P. If the vector fields are flows of water, then a paddle wheel at P would rotate when the curl of the field at P is not zero, and vice versa.

where a, b are the end points of a curve C. This is a generalization of Eq. (1.23) to higher dimension.

2. **Divergence theorem**

The integral of a divergence $\nabla \cdot \mathbf{V}$ over a volume V can be written as a surface integral of flux,

$$\int_V dv \nabla \cdot \mathbf{V}(\mathbf{r}) = \int_S d\mathbf{s} \cdot \mathbf{V}(\mathbf{r}), \tag{1.54}$$

where S is the surface of V, and $d\mathbf{s}$ points *out of* volume V. This can be understood as follows: First, divide the volume V into boxes (Fig. 7(a)). Then (see Eq. (1.22))

$$\int_V dv \nabla \cdot \mathbf{V}(\mathbf{r}) \simeq \sum_i dv \nabla \cdot \mathbf{V}(\mathbf{r}_i). \tag{1.55}$$

This becomes an equality when $dv \to 0$. For each box, Eq. (1.36) applies, so that

$$\sum_i dv \nabla \cdot \mathbf{V}(\mathbf{r}_i) = \sum_i \int_{S_i} d\mathbf{s} \cdot \mathbf{V}(\mathbf{r}) \tag{1.56}$$
$$= \int_{\sum_i S_i} d\mathbf{s} \cdot \mathbf{V}(\mathbf{r}), \tag{1.57}$$

where S_i is the surface of box-i (with normal vectors pointing *outward*). But since the sum of the surfaces of two boxes equals their outer surface (Fig. 7(d)), so eventually $\sum_i S_i = S$, and Eq. (1.54) follows. That is, the divergence theorem is the macroscopic version of Eq. (1.36).

3. **Curl theorem** (aka **Stokes theorem**)

The integral of a curl $\nabla \times \mathbf{V}$ over a surface S can be written as a line integral of circulation,

$$\int_S d\mathbf{s} \cdot \nabla \times \mathbf{V}(\mathbf{r}) = \oint_C d\mathbf{r} \cdot \mathbf{V}(\mathbf{r}), \tag{1.58}$$

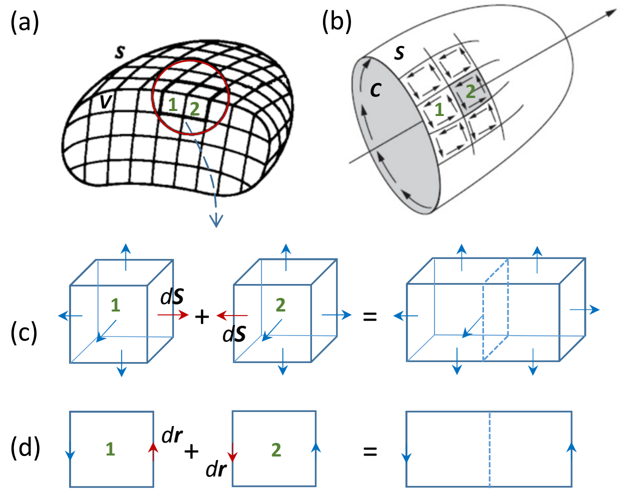

FIG. 7 (a) A finite volume V with surface S can be divided into many small volume elements dv. (b) A finite surface S with boundary C can be divided into many surface elements $d\mathbf{s}$. (c) At the interface between adjacent boxes, the normal vectors (in red) from these two boxes are opposite. (d) At the boundary between adjacent rectangles, the circulations (in red) from these two rectangles are opposite.

where C is the boundary of S, and the orientation of C is determined by the direction of $d\mathbf{s}$ (see the Note below). This can be understood as follows: First, divide the surface S into rectangles (Fig. 7(b)). Then

$$\int_S d\mathbf{s} \cdot \nabla \times \mathbf{V}(\mathbf{r}) \simeq \sum_i d\mathbf{s} \cdot \nabla \times \mathbf{V}(\mathbf{r}_i). \tag{1.59}$$

This becomes an equality when $d\mathbf{s} \to 0$. For each rectangle, Eq. (1.36) applies, so that

$$\sum_i d\mathbf{s} \cdot \nabla \times \mathbf{V}(\mathbf{r}_i) = \sum_i \int_{C_i} d\mathbf{r} \cdot \mathbf{V}(\mathbf{r}) \tag{1.60}$$

$$= \int_{\sum_i C_i} d\mathbf{r} \cdot \mathbf{V}(\mathbf{r}), \tag{1.61}$$

where C_i is the boundary of rectangle-i (with *right-hand* circulation). But since the sum of the boundaries of two rectangles equals their outer boundary (Fig. 7(c)), so eventually $\sum_i C_i = C$, and Eq. (1.58) follows. That is, the curl theorem is the macroscopic version of Eq. (1.40).

Note: For an *open* surface S with boundary C, there are two possible choices of $\hat{\mathbf{n}}$'s: it either points up or points down. *Once $\hat{\mathbf{n}}$ is chosen, the direction of C is determined by the right-hand rule*: $\hat{\mathbf{n}}$ is along the thumb, and the direction of C is along four curved fingers.

E. Some useful symbols and identities

1. First, two symbols ($i, j, k = 1, 2, 3$, or $x.y.z$):

Kronecker delta symbol:

$$\delta_{ij} \equiv \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{1.62}$$

Levi-Civita symbol:

$$\epsilon_{ijk} \equiv \begin{cases} 0 \text{ if any two subscripts are the same} \\ +1 \text{ if } (i,j,k) = (1,2,3), (2,3,1), (3,1,2) \\ -1 \text{ if } (i,j,k) = (2,1,3), (1,3,2), (3,2,1) \end{cases} \tag{1.63}$$

It follows that $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$.

For example, if $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, then when written in components, one has

$$c_i = \epsilon_{ijk} a_j b_k. \tag{1.64}$$

We have used **Einstein's summation convention**: *repeated subscripts are automatically summed.* Also, when written in components,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk} a_i b_j c_k. \tag{1.65}$$

It's not difficult to see that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}. \tag{1.66}$$

It is helpful to know that

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}. \tag{1.67}$$

Pf: First, if any two numbers in the triplet (i, j, k) or the triplet (l, m, n) are the same, then the left-hand side (LHS) is zero (see Eq. (1.63)). The right-hand side (RHS) is also zero since two rows or two columns in the determinant are the same. So the equality is valid.

Next, consider the cases when the numbers in a triplet are different. If $(i, j, k) = (1, 2, 3)$ and $(l, m, n) = (1, 2, 3)$, then it's obvious that the LHS equals the RHS. Now if you exchange any two numbers in the first triplet or the second triplet, then the LHS changes sign (see Eq. (1.63)). The RHS also changes sign since two rows or two columns in the determinant are exchanged. So the equality remains valid. It's not difficult to see that this applies to other permutations of the triplets. QED.

A special case:

$$\epsilon_{ijk}\epsilon_{imn} = \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix}. \tag{1.68}$$

Note that the subscript i is repeated and needs be summed. It is a *dummy index* that would not appear in the result. The proof of this equation is left as an exercise.

2. A frequently used identity is,

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \tag{1.69}$$

This is called the **BAC-CAB rule**:
Pf: When written in components,

$$
\begin{aligned}
[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]_i &= \epsilon_{ijk}a_j(\mathbf{b}\times\mathbf{c})_k && (1.70)\\
&= \underbrace{\epsilon_{ijk}\epsilon_{mnk}}_{\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}} \; a_j b_m c_n && (1.71)\\
&= a_j b_i c_j - a_j b_j c_i && (1.72)\\
&= b_i\mathbf{a}\cdot\mathbf{c} - c_i\mathbf{a}\cdot\mathbf{b}. && (1.73)
\end{aligned}
$$

QED.
Note that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.
3. Gradient
Assume that function $f(r)$ depends only on $r = |\mathbf{r}|$, then

$$\nabla f(r) = \frac{df(r)}{dr}\hat{\mathbf{r}}, \text{ or } f'(r)\hat{\mathbf{r}}. \qquad (1.74)$$

Furthermore, let $\mathbf{R} \equiv \mathbf{r}-\mathbf{r}'$, $R = |\mathbf{R}|$, then

$$
\begin{aligned}
\nabla f(R), \text{ or } \left.\frac{\partial f(R)}{\partial \mathbf{r}}\right|_{\mathbf{r}' \text{ fixed}} &= f'(R)\hat{\mathbf{R}}, && (1.75)\\
\nabla' f(R), \text{ or } \left.\frac{\partial f(R)}{\partial \mathbf{r}'}\right|_{\mathbf{r} \text{ fixed}} &= -f'(R)\hat{\mathbf{R}}, && (1.76)
\end{aligned}
$$

in which $f'(R) = df(R)/dR$.
For example, $f(R) = 1/|\mathbf{r}-\mathbf{r}'|$, then

$$
\begin{aligned}
\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, && (1.77)\\
\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= +\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. && (1.78)
\end{aligned}
$$

In electrodynamics, $\mathbf{r}$ and $\mathbf{r}'$ often refer to the distributions of *field* and *source* respectively, and R is the distance between source and field (observation point). Sometimes we need to take a derivative with respect to $\mathbf{r}$, sometimes $\mathbf{r}'$, and the results differ by a sign, $\nabla' f(R) = -\nabla f(R)$.

F. Dirac delta function

Dirac delta function looks like a spike,

$$\delta(x-x') = \begin{cases} 0 & \text{if } x\neq x' \\ +\infty & \text{if } x = x' \end{cases} \qquad (1.79)$$

It is an even function, $\delta(-x) = \delta(x)$. You may think of it as a very sharp **Gaussian distribution** (Fig. 8),

$$\delta(x-x') = \lim_{w\to 0}\frac{1}{\sqrt{2\pi}w}e^{-(x-x')^2/2w^2}. \qquad (1.80)$$

In addition, the Dirac delta function has to satisfy

$$
\begin{aligned}
\int_{-\infty}^{\infty} dx\, \delta(x-x') &= 1, && (1.81)\\
\int_{-\infty}^{\infty} dx f(x)\delta(x-x') &= f(x'). && (1.82)
\end{aligned}
$$

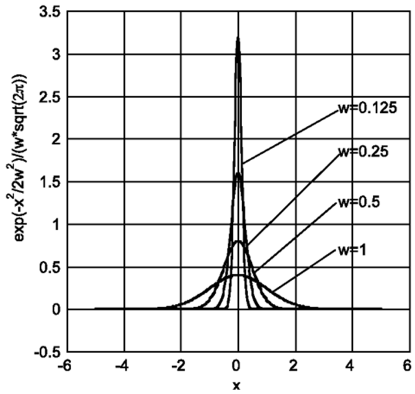

FIG. 8 A Dirac delta function can be considered as a Gaussian function with zero width and infinite height.

It's almost always a good news to have the delta function inside an integral, since the integration then becomes trivial.
If c is a nonzero constant, then

$$\delta[c(x-x')] = \frac{1}{|c|}\delta(x-x'). \qquad (1.83)$$

The "$|\ |$" is required since the delta function is always positive. If a function $f(x)$ has roots at $x = x_i$, then

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{\left|\frac{df(x)}{dx_i}\right|}. \qquad (1.84)$$

For example,

$$\delta(x^2-a^2) = \frac{1}{2|a|}\left[\delta(x-a)+\delta(x+a)\right]. \qquad (1.85)$$

The delta function is the **Fourier transformation** of "1",

$$
\begin{aligned}
\delta(x) &= \int_{-\infty}^{\infty}\frac{dk}{2\pi}\, e^{ikx}, && (1.86)\\
\text{or } \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} &= 2\pi\delta(x-x'). && (1.87)
\end{aligned}
$$

The delta function can be generalized to higher dimensions. In three dim,

$$\delta(\mathbf{r}-\mathbf{r}') \equiv \delta(x-x')\delta(y-y')\delta(z-z'). \qquad (1.88)$$

It is zero everywhere in space, except being infinite at a single point $\mathbf{r}'$. Also (*all* means *all space*),

$$
\begin{aligned}
\int_{all} dv\, \delta(\mathbf{r}-\mathbf{r}') &= 1, && (1.89)\\
\int_{all} dv f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') &= f(\mathbf{r}'). && (1.90)
\end{aligned}
$$

The 3-dim generalization of Eqs. (1.86) and (1.87) are

$$\delta(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{1.91}$$

$$\text{or} \quad \int d^3k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} = (2\pi)^3\delta(\mathbf{r}-\mathbf{r}'). \tag{1.92}$$

This is the **orthogonal relation for plane waves** $e^{i\mathbf{k}\cdot\mathbf{r}}$.

The Dirac delta function is ideal for describing a *point charge*. If there is a point charge q at location $\mathbf{r}'$, then its *charge density* can be described as,

$$\rho(\mathbf{r}) = q\delta(\mathbf{r}-\mathbf{r}'). \tag{1.93}$$

It is zero everywhere in space, except being infinite at a single point $\mathbf{r}'$. After integration, we get the total charge,

$$\int_{all} dv\rho(\mathbf{r}) = q\int_{all} dv\delta(\mathbf{r}-\mathbf{r}') = q, \tag{1.94}$$

as it should be.

Finally, we show that

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}'). \tag{1.95}$$

Pf: First, since

$$\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{1.96}$$

it follows that

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\nabla\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = 0. \tag{1.97}$$

However, this is valid only if $\mathbf{r}' \neq \mathbf{r}$. When $\mathbf{r}' = \mathbf{r}$, the function diverges and its derivative cannot be taken. However, if we integrate the $\nabla^2(1/R)$ over a tiny sphere V centered at the point $\mathbf{r}'$, then with the divergence theorem, one has

$$\begin{aligned} \int_V dv\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \int_S d\mathbf{s}\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} && (1.98)\\ &= -\int_{S_0} d\mathbf{s}\cdot\frac{\mathbf{r}}{|\mathbf{r}|^3},\ d\mathbf{s} = r^2\sin\theta d\theta d\phi\ \hat{\mathbf{r}} \\ &= -4\pi, && (1.99)\end{aligned}$$

where the center of S_0 is at 0. We get a finite result -4π no matter how small V is, as long as it encloses $\mathbf{r}'$. This shows that $\nabla^2\frac{1}{R}$ is a delta function with strength -4π, thus Eq. (1.95) follows. QED.

This mathematical identity is consistent with the fact that, if there is a point charge q at $\mathbf{r}'$, then its Coulomb potential is $\phi(\mathbf{r}) = q/4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|$, its charge density is given as Eq. (1.93). From the **Poission equation** in electrostatics,

$$\nabla^2\phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}, \tag{1.100}$$

we can also reach Eq. (1.95).

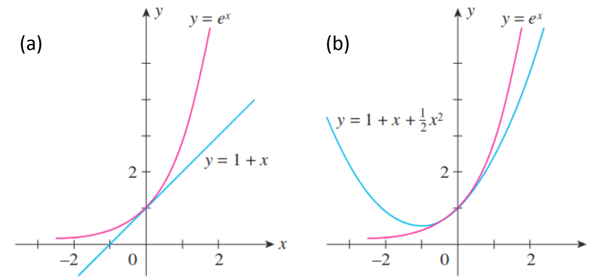

FIG. 9 Use Taylor expansion to approximate e^x to first order (a) and second order (b).

G. Series expansion

Series expansions are really useful for approximations. Here we mention two of them:

1. **Binomial expansion**

If $|x| < 1$, and α is a real number, then

$$(1+x)^\alpha = 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3\cdots. \tag{1.101}$$

For example, when $|x| \ll 1$,

$$\frac{1}{\sqrt{2-(1+x)^2}} \simeq 1+x+2x^2+O(x^3). \tag{1.102}$$

2. **Taylor expansion**

For small a, we have

$$f(x+a) = f(x) + a\frac{df}{dx} + \frac{a^2}{2!}\frac{d^2f}{dx^2} + \cdots. \tag{1.103}$$

An alternative form is,

$$f(a+x) = f(a) + x\left.\frac{df}{dx}\right|_{x=a} + \frac{x^2}{2!}\left.\frac{d^2f}{dx^2}\right|_{x=a} + \cdots, \tag{1.104}$$

in which x is small.

For example, expand $f(x) = e^x$ with respect to $x = 0$, one has (see Fig. 9)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{1.105}$$

Eq. (1.103) is sometimes written as,

$$f(x+a) = e^{a\frac{d}{dx}}f(x), \tag{1.106}$$

in which $e^{a\frac{d}{dx}}$ is expanded as in Eq. (1.105).

You may also check that the binomial expansion is a special case of the Taylor expansion, if we expand $f(1+x) = (1+x)^\alpha$ with respect to $x = 0$.

In three dimension, we have

$$\begin{aligned} f(\mathbf{r}+\mathbf{a}) &= e^{a_x\frac{\partial}{\partial x}}e^{a_y\frac{\partial}{\partial y}}e^{a_z\frac{\partial}{\partial z}}f(\mathbf{r}) && (1.107)\\ &= e^{\mathbf{a}\cdot\frac{\partial}{\partial\mathbf{r}}}f(\mathbf{r}) && (1.108)\\ &= f(\mathbf{r}) + \mathbf{a}\cdot\frac{\partial f}{\partial\mathbf{r}} + \frac{1}{2!}\left(\mathbf{a}\cdot\frac{\partial}{\partial\mathbf{r}}\right)^2 f + \cdots. \end{aligned}$$

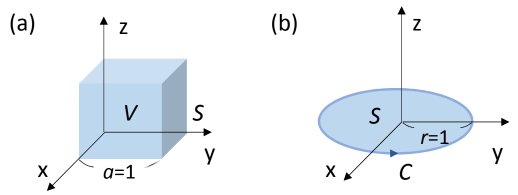

FIG. 10 (a) The cubic box for Prob. 2(a). (b) The circle for Prob. 2(b).

More mathematical tools can be found in the first chapter of Zangwill, 2013. In addition to the mathematics reviewed here, we will pick up and learn some more along the way when they are needed.

Problems:

1. Draw the following vector fields,

$$\mathbf{V}_1(\mathbf{r}) = (x, y, 0), \quad (1.109)$$
$$\mathbf{V}_2(\mathbf{r}) = (0, x, 0), \quad (1.110)$$
$$\mathbf{V}_3(\mathbf{r}) = (-y, x, 0). \quad (1.111)$$

Calculate their divergence and curl. Which one has non-zero divergence, and which one has non-zero curl?

2. (a) Calculate the *flux* of the vector field $\mathbf{V}(\mathbf{r}) = (0, x, 0)$ out of a cubic box with side length 1 in the first octant (See Fig. 10(a)).
(b) Calculate the *circulation* of the vector field $\mathbf{V}(\mathbf{r}) = (-y, x, 0)$ around a circle with radius 1 lying on the $x-y$ plane centered at the origin (See Fig. 10(b)).

3. Evaluate the following expressions,

$$(a) \quad \delta_{ii} \quad (1.112)$$
$$(b) \quad \delta_{ij}\epsilon_{ijk} \quad (1.113)$$
$$(c) \quad \epsilon_{ijk}\epsilon_{imn} \quad (1.114)$$
$$(d) \quad \epsilon_{ijk}\epsilon_{\ell jk} \quad (1.115)$$

A repeated index is summed, according to the Einstein summation convention.

4. Prove the following identities,

$$(a) \quad \nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b} \quad (1.116)$$
$$(b) \quad \nabla \times \nabla \times \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \quad (1.117)$$

Hint: Write $\nabla \cdot (\mathbf{a} \times \mathbf{b})$ in components $\epsilon_{ijk}\frac{\partial}{\partial x_i}(a_j b_k)$ and differentiate it, similarly for $\nabla \times \nabla \times \mathbf{a}$.

5. With the help of the divergence theorem and the Stokes theorem, prove that

$$(a) \quad \int_V dv \nabla \times \mathbf{v}(\mathbf{r}) = \int_S d\mathbf{s} \times \mathbf{v}(\mathbf{r}) \quad (1.118)$$
$$(b) \quad \int_S d\mathbf{s} \times \nabla f(\mathbf{r}) = \oint_C d\mathbf{r} f(\mathbf{r}) \quad (1.119)$$

Hint: Let the vector fields in the divergence theorem and the Stokes theorem be $\mathbf{V}(\mathbf{r}) = \mathbf{v}(\mathbf{r}) \times \mathbf{c}$ or $\mathbf{V}(\mathbf{r}) = \mathbf{c} f(\mathbf{r})$, $\mathbf{c}$ is a constant vector.

II. THE MAXWELL EQUATIONS

In this chapter, we outline the fundamental equations in electrodynamics.

A. Charge and current

1. Charge density

Consider a distribution of charge inside a volume V. If in a volume element dv near point $\mathbf{r}$, there is charge dQ, then the **charge density** at this location is

$$\rho(\mathbf{r}) \equiv \frac{dQ}{dv}. \quad (2.1)$$

By the integration of $\rho(\mathbf{r})$, we can have the total charge Q inside a volume V,

$$Q = \int_V dv \rho(\mathbf{r}). \quad (2.2)$$

As we have mentioned in Chap 1, for a point charge q at $\mathbf{r}_1$, its charge density is,

$$\rho(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}_1). \quad (2.3)$$

If there are point charges $q_1, q_1, \cdots, q_N$ at locations $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$, then the charge density of this system is,

$$\rho(\mathbf{r}) = \sum_{i=1}^{N} q_i \delta(\mathbf{r} - \mathbf{r}_i). \quad (2.4)$$

The total charge inside a volume V that encloses these charges is,

$$Q = \int_V dv \rho(\mathbf{r}) = \sum_{i=1}^{N} q_i \int_V dv \delta(\mathbf{r} - \mathbf{r}_i) \quad (2.5)$$
$$= \sum_{i=1}^{N} q_i. \quad (2.6)$$

Given a distribution of charges on a surface S. If on a surface element ds near point $\mathbf{r}$, there is charge dQ, then the **surface charge density** at this location is

$$\sigma(\mathbf{r}) \equiv \frac{dQ}{ds}. \quad (2.7)$$

By the integration of $\sigma(\mathbf{r})$, we can have the total charge Q on a surface S,

$$Q = \int_S ds\ \sigma(\mathbf{r}). \quad (2.8)$$

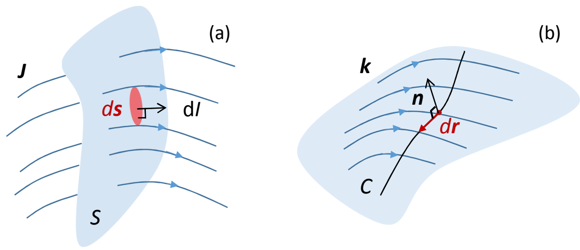

FIG. 11 (a) Current density $\mathbf{J}$ is the current flowing through a unit area ds. (b) Surface current density $\mathbf{K}$ is the surface current flowing pass a unit length dr.

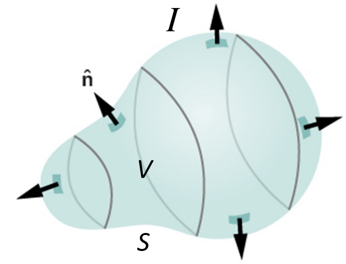

FIG. 12 Charges I flowing out of the surface S of volume V.

2. Current density

Electric current passing through a surface S is defined as the amount of charge passing through S per unit time. Current density is the current per unit area. Its dimension is [current]/[area], the dimension of current divided by the dimension of area. If there is current dI passing through a surface element $d\mathbf{s} = ds\hat{\mathbf{n}}$, then (Fig. 1(a))

$$dI = \mathbf{J}(\mathbf{r}) \cdot d\mathbf{s} = J_{\|}(\mathbf{r})ds, \tag{2.9}$$

where $\mathbf{J}(\mathbf{r})$ is the **current density** along the direction of charge motion, and $J_{\|} = \mathbf{J} \cdot \hat{\mathbf{n}}$ is its component along the surface normal $\hat{\mathbf{n}}$.

After integration, we can find out the total current passing through surface S,

$$I = \int_S d\mathbf{s} \cdot \mathbf{J}(\mathbf{r}). \tag{2.10}$$

If a small packet of charge dQ is moving with velocity $\mathbf{v}$, then within a time dt, the charges passing through $d\mathbf{s}$ have spanned a volume $dv = (\mathbf{v}dt) \cdot d\mathbf{s}$. Inside this volume,

$$dQ = \rho dv = \rho(\mathbf{v}dt) \cdot d\mathbf{s}, \tag{2.11}$$

which delivers a current,

$$dI = \frac{dQ}{dt} = \rho\mathbf{v} \cdot d\mathbf{s}. \tag{2.12}$$

Compared with Eq. (2.9), one has

$$\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}(\mathbf{r}). \tag{2.13}$$

For point charges, with Eq. (2.4), one has

$$\mathbf{J}(\mathbf{r}) = \sum_{i=1}^{N} q_i \mathbf{v}_i \delta(\mathbf{r} - \mathbf{r}_i), \tag{2.14}$$

where $\mathbf{v}_i$ is the velocity of charge i.

Next, consider the current flowing on a surface. The surface has normal vector $\hat{\mathbf{n}}$, and there is a line element $d\mathbf{r} \perp \hat{\mathbf{n}}$ on the surface (see Fig. 1(b)). The vector $\hat{\mathbf{n}} \times d\mathbf{r}$ is tangent to the surface and perpendicular to $d\mathbf{r}$. The current dI passes through $d\mathbf{r}$ is,

$$dI = \mathbf{K}(\mathbf{r}) \cdot (\hat{\mathbf{n}} \times d\mathbf{r}), \tag{2.15}$$

where $\mathbf{K}(\mathbf{r})$ is the **surface current density** along the direction of charge motion. Its dimension is [current]/[length].

After integration, we can find out the total current passing through a curve C on the surface,

$$I = \int_C \mathbf{K}(\mathbf{r}) \cdot \hat{\mathbf{n}} \times d\mathbf{r} = \int_C \mathbf{K}(\mathbf{r}) \times \hat{\mathbf{n}} \cdot d\mathbf{r} \tag{2.16}$$

3. Conservation of charge

Suppose the charge Q inside a volume V is leaking through its surface S to the outside (Fig. 2). The leaking current through S is,

$$I = -\frac{dQ}{dt}. \tag{2.17}$$

With (2.10), we have

$$I = \int_S d\mathbf{s} \cdot \mathbf{J} = \int_V dv \nabla \cdot \mathbf{J}, \tag{2.18}$$

and from Eqs. (2.2),

$$\frac{dQ}{dt} = \frac{d}{dt}\int_V dv\rho(\mathbf{r}, t) = \int_V dv \frac{\partial \rho(\mathbf{r}, t)}{\partial t}, \tag{2.19}$$

in which the region V of integration is fixed. Hence,

$$\int_V dv \nabla \cdot \mathbf{J} = -\int_V dv \frac{\partial \rho(\mathbf{r}, t)}{\partial t} \tag{2.20}$$

$$\text{or} \quad \int_V dv \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right) = 0. \tag{2.21}$$

Since the charge should be conserved for any dv in any location, so we can choose V to be one of the dv, then

$$\int_V dv \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right) \simeq dv \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right) \tag{2.22}$$

$$\to \nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0, \text{ at any } \mathbf{r}. \tag{2.23}$$

This is **equation of continuity**, which is valid *if and only if* charge is conserved.

B. Maxwell equations in vacuum

1. Electrostatics

According to **Coulomb's law**, the electric force between two charges q, q_1 at positions $\mathbf{r}, \mathbf{r}_1$ is,

$$\mathbf{F} = \frac{qq_1}{4\pi\varepsilon_0}\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|^3}, \tag{2.24}$$

where the **electric permittivity** of free space $\varepsilon_0 = 8.8542 \times 10^{-12}\ \mathrm{C}^2/\mathrm{Nm}^2$.

If there are N charges $q_1, q_2, \cdots, q_N$ at positions $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$, then a test charge charge q at $\mathbf{r}$ feels a force,

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N} qq_i\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}. \tag{2.25}$$

The electric field $\mathbf{E}$ from these N charges is given as,

$$\mathbf{E}(\mathbf{r}) \equiv \frac{\mathbf{F}}{q} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N} q_i\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}. \tag{2.26}$$

A continuous charge distribution can be divided into small packets with charges $\rho(\mathbf{r}')dv'$ (Fig. 3). Identify q_i with $\rho(\mathbf{r}')dv'$ and replace the summation with an integral, one then has

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{2.27}$$

This form is valid for all kinds of charge distribution, continuous or discrete. You may check that with Eq. (2.4), Eq. (2.27) reduces to Eq. (2.26).

We can rewrite

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{2.28}$$

so that

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -\frac{1}{4\pi\varepsilon_0}\int dv'\rho(\mathbf{r}')\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}, && (2.29)\\ &= -\nabla\phi(\mathbf{r}), && (2.30)\end{aligned}$$

with **electric potential**,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.31}$$

Note that the order of $\int dv'$ and ∇ can be exchanged, since $\mathbf{r}'$ and $\mathbf{r}$ are independent variables.

Also, remember that

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}'). \tag{2.32}$$

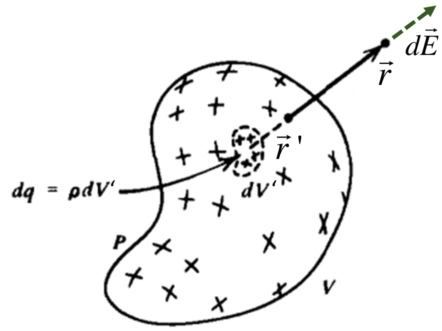

FIG. 13 The electric field $\mathbf{E}(\mathbf{r})$ at point $\mathbf{r}$ is the sum of the electric fields $d\mathbf{E}$ produced by charges $\rho(\mathbf{r}')dv'$ in volume elements.

Thus,

$$\begin{aligned}\nabla\cdot\mathbf{E}(\mathbf{r}) &= -\frac{1}{4\pi\varepsilon_0}\int dv'\rho(\mathbf{r}')\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} && (2.33)\\ &= \frac{1}{\varepsilon_0}\int dv'\rho(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') && (2.34)\\ &= \frac{\rho(\mathbf{r})}{\varepsilon_0}. && (2.35)\end{aligned}$$

This is **Gauss's law**. When written in electric potential, we have

$$\nabla^2\phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}. \tag{2.36}$$

This is the **Poission equation** that has been mentioned in Chap 1.

Furthermore, since the curl of divergence is zero, so

$$\nabla\times\mathbf{E}(\mathbf{r}) = -\nabla\times\nabla\phi = 0. \tag{2.37}$$

In short, the fundamental equations of electrostatics are

$$\begin{aligned}\nabla\cdot\mathbf{E}(\mathbf{r}) &= \frac{\rho(\mathbf{r})}{\varepsilon_0}, && (2.38)\\ \nabla\times\mathbf{E}(\mathbf{r}) &= 0. && (2.39)\end{aligned}$$

If we integrate Eq. (2.38) over a region V enclosed by surface S, then

$$\begin{aligned}\int_v dv\nabla\cdot\mathbf{E}(\mathbf{r}) &= \frac{1}{\varepsilon_0}\int_V dv\rho(\mathbf{r}), && (2.40)\\ \text{or}\quad \int_S d\mathbf{s}\cdot\mathbf{E}(\mathbf{r}) &= \frac{Q}{\varepsilon_0}, && (2.41)\end{aligned}$$

where Q is the total amount of charge inside V. This is the integral form of the Gauss's law.

If we integrate Eq. (2.39) over a surface S with boundary C, then

$$\begin{aligned}\int_S d\mathbf{s}\cdot\nabla\times\mathbf{E}(\mathbf{r}) &= 0, && (2.42)\\ \text{or}\quad \oint_C d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) &= 0. && (2.43)\end{aligned}$$

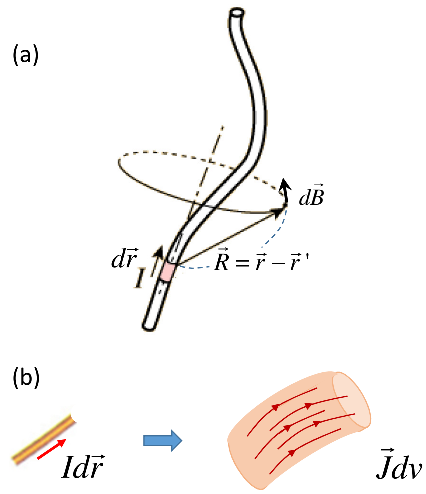

FIG. 14 (a) The magnetic field $d\mathbf{B}$ produced by a segment $d\mathbf{r}$ of a current-carrying wire. (b) A thin wire is replaced by a body (or a region) with volume element dv.

2. Magnetostatics

According to **Biot-Savart law**, the magnetic field produced by a short segment $d\mathbf{r}'$ of a thin wire carrying current I is (see Fig. 4(a)),

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{2.44}$$

where the **magnetic permeability** in vacuum $\mu_0 = 4\pi\times 10^{-7}$ N/A^2. For a closed loop C of thin wire,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint_C I d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{2.45}$$

Given a general current distribution, just (see Fig. 4(b))

$$\text{replace} \quad I d\mathbf{r}' \quad \text{with} \quad \mathbf{J}(\mathbf{r}')dv', \tag{2.46}$$

so that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V dv' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{2.47}$$

This is the most general form of the Biot-Savart law that applies to all kinds of current distribution.

Again we can rewrite

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.48}$$

With the identity,

$$\nabla\times(f\mathbf{v}) = \nabla f \times \mathbf{v} + f\nabla\times\mathbf{v}, \tag{2.49}$$

we have

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= -\frac{\mu_0}{4\pi} \int dv' \mathbf{J}(\mathbf{r}') \times \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} && (2.50) \\ &= \nabla\times\mathbf{A}(\mathbf{r}), && (2.51) \end{aligned}$$

with the **vector potential**,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dv' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.52}$$

For a thin wire, it reduces to

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \oint_C d\mathbf{r}' \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.53}$$

Since the divergence of curl is zero, so

$$\nabla\cdot\mathbf{B}(\mathbf{r}) = \nabla\cdot\nabla\times\mathbf{A}(\mathbf{r}) = 0. \tag{2.54}$$

This is **Gauss's law in magnetism**. Also, if we take the curl of $\mathbf{B}$, then

$$\nabla\times\mathbf{B}(\mathbf{r}) = \mu_0\mathbf{J}(\mathbf{r}). \tag{2.55}$$

This is **Ampère's law**.

Pf: First, we can show that for the steady case $\nabla\cdot\mathbf{J}=0$, one has $\nabla\cdot\mathbf{A}=0$. This is because

$$\begin{aligned} \nabla\cdot\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int dv' \mathbf{J}(\mathbf{r}')\cdot\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} && (2.56) \\ &= -\frac{\mu_0}{4\pi} \int dv' \mathbf{J}(\mathbf{r}')\cdot\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} && (2.57) \\ &= \frac{\mu_0}{4\pi} \int dv' \underbrace{\nabla'\cdot\mathbf{J}(\mathbf{r}')}_{=0} \frac{1}{|\mathbf{r}-\mathbf{r}'|} && (2.58) \\ &= 0, && (2.59) \end{aligned}$$

where we have used the identity,

$$\nabla\cdot(f\mathbf{v}) = \nabla f\cdot\mathbf{v} + f\nabla\cdot\mathbf{v}. \tag{2.60}$$

Also, a surface term (for the surface at infinity) has been dropped.

Second,

$$\begin{aligned} \nabla\times\mathbf{B}(\mathbf{r}) &= \nabla\times(\nabla\times\mathbf{A}) && (2.61) \\ &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} && (2.62) \\ &= -\nabla^2\mathbf{A}(\mathbf{r}) \quad \because \ \nabla\cdot\mathbf{A}=0 && (2.63) \\ &= -\frac{\mu_0}{4\pi} \int dv' \mathbf{J}(\mathbf{r}') \underbrace{\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|}}_{=-4\pi\delta(\mathbf{r}-\mathbf{r}')} && (2.64) \\ &= \mu_0\mathbf{J}(\mathbf{r}). \qquad QED && (2.65) \end{aligned}$$

When written in potential, we have

$$\nabla^2\mathbf{A}(\mathbf{r}) = -\mu_0\mathbf{J}(\mathbf{r}). \tag{2.66}$$

This is the **vector Poission equation** in magnetostatics.

In short, the fundamental equations of magnetostatics are

$$\begin{aligned} \nabla\cdot\mathbf{B}(\mathbf{r}) &= 0, && (2.67) \\ \nabla\times\mathbf{B}(\mathbf{r}) &= \mu_0\mathbf{J}(\mathbf{r}). && (2.68) \end{aligned}$$

If we integrate Eq. (2.67) over a region V enclosed by surface S, then

$$\int_v dv \nabla \cdot \mathbf{B}(\mathbf{r}) = \int_S d\mathbf{s} \cdot \mathbf{B}(\mathbf{r}) = 0, \tag{2.69}$$

This shows that the magnetic flux through a closed surface is always zero. The existence of a magnetic monopole would contradict this result, but no magnetic monopole has been found so far.

If we integrate Eq. (2.68) over a surface S with boundary C, then

$$\int_S d\mathbf{s} \cdot \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \int_S d\mathbf{s} \cdot \mathbf{J}(\mathbf{r}), \tag{2.70}$$

$$\text{or} \quad \oint_C d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = \mu_0 I, \tag{2.71}$$

where I is the total current flowing through S. This is the integral form of the Ampère's law.

3. Dynamic electromagnetic field

Eqs. (2.38), (2.39), (2.67), and (2.68) are the Maxwell equations for static electromagnetic field. For dynamics fields, we need to add two new terms,

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \frac{\rho(\mathbf{r}, t)}{\varepsilon_0}, \tag{2.72}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{2.73}$$

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t), \tag{2.74}$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \mathbf{J}(\mathbf{r}, t) + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r}, t). \tag{2.75}$$

The charge density and the current density are related by the equation of continuity,

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) + \frac{\partial \rho(\mathbf{r}, t)}{\partial t} = 0. \tag{2.76}$$

The first change is that in Eq. (2.74), the right hand side (RHS) is no longer zero. This is **Faraday's law**:*a time-changing magnetic field produces an electric field.*

The second change is that there is an extra term on the RHS of Eq. (2.75). This is the famous **displacement current** added by Maxwell: *a time-changing electric field produces a magnetic field.* This modified equation is called **Ampère-Maxwell's law**.

When the fields are static, the Maxwell's equations decouple into two sets of equations: two for electric field, and two for magnetic field. Thus, electrostatics and magnetostatics are independent of each other.

Integrating a divergence (e.g., $\nabla \cdot \mathbf{E}$) over a volume V or a curl (e.g., $\nabla \times \mathbf{E}$) over a surface, and using the divergence theorem or the Stokes theorem, we have the *integral form* of the Maxwell equations (Fig. 5):

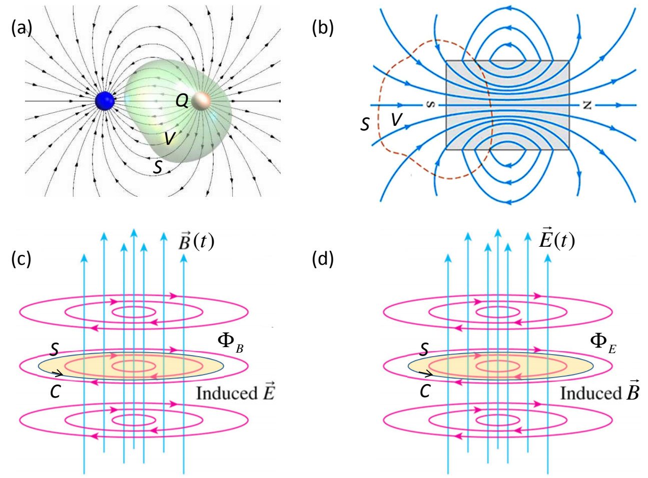

FIG. 15 Illustration of the Maxwell's equations: (a) Gauss's law. (b) Gauss's law in magnetism. (c) Faraday's law. (d) Ampère-Maxwell's law (with $I = 0$). Figs. from the web.

$$\int_S d\mathbf{s} \cdot \mathbf{E}(\mathbf{r}, t) = \frac{Q}{\varepsilon_0}, \tag{2.77}$$

$$\int_S d\mathbf{s} \cdot \mathbf{B}(\mathbf{r}, t) = 0, \tag{2.78}$$

$$\oint_C d\mathbf{r} \cdot \mathbf{E}(\mathbf{r}, t) = -\frac{d\Phi_B}{dt}, \tag{2.79}$$

$$\oint_C d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}, t) = \mu_0 I + \frac{1}{c^2}\frac{d\Phi_E}{dt}, \tag{2.80}$$

in which

$$\Phi_B \equiv \int_S d\mathbf{s} \cdot \mathbf{B}, \tag{2.81}$$

$$\Phi_E \equiv \int_S d\mathbf{s} \cdot \mathbf{E}. \tag{2.82}$$

They are the **magnetic flux** and the **electric flux** passing through surface S. Eq. (2.79) (Eq. (2.80)) tells us that a changing magnetic (electric) flux through surface S would induce electric (magnetic) circulation around the boundary C of S.

Note: The first order derivatives of a vector $\mathbf{V}(\mathbf{r})$ have 9 components, $\partial V_i/\partial x_j$ $(i, j = 1, 2, 3)$. The Maxwell equations are written in terms of divergence and curl of $\mathbf{E}$ (or $\mathbf{B}$), which does not exhaust the possibilities just mentioned. This is all right since according to the **Helmholtz theorem**, a vector field $\mathbf{V}(\mathbf{r})$ that vanishes at infinity is completely determined by giving its divergence and curl everywhere in space.

C. Some history

In 1873, James C. Maxwell published " *Treatise on electricity and magnetism*" (Maxwell, 1891), in which he con-

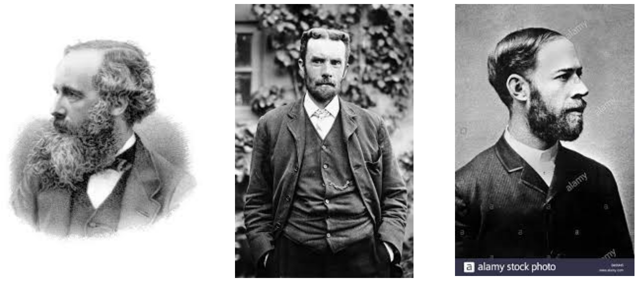

FIG. 16 From left to right, Maxwell, Heaviside, and Hertz.

structed a mathematical framework to describe the phenomena of electromagnetism. It has all the essence included but it's hard to find "Maxwell equations" in the *Treatise*, since they are written as 20 equations in 20 variables scattered through the monograph. Some of the equations describe things like $\mathbf{D} = \varepsilon\mathbf{E}$, or $\mathbf{B} = \nabla \times \mathbf{A}$. It's a pity that Maxwell died six years later at the age of 48, and was unable to pursue this subject further.

The four Maxwell equations we are familiar with nowadays are mainly the works of Oliver Heaviside and, independently, Heinrich R. Hertz (Fig. 6). It's interesting to know that when the *Treatise* was just published, Heaviside (then 24 years old) flipped through it in library and immediately saw the "prodigious possibilities in its power". He then "determined to master the book". (Mahon, 2017) Remember that at that time Maxwell is still not "the Maxwell" and not many people trust his obscure, sometimes unintelligible theory of electromagnetism.

Heaviside has no college education, and has forgotten most of the algebra and trigonometry learned in school. Thus, he quit his job that has a decent pay, stayed at home with his far-from-rich parents and started studying the *Treatise*. He remained "self-employed" ever since and never to get a job again. Heaviside has to learn all of the difficult mathematics of divergence, curl, and related theorems on his own, without friendly textbooks to ease the job. In his later years, Heaviside recalls that "It took me several years before I could understand as much as I possibly could. Then I set Maxwell aside and followed my own course."

The effort and sacrifice pay off. With his own formulation of Maxwell equations, Heaviside discovered things like **electric inductance**, contraction of the electric field of a moving charge (**Heaviside ellipsoid**), and magnetic-like field of gravity (**gravito-magnetism**).

In 1888, to the surprise of everybody, Hertz generated and detected electromagnetic wave in free space. This is the strongest boost to the status of Maxwell's electromagnetic theory since at that time there was no other theory of electromagnetism that predicted the existence of EM wave. Afterwards, optics becomes a branch of electromagnetism.

More progress followed, such as the discovery of **electron** (the source of electric field) by J. J. Thomson in 1897, the theory of **thermal radiation** (randomized EM field) by Ludwig E. Boltzmann and others. The latter pursuit eventually leads to Max Planck's important discovery of **energy quantum** at 1900.

Furthermore, in an attempt to resolve a paradox regarding motional electromotive force, Einstein discovered the theory of **special relativity** in 1905. As a result, Newton's theory of mechanics needs to be revised. Nevertheless, *Maxwell's theory remains intact*, since it is based on experimental observations that have already included relativistic effects.

Problem:

1. The electric potential of an atom is given by

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r}, \tag{2.83}$$

where $q(>0), \alpha$ are constants.
(a) Find out the electron charge density $\rho(\mathbf{r})$ outside the nucleus.
(b) Find out the total charge of this charge distribution.
Hint: Poission equation.

2. (a) Show that Eqs. (2.72) and (2.75) are consistent with the equation of continuity in Eq. (2.76).
Hint: Take the time derivative of ρ on the right-hand side of Eq. (2.72), and the divergence of $\mathbf{J}$ on the right-hand side of Eq. (2.75).
(b) Suppose there are magnetic monopoles, such that

$$\nabla \cdot \mathbf{B} = \mu_o \rho_m, \tag{2.84}$$

where ρ_m is the magnetic charge density Similar to electric charges, the equation of continuity of magnetic charges is,

$$\nabla \cdot \mathbf{J}_m + \frac{\partial \rho_m}{\partial t} = 0, \tag{2.85}$$

where $\mathbf{J}_m$ is the magnetic current density. What type of term should be added to the right-hand-side of Eq. (2.74), so that new Maxwell equations can be consistent with the equation of continuity above?

III. ELECTROSTATICS

A. Introduction

There are several ways to find out an electric field. First, if we have the complete information of charge distribution $\rho(\mathbf{r})$, then one just needs to evaluate the Coulomb integral,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{3.1}$$

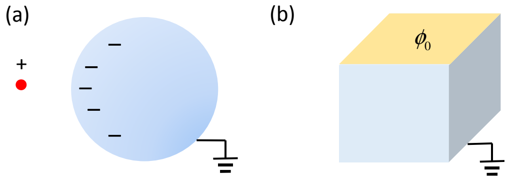

FIG. 17 (a) When a positive charge is near a grounded metal sphere, there are negative induced charges on the surface of the sphere. (b) The metal box is grounded, except the top surface, which is maintained at potential ϕ_0.

where $\varepsilon_0 = 8.8542 \times 10^{-12}$ C^2/Nm^2. Or, one may calculate the electric potential first,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int dv' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{3.2}$$

then take its gradient to get $\mathbf{E} = -\nabla\phi$.
Note: when we write $\int$ instead of $\int_V$, an integration over the whole space is often implied.

The problem with the method above is that the charge distribution is not always known. For example, when you place a point charge near a grounded metal sphere, the induced charge is not known beforehand (Fig. 1). Or, a metal box is grounded for five of its surface, except that the top surface is maintained at potential ϕ_0. The charges on metal box redistribute themselves to meet this condition, but their distribution unknown. For these cases, we need Gauss's law,

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0}, \tag{3.3}$$

or Poisson equation,

$$\nabla^2\phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0}. \tag{3.4}$$

We need to solve it, given the boundary condition (BC) for ϕ. Afterwards, we can get the electric field $\mathbf{E} = -\nabla\phi$. The distribution of charges can be determined after the field is known.

When a system has simple geometry, such as a cylinder or a sphere, it is convenient to find $\mathbf{E}$ using the integral form of Gauss's law,

$$\int_S d\mathbf{s} \cdot \mathbf{E}(\mathbf{r}) = \frac{Q}{\varepsilon_0}. \tag{3.5}$$

In this course, we avoid using the second method. Not because it's not important or less used, but because we'd like to focus more on physics, less on solving partial differential equations and wielding special functions.

B. Coulomb's law

Let's practice the first, direct integration method with an example.

Example:
Find the electric field along the central axis of (a) a charged ring, (b) a charged disk, and (c) a charged plane. All of them uniformly charged.
Sol'n:
(a) Suppose a ring with radius r has charge Q, then its charge density per unit length $\lambda = Q/2\pi r$. A short segment $d\ell$ with charge $dQ = \lambda d\ell$ produces an electric field $d\mathbf{E}$ (Fig. 2(a)). Along the central axis at a distance z away,

$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{r^2+z^2}\cos\alpha,\ \cos\alpha = \frac{z}{\sqrt{r^2+z^2}}. \tag{3.6}$$

After integration,

$$\begin{aligned} E_z(z) &= \frac{1}{4\pi\varepsilon_0}\oint_C \frac{\lambda d\ell}{r^2+z^2}\cos\alpha && (3.7)\\ &= \frac{Q}{4\pi\varepsilon_0}\frac{z}{(r^2+z^2)^{3/2}}. && (3.8)\end{aligned}$$

The components $E_{x,y}$ cancels away after integration, thus $\mathbf{E}(z) = E_z(z)\hat{\mathbf{z}}$. If you are interested in the potential away from the central axis, which is a more difficult problem, see Chap 3 of Jackson, 1998.

(b) A disk can be considered as a collection of rings (Fig. 2(b)). Suppose it has radius R and charge Q, then its surface charge density $\sigma = Q/\pi R^2$. A ring with radius r and width dr has charge

$$dQ = \sigma\ 2\pi r dr. \tag{3.9}$$

According to Eq. (3.8), along the central axis at a distance z away,

$$dE_z = \frac{dQ}{4\pi\varepsilon_0}\frac{z}{(r^2+z^2)^{3/2}}. \tag{3.10}$$

Integrate along the radial direction to get

$$\begin{aligned} E_z(z) &= \frac{1}{2\varepsilon_0}\int_0^R \sigma r dr \frac{z}{(r^2+z^2)^{3/2}} && (3.11)\\ &= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right). && (3.12)\end{aligned}$$

Finally, $\mathbf{E}(z) = E_z(z)\hat{\mathbf{z}}$.

(c) To get the electric field of an infinite charged plane, just let the R in Eq. (3.12) be infinite,

$$\mathbf{E}(z>0) = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}. \tag{3.13}$$

On the other side of the plane, obviously we have

$$\mathbf{E}(z<0) = -\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}. \tag{3.14}$$

The electric field is discontinuous across the plate,

$$\mathbf{E}(0^+) - \mathbf{E}(0^-) = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{z}}. \tag{3.15}$$

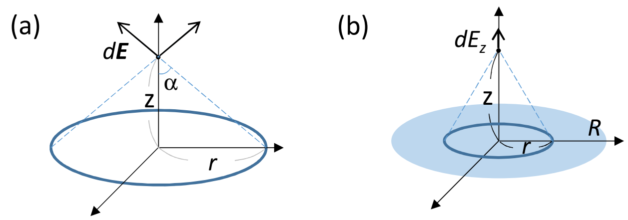

FIG. 18 (a) A charged ring. (b) A charged disk.

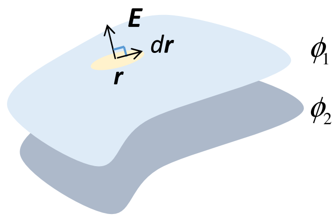

FIG. 19 Electric field is perpendicular to equipotential surface.

C. Electric potential

The following three equations state the same fact about the electrostatic field,

$$1.\ \ \mathbf{E} = -\nabla\phi, \tag{3.16}$$

$$2.\ \ \nabla \times \mathbf{E} = 0, \tag{3.17}$$

$$3.\ \ \oint d\mathbf{r}\cdot\mathbf{E} = 0, \tag{3.18}$$

1 implies 2 since the curl of gradient is zero. Conversely, 2 implies 1 since if a vector field is curless, then it can be written as a gradient (see Chap 1). Also, 2 and 3 are simply the differential form and the integral form of the same Maxwell equation (see Chap 2).

1. Equipotential surface

The equation $\phi(\mathbf{r}) = \phi_0$, where ϕ_0 is a constant, defines an **equipotential surface** S_0. If $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ are both located on S_0, then moving a charge from $\mathbf{r}$ to $\mathbf{r}+d\mathbf{r}$ requires no work,

$$dW = q\mathbf{E}\cdot d\mathbf{r} = 0. \tag{3.19}$$

This is valid for any tangent vector $d\mathbf{r}$ emanating from $\mathbf{r}$. Thus, $\mathbf{E}(\mathbf{r})$ is perpendicular to the tangent plane of S_0 at $\mathbf{r}$ (Fig. 3). That is, *the steepest descent $-\nabla\phi$ is perpendicular to the equipotential surface.*

Example:

Find out the electric potential of a uniformly charged wire with length $2L$ and linear charge density λ.

Sol'n:

Suppose the wire is lying on the z-axis, as in Fig. 4(a). Since there is rotational symmetry around the wire, it is convenient to use the cylindrical coordinate. The potential at a point with coordinate z, ρ is,

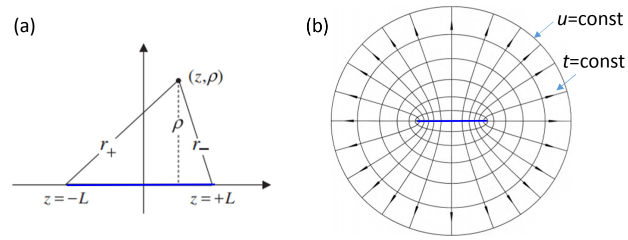

FIG. 20 (a) A charged wire. (b) Equipotential surfaces and field lines of a charged wire. (Fig. from Zangwill)

$$\begin{aligned}\phi(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int_{\text{wire}}\frac{\lambda dr'}{|\mathbf{r}-\mathbf{r}'|} &(3.20)\\ &= \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{\lambda dz'}{\sqrt{(z'-z)^2+\rho^2}} &(3.21)\\ &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{\sqrt{(L-z)^2+\rho^2}+L-z}{\sqrt{(L+z)^2+\rho^2}-L-z}\right), &(3.22)\end{aligned}$$

where we have used

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \ln\left(\sqrt{x^2+a^2}+x\right). \tag{3.23}$$

When the observation point is far away from the wire, $z, \rho \gg L$, and $r = \sqrt{z^2+\rho^2} \gg z, L$, one has

$$\phi(\mathbf{r}) \simeq \frac{1}{4\pi\varepsilon}\frac{Q}{r},\ \ Q = 2L\lambda. \tag{3.24}$$

It is similar to the potential of a point charge.

On the other hand, if the observation point is close to the center of the wire, $\rho \ll L, z = 0$, then expand the potential to the second order of ρ/L to get

$$\phi(\rho) \simeq -\frac{\lambda}{2\pi\varepsilon_0}\ln\rho + \frac{\lambda}{2\pi\varepsilon_0}\ln(2L). \tag{3.25}$$

Note that it diverges if $\rho \to 0$. Its gradient gives the electric field,

$$\mathbf{E}(\rho) = -\nabla\phi \simeq \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\rho}. \tag{3.26}$$

Further analysis of the result:

Instead of z, ρ, we can use r_+, r_- as coordinates (see Fig. 4(a)),

$$r_\pm \equiv \sqrt{(L\pm z)^2+\rho^2}. \tag{3.27}$$

Note that

$$r_+^2 - r_-^2 = 4Lz \ \ \to\ \ z = \frac{1}{4L}\left(r_+^2 - r_-^2\right). \tag{3.28}$$

With the two relations above, we can write the potential in new coordinate $\phi(r_+, r_-)$.

The third choice of coordinate is u, t, where

$$\begin{cases} u = \frac{1}{2}(r_- + r_+), \\ t = \frac{1}{2}(r_- - r_+) \end{cases} \leftrightarrow \begin{cases} r_- = u + t, \\ r_+ = u - t \end{cases} \tag{3.29}$$

Note that the equation u =constant draws out an ellipse, and t =constant an hyperbola. Thus the new coordinate is called **elliptic coordinate**, which is an *orthogonal* coordinate since at the intersection of coordinate curves, the tangents are perpendicular to each other.

Now,

$$ut = -zL \quad \rightarrow \quad z = -\frac{ut}{L}. \tag{3.30}$$

Thus,

$$\phi(u,t) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{u+t+L-z}{u-t-L-z}\right) \tag{3.31}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{u+L}{u-L}\right), \tag{3.32}$$

which is independent of t. Hence, the potential is a constant when u is fixed. That is, the equipotential surface is an ellipse (Fig. 4(b)), or an ellipsoid after revolving around the z-axis. Furthermore, since the curves of fixed t's describe electric field lines, since they are perpendicular to the equipotential surfaces.

2. Earnshaw's theorem

Inside a region V without any charge, the electric potential *cannot* have any local minimum or local maximum. This is called **Earnshaw's theorem**, which is true for *electrostatic* field.

Pf: We'll prove this by contradiction. Suppose the potential $\phi(\mathbf{r})$ has a local minimum at point P inside V. Then, when one moves away from P, the potential increases (Fig. 5).

Surround the point P with a small spherical surface S. Then on surface S, the gradient $\nabla\phi$, which is along the steepest ascent, points outward. That is, if $\hat{\mathbf{n}}$ is the normal vector of S (pointing outward), then

$$\hat{\mathbf{n}} \cdot \nabla\phi > 0. \tag{3.33}$$

for *every* point on S.

Thus, after integration,

$$\int_S ds\hat{\mathbf{n}} \cdot \nabla\phi > 0. \tag{3.34}$$

With the help of divergence theorem, the LHS can be written as,

$$\int_S d\mathbf{s} \cdot \nabla\phi = -\int_V dv \nabla \cdot \mathbf{E} = 0, \tag{3.35}$$

It is zero because there is no charge inside V. Thus, we have a contradiction. The same contradiction occurs if P

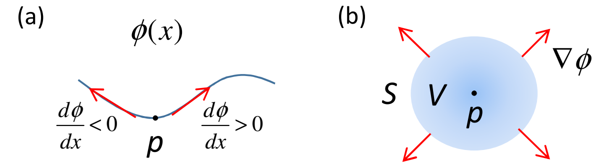

FIG. 21 (a) Potential and its slope in one dimension. (b) Potential and its gradient in three dimension.

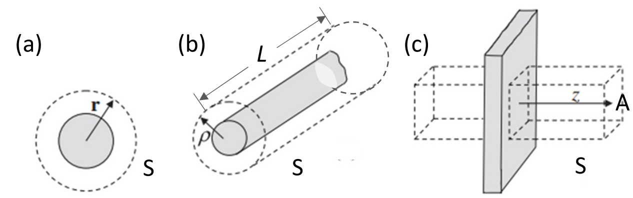

FIG. 22 Charge distribution with (a) spherical symmetry, (b) cylindrical symmetry, and (c) planar symmetry. (Fig. from Zangwill)

is a local maximum. Hence, neither local minimum nor maximum is allowed inside V. QED.

Alternatively speaking, the location of local max or local min of ϕ always hosts positive or negative charges.

D. Gauss's law

As we have mentioned in Sec. III.A, when a system has a simple geometry, we can use the integral form of the Gauss's law to find electric field,

$$\int_S d\mathbf{s} \cdot \mathbf{E}(\mathbf{r}) = \frac{Q}{\varepsilon_0}. \tag{3.36}$$

Example:
Find out the electric field for systems with (Fig. 6)
(a) Spherical symmetry: $\rho(r,\theta,\phi) = \rho(r)$.
(b) Cylindrical symmetry: $\rho_e(\rho,\phi,z) = \rho_e(\rho)$.
(c) Planar symmetry: $\rho(x,y,z) = \rho(z)$. Furthermore, assume $\rho(-z) = \rho(z)$.
Sol'n:
(a) We expect the electric field to be radial and depend only on r, $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$. Choose S to be a spherical surface with radius r, then Eq. (3.36) gives

$$\int_S d\mathbf{s} \cdot \mathbf{E}(\mathbf{r}) = 4\pi r^2 E(r) = \frac{Q(r)}{\varepsilon_0}, \tag{3.37}$$

where $Q(r)$ is the charge inside the surface S. Hence,

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r^2}. \tag{3.38}$$

If all of the charges Q_0 are confined within radius R, then when $r \geq R$,

$$E(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q_0}{r^2}, \tag{3.39}$$

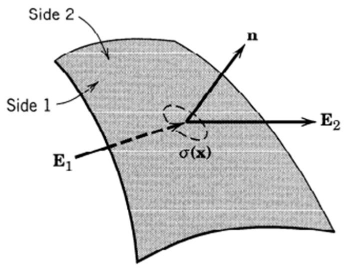

FIG. 23 A charge surface has different electric fields on two sides. (Fig. from Jackson, 1998)

same as the field of a point charge Q_0 at the origin.

(b) We expect the electric field to be radial and depend only on ρ, $\mathbf{E}(\mathbf{r}) = E(\rho)\hat{\boldsymbol{\rho}}$. Choose S to be a cylindrical surface with radius ρ and height L, then Eq. (3.36) gives

$$\int_S d\mathbf{s} \cdot \mathbf{E}(\mathbf{r}) = 2\pi\rho L E(\rho) = \frac{Q(\rho)}{\varepsilon_0}, \tag{3.40}$$

where $Q(\rho)$ is the charge inside the surface S. Hence,

$$E(\rho) = \frac{1}{2\pi\varepsilon_0}\frac{Q(\rho)/L}{\rho}. \tag{3.41}$$

(c) We expect the electric field to be along z and depend only on z,

$$\mathbf{E}(\mathbf{r}) = \begin{cases} E(z)\hat{\mathbf{z}}, & \text{for } z > 0 \\ -E(z)\hat{\mathbf{z}}, & \text{for } z < 0 \end{cases} \tag{3.42}$$

Choose S to be a box surface (bisected by the x-y plane) with area A and height $2z$, then Eq. (3.36) gives

$$\int_S d\mathbf{s} \cdot \mathbf{E}(\mathbf{r}) = 2AE(z) = \frac{Q(z)}{\varepsilon_0}, \tag{3.43}$$

where $Q(z)$ is the charge inside the box S. Hence,

$$E(z) = \frac{Q(z)/A}{2\varepsilon_0}. \tag{3.44}$$

If all of the charges are confined within $|z| < Z$, then when $z \geq Z$,

$$E(z) = \frac{\sigma_0}{2\varepsilon_0}, \tag{3.45}$$

where $\sigma_0 = Q(Z)/A$ is the surface charge density. In general, for $|z| \geq Z$

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma_0}{2\varepsilon_0}\mathrm{sgn}(z)\hat{\mathbf{z}}. \tag{3.46}$$

E. Boundary condition for E

In general, the electric fields on opposite sides of a charged surface are not the same. Their difference is caused by the charges on the surface. Suppose a surface has surface charge density $\sigma(\mathbf{r})$. At a point $\mathbf{r}$ on the surface, the electric fields on opposite sides are $\mathbf{E}_1(\mathbf{r})$ and $\mathbf{E}_2(\mathbf{r})$ (Fig. 7). What's the relation between this two electric fields?

First, divide the surface S into a small disk $\circ$ and a surface S' (S with $\circ$ removed),

$$S = \circ + S'. \tag{3.47}$$

The disk is microscopically large, but macroscopically small (say, with a radius of 1 μm). The field, $\mathbf{E}_1(\mathbf{r})$ or $\mathbf{E}_2(\mathbf{r})$, is the superposition of the fields produced by $\circ$ and S'.

When one approaches the center of the disk, the field is close to the field of an infinite plane, $\mathbf{E}(\mathbf{r}) = \frac{\sigma}{2\varepsilon_0}\mathrm{sgn}(z)\hat{\mathbf{z}}$. Suppose the field produced by S' is $\mathbf{E}_S$, then

$$\mathbf{E}_1 = \mathbf{E}_S - \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}, \tag{3.48}$$

$$\mathbf{E}_2 = \mathbf{E}_S + \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}, \tag{3.49}$$

where $\hat{\mathbf{n}}$ is the normal vector pointing *from region 1 to region 2.*

Even though $\mathbf{E}_S$ remains unknown, we can substrate the field to get

$$\mathbf{E}_2(\mathbf{r}) - \mathbf{E}_1(\mathbf{r}) = \frac{\sigma(\mathbf{r})}{\varepsilon_0}\hat{\mathbf{n}}. \tag{3.50}$$

This is the BC for fields near a charged surface. Sometimes it is written as,

$$\hat{\mathbf{n}} \cdot (\mathbf{E}_2 - \mathbf{E}_1) = \frac{\sigma}{\varepsilon_0}, \tag{3.51}$$

$$\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0. \tag{3.52}$$

1. Force on charged surface

Following the example above, the force $d\mathbf{F}$ on disk $\circ$ is due to the charges on S'. *The disk exerts no force on itself.* If the disk has area ds, then

$$d\mathbf{F} = (\sigma ds)\mathbf{E}_S. \tag{3.53}$$

The force per unit area (or pressure), is

$$\mathbf{f} \equiv \frac{d\mathbf{F}}{ds} = \sigma\mathbf{E}_S. \tag{3.54}$$

Since $\mathbf{E}_S = (\mathbf{E}_1 + \mathbf{E}_2)/2$, we have

$$\mathbf{f} = \frac{\sigma}{2}(\mathbf{E}_1 + \mathbf{E}_2). \tag{3.55}$$

For example, for a closed metallic surface, the electric fields on the inside and outside are (Fig. 8),

$$\mathbf{E}_1 = 0, \quad \mathbf{E}_2 = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}. \tag{3.56}$$

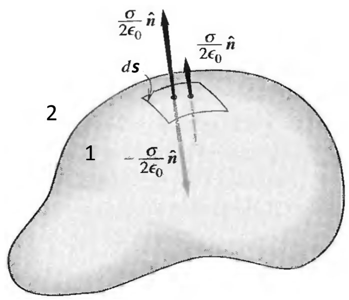

FIG. 24 The electric fields produced by an area element $d\mathbf{s}$ and the surface with a hole (at $d\mathbf{s}$). (Fig. from Lorrain and Corson)

Hence, according to Eq. (3.55),

$$\mathbf{f} = \frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{n}}, \tag{3.57}$$

where $\hat{\mathbf{n}}$ points out of the sphere.

Note that if one calculates the force via

$$\mathbf{f} = \sigma\mathbf{E}_2 = \frac{\sigma^2}{\varepsilon_0}\hat{\mathbf{n}}, \tag{3.58}$$

then the result is wrong by a factor of two, since it has wrongly included the force exerted by the disk on itself.

F. Solid angle

The solid angle spanned by an area element $d\mathbf{s} = ds\hat{\mathbf{n}}$ located at $\mathbf{r}$ with respect to the origin (Fig. 9(a)) is defined as,

$$d\Omega \equiv \frac{\hat{\mathbf{r}}\cdot d\mathbf{s}}{r^2} = \frac{\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}ds}{r^2}. \tag{3.59}$$

Since $\hat{\mathbf{r}}\cdot d\mathbf{s}$ is the area of ds projected onto a sphere with radius r, so $d\Omega$ equals *the projected area on a unit sphere* centered at $\mathbf{r}=0$. The solid angle $d\Omega$ can be negative if $\hat{\mathbf{r}}\cdot\hat{\mathbf{n}} < 0$.

In spherical coordinate,

$$d\mathbf{s} = r^2\sin\theta d\theta d\phi\ \hat{\mathbf{n}}, \tag{3.60}$$

hence

$$d\Omega = \hat{\mathbf{r}}\cdot\hat{\mathbf{n}}\sin\theta d\theta d\phi. \tag{3.61}$$

For a sphere with radius r, the solid angle extended by an area $ds\hat{\mathbf{n}}$ ($\hat{\mathbf{n}}=\hat{\mathbf{r}}$) on its surface is,

$$\begin{aligned} d\Omega &= \frac{ds}{r^2} = \sin\theta d\theta d\phi, & (3.62)\\ \text{or}\quad ds &= r^2 d\Omega. & (3.63)\end{aligned}$$

The total solid angle of a sphere is

$$\Omega = \int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi = 4\pi. \tag{3.64}$$

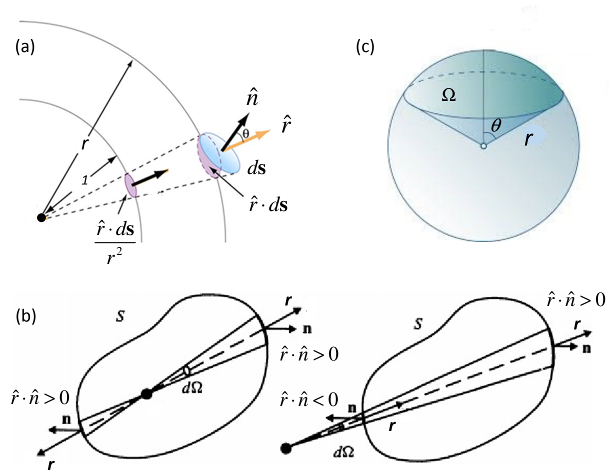

FIG. 25 (a) The areas $d\mathbf{s}$, $\hat{\mathbf{r}}\cdot d\mathbf{s}$, and $\hat{\mathbf{r}}\cdot d\mathbf{s}/r^2$. (b) The origin is inside (left) or outside (right) S. When it's outside, two projected areas with equal magnitude but opposite signs cancel with each other. (c) A spherical cap.

If S is a *closed* surface surrounding the origin, then its projected image covers the unit sphere centered at the origin once. If the origin is outside S, then part of S has "positive" image on the unit sphere, the other part has negative image, and the two parts cancel with each other. Thus, the total solid angle Ω is zero (Fig. 9(b)). That is,

$$\Omega = \begin{cases} 4\pi & \text{if the origin is } inside\ S,\\ 0 & \text{if the origin is } outside\ S.\end{cases} \tag{3.65}$$

In general, for a surface S described by coordinate $\mathbf{r}$, its solid angle with respect to a point $\mathbf{r}_s$ is,

$$\Omega = \int d\Omega = \int_S \frac{\mathbf{r}-\mathbf{r}_s}{|\mathbf{r}-\mathbf{r}_s|^3}\cdot d\mathbf{s}. \tag{3.66}$$

We have just replaced the $\mathbf{r}$ in Eq. (3.59) by $\mathbf{R} = \mathbf{r}-\mathbf{r}_s$.

<u>Example:</u>

Find out the solid angle of a spherical cap with respect to the origin, as shown in Fig. 9(c).

Sol'n:

$$\begin{aligned} \Omega &= \int\frac{\hat{\mathbf{r}}\cdot d\mathbf{s}}{r^2} & (3.67)\\ &= \int_0^\theta \sin\theta d\theta\int_0^{2\pi} d\phi & (3.68)\\ \text{or}\quad &= \int_{\cos\theta}^1 d\cos\theta\int_0^{2\pi} d\phi & (3.69)\\ &= 2\pi(1-\cos\theta). & (3.70)\end{aligned}$$

When the cap covers the whole sphere ($\theta=\pi$), $\Omega = 4\pi$, as it should be.

<u>Application</u>

There is a point charge $q(>0)$ at the origin in a uniform

electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$ ($E_0 > 0$). Find out the electric field lines of this system.
Sol'n:
It's easy to get the electric field of this system,

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r^2} + E_0\hat{\mathbf{z}}. \tag{3.71}$$

However, we are interested in field lines, not $\mathbf{E}$, which is the tangent of a field line.

To obtain the mathematical expression of field lines, let's adopt the following method. In Fig. 10(a) we see that the electric flux is conserved along the flow of field lines,

$$\Phi_E(S) = \Phi_E(S'), \tag{3.72}$$

where S and S' are flat disks perpendicular to the z-axis. If we can write Φ_E as a function of r, θ (no ϕ because of the rotation symmetry around the z-axis), then flux conservation should give us an equation of field lines.

Note that the flux through S is the same as the flux through the cap S_c in Fig. 10(a), thus with spherical coordinate,

$$\begin{aligned}
\Phi_E(S) &= \int_{S_c} \mathbf{E}\cdot d\mathbf{s}, \quad d\mathbf{s} = r^2\hat{\mathbf{r}}d\Omega &(3.73)\\
&= \frac{q}{4\pi\varepsilon_0}\int_{S_c}\frac{\hat{\mathbf{r}}}{r^2}\cdot d\mathbf{s} + \int_{S_c}\mathbf{E}_0\cdot d\mathbf{s} &(3.74)\\
&= \frac{q}{4\pi\varepsilon_0}\Omega(S_c) + E_0\int_{\cos\theta}^{1}\cos\theta r^2 d\cos\theta d\phi &\\
&= \frac{q}{4\pi\varepsilon_0}2\pi(1-\cos\theta) + E_0\pi r^2\sin^2\theta &(3.75)\\
&= \text{constant } \alpha, &(3.76)
\end{aligned}$$

in which $\Omega(S_c)$ is the solid angle of S_c. Note that the choice of S_c (instead of S) makes the second term harder to calculate. However, if we choose S, then the first term would be even harder to calculate.

Finally, one can write r in terms of θ, and different constants give different field lines (Fig. 10(b)),

$$r^2 = \frac{\alpha - \frac{q}{\varepsilon_0}\sin^2\frac{\theta}{2}}{E_0\pi\sin^2\theta}, \quad \theta \neq 0. \tag{3.77}$$

G. Electric potential energy

Electric potential energy is the potential energy of charges in an *external* electric potential. Suppose there are two sets of charge distribution $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$. They can be spatially separated or mixed (but remain different sets). The first set produces electric potential,

$$\phi_1(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho_1(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{3.78}$$

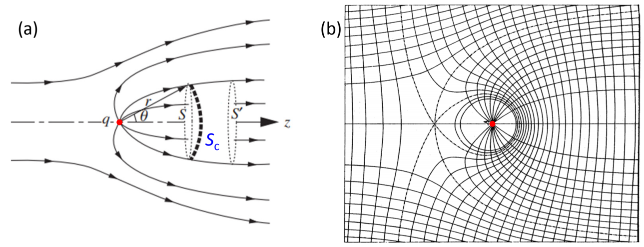

FIG. 26 (a) A point charge in a uniform electric field (Fig. from Zangwill). (b) The field lines and equipotential lines of the system in (a). Fig. from Maxwell, 1891.

similarly for the second set. Then $\rho_2(\mathbf{r})$ in $\phi_1(\mathbf{r})$ has the electric potential energy,

$$\begin{aligned}
V_E &= \int dv\rho_2(\mathbf{r})\phi_1(\mathbf{r}) &(3.79)\\
&= \frac{1}{4\pi\varepsilon_0}\int dvdv'\frac{\rho_2(\mathbf{r})\rho_1(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} &(3.80)\\
&= \int dv'\rho_1(\mathbf{r}')\phi_2(\mathbf{r}'). &(3.81)
\end{aligned}$$

That is, the potential energy of $\rho_2(\mathbf{r})$ in $\phi_1(\mathbf{r})$ is the same as that of $\rho_1(\mathbf{r})$ in $\phi_2(\mathbf{r})$. This is called **Green's reciprocity relation**.
Application:
In a finite region without any charge, the average of potential $\phi(\mathbf{r})$ over a spherical surface S is equal to its value at the center of the sphere (Fig. 3-11). That is, if the radius of the fictitious sphere S is R (which does not need to be small), then

$$\langle\phi_1(\mathbf{r})\rangle_S \equiv \frac{1}{4\pi R^2}\int_S ds\phi(\mathbf{r}) = \phi(0). \tag{3.82}$$

This is called the **mean value theorem** of electrostatic potential.
Pf: The are more than one way to prove this theorem. Here we use a trick using Green's reciprocity relation. Suppose that the charge density that produces the potential is $\rho(\mathbf{r})$, which is *outside* S. Let

$$\rho_1(\mathbf{r}) = \rho(\mathbf{r}), \phi_1(\mathbf{r}) = \phi(\mathbf{r}). \tag{3.83}$$

In order to select the potential on the surface of S, choose

$$\rho_2(\mathbf{r}) = \delta(r-R). \tag{3.84}$$

It has a total charge

$$Q_2 = \int dv\delta(r-R) = 4\pi R^2. \tag{3.85}$$

The charge $\rho_2(\mathbf{r})$ produces a potential $\phi_2(\mathbf{r})$,

$$\begin{aligned}
\phi_2(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0}\int dv'\frac{\rho_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} &(3.86)\\
&= \begin{cases}\frac{1}{4\pi\varepsilon_0}\frac{Q_2}{r}, & r \geq R,\\ \frac{1}{4\pi\varepsilon_0}\frac{Q_2}{R}, & r \leq R.\end{cases} &(3.87)
\end{aligned}$$

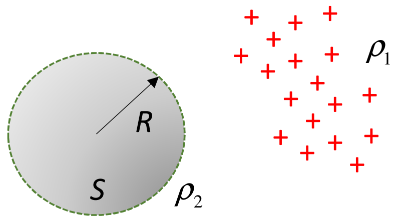

FIG. 27 A fictitious sphere outside the distribution ρ_1 of charges. The charges ρ_2 on the surface of the sphere are employed to prove the mean value theorem.

Be aware that $\rho_2(\mathbf{r})$ is simply a mathematical apparatus and does not really physically coexist with $\rho_1(\mathbf{r})$.

According to Green's reciprocity relation, one has

$$\int dv\delta(r-R)\phi_1(\mathbf{r}) = \int dv\rho_1(\mathbf{r})\phi_2(\mathbf{r}), \tag{3.88}$$

The integration is over the whole space. First, the LHS gives

$$\begin{aligned} LHS &= \int r^2 dr d\Omega \delta(r-R)\phi_1(\mathbf{r}) && (3.89)\\ &= \int_S R^2 d\Omega \phi_1(R),\ R^2 d\Omega = ds && (3.90)\\ &= \int_S ds\phi_1(\mathbf{r}). && (3.91)\end{aligned}$$

Since the charge density ρ is outside S, the integrand of the RHS is nonzero only when $r > R$,

$$\begin{aligned} RHS &= \int_{r>R} dv\rho_1 \frac{1}{4\pi\varepsilon_0}(\mathbf{r})\frac{Q_2}{r} && (3.92)\\ &= \frac{Q_2}{4\pi\varepsilon_0}\int_{r>R} dv\frac{\rho_1(\mathbf{r})}{r} && (3.93)\\ &= \underbrace{4\pi R^2}_{Q_2}\phi_1(0). && (3.94)\end{aligned}$$

Equate the LHS with the RHS, we have

$$\langle\phi_1(\mathbf{r})\rangle_S = \phi_1(0), \quad \text{QED.} \tag{3.95}$$

Note that the mean value theorem implies Earnshaw's theorem: If there is a local min or max in a charge-free region, then the mean value theorem would no longer be true. Thus, in order for the later to be true, there cannot be local min/max in a charge-free region.

H. Electrostatic energy

The electrostatic energy of a charge distribution equals the total work required to assemble these charges, starting from an initial state with energy zero, when all of the charges are dispersed far away from each other. First, consider two point charges q_1, q_2 at $\mathbf{r}_1$, $\mathbf{r}_2$. The electrostatic energy is (ignoring the self-energy of point charges),

$$U_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \tag{3.96}$$

which is the same as the potential energy of q_2 in the field produced by q_1, or vice versa.

If there are N charges $q_1, \cdots, q_N$ at $\mathbf{r}_1, \cdots, \mathbf{r}_N$, then the electrostatic energy is (again ignoring the self-energy),

$$\begin{aligned} U_E = \sum_{i<j} U_{ij} &= \frac{1}{2}\sum_{i,j=1}^{N} U_{ij} && (3.97)\\ &= \frac{1}{8\pi\varepsilon_0}\sum_{i,j=1}^{N}\frac{q_i q_j}{|\mathbf{r}_i - \mathbf{r}_j|} && (3.98)\\ &= \frac{1}{2}\sum_{i=1}^{N} q_i\phi(\mathbf{r}_i), && (3.99)\end{aligned}$$

where

$$\phi(\mathbf{r}_i) = \frac{1}{4\pi\varepsilon_0}\sum_{j=1}^{N}\frac{q_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{3.100}$$

It is half (to avoid double counting) of the sum of the potential energy from each charge.

A continuous charge distribution can be divided into volume elements with charges $q_i = \rho(\mathbf{r}_i)dv_i$. Thus, just replace the q_i in Eq. (3.98) with $\rho(\mathbf{r}_i)dv_i$, and replace the summation with integral to get

$$\begin{aligned} U_E &= \frac{1}{8\pi\varepsilon_0}\int dvdv'\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} && (3.101)\\ &= \frac{1}{2}\int dv\rho(\mathbf{r})\phi(\mathbf{r}). && (3.102)\end{aligned}$$

We can rewrite this expression as,

$$U_E = \frac{\varepsilon_0}{2}\int dv|\mathbf{E}|^2. \tag{3.103}$$

Pf: The charge density can be related to field using Gauss's law,

$$\rho(\mathbf{r}) = \varepsilon_0\nabla\cdot\mathbf{E}. \tag{3.104}$$

With integration by parts, Eq. (3.102) becomes

$$\begin{aligned} U_E &= \frac{\varepsilon_0}{2}\int dv\nabla\cdot\mathbf{E}\ \phi(\mathbf{r}) && (3.105)\\ &= -\frac{\varepsilon_0}{2}\int dv\mathbf{E}\cdot\nabla\phi + \text{surface term} && (3.106)\\ &= \frac{\varepsilon_0}{2}\int dv|\mathbf{E}|^2. && (3.107)\end{aligned}$$

The surface term can be dropped since the surface (of the whole space) is at infinity. The integrand above is the **energy density** of electric field,

$$u_E = \frac{\varepsilon_0}{2}|\mathbf{E}|^2 \tag{3.108}$$

Note that the electrostatic energy in Eq. (3.101) is always positive but the one in Eq. (3.98) can be positive or negative. This is because the self-energy of point charge, which is positive and infinite, is not included in Eq. (3.98).

To illustrate this, consider two different charge distributions ρ_1 and ρ_2. The electrostatic energy of the whole system with $\rho(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho(\mathbf{r})$ is, according to Eq. (3.101),

$$U_E = U_1 + U_2 + \frac{1}{4\pi\varepsilon_0}\int dv dv' \frac{\rho_1(\mathbf{r})\rho_2(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}, \tag{3.109}$$

where $U_{1,2}$ are the "self-energies" of $\rho_{1,2}$. U_E is always positive, but the "interaction energy" (the last term above) can be either positive or negative.

Example:

Calculate the electrostatic energy of a uniformly charged ball with radius a and charge Q_0.

Sol'n:

Instead of using Eq. (3.107), let's calculate U_E with the work required to build this charged ball. Write the charge density of the ball as ρ_0. A ball with radius r has charge $Q(r) = \rho_0(4\pi r^3/3)$, $Q(a) = Q_0$. The work required to add an additional layer with thickness dr is $dW = dQ\ \phi_s$, where ϕ_s is the potential at the surface,

$$dQ = \rho_0 4\pi r^2 dr, \tag{3.110}$$

$$\phi_s = \frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r} = \frac{\rho_0}{3\varepsilon_0}r^2. \tag{3.111}$$

Thus,

$$dW = dQ\ \phi_s = \frac{4\pi}{3\varepsilon_0}\rho_0^2 r^4 dr, \tag{3.112}$$

and

$$U_E = \int dW \tag{3.113}$$

$$= \frac{4\pi}{3\varepsilon_0}\rho_0^2\int_0^a r^4 dr,\ \ \rho_0 = \frac{Q_0}{4\pi a^3/3} \tag{3.114}$$

$$= \frac{3}{5}\frac{Q_0^2}{4\pi\varepsilon_0 a}. \tag{3.115}$$

You may also calculate U_E using Eq. (3.107). This is left as an exercise.

Problem:

1. Starting from the electric potential for a finite, charged wire in Eq. (3.22), verify that (a) at large distance it reduces to Eq. (3.24); (b) at short distance, it reduces to (3.25).

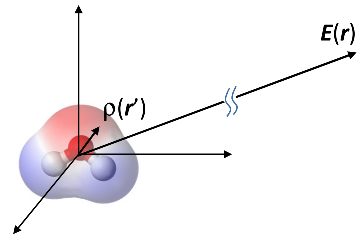

FIG. 28 Observe a localized charge distribution at a distance far away.

2. Suppose a metallic, spherical shell with radius 1 m has total charge $Q = 10^{-3}$ C.
(a) Find out its surface charge density σ.
(b) Find out magnitude and direction of the pressure $\mathbf{f}$ (due to the electric field) on the wall of the spherical shell.

3. Two concentric, spherical metal *shells* have radii a and b ($b > a$). The inner shell and the outer shell have charges Q and $-Q$ respectively. Two shells are separated by vacuum.
(a) What is the electric field inside and outside the two shells?
(b) What is the total electrostatic energy of this system?

IV. ELECTRIC MULTIPOLES

A. Multipole expansion

Electric multipoles are useful if 1). the charge distribution $\rho(\mathbf{r})$ is localized within a finite region, and 2). the location of observation is far away (Fig. 1).

In general, the electric potential is given as,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{4.1}$$

If the condition above is satisfied, $r \gg r'$, then we can use the binomial expansion to have

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \simeq \frac{1}{r} + \frac{\hat{\mathbf{r}}}{r^2}\cdot\mathbf{r}' + \frac{1}{2r^3}\left[3(\hat{\mathbf{r}}\cdot\mathbf{r}')^2 - |\mathbf{r}'|^2\right]. \tag{4.2}$$

It follows that,

$$\begin{aligned}\phi(\mathbf{r}) \simeq\ & \frac{1}{4\pi\varepsilon_0}\left[\int dv' \rho(\mathbf{r}')\right]\frac{1}{r} \\ &+ \frac{1}{4\pi\varepsilon_0}\left[\int dv' \rho(\mathbf{r}')\mathbf{r}'\right]\cdot\frac{\mathbf{r}}{r^3} \\ &+ \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{2}\int dv' \rho(\mathbf{r}')\left(3r_i'r_j' - r'^2\delta_{ij}\right)\right]\frac{r_i r_j}{r^5}.\end{aligned} \tag{4.3}$$

The Einstein summation convention has been used. Inside the square brackets are electric monopole moment (electric charge), **electric dipole moment**, and **elec-**

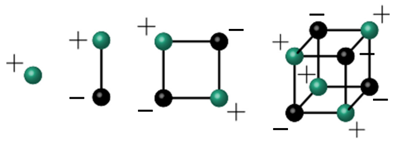

FIG. 29 From left to right, sets of point charges with electric monopole, dipole, quadrupole, and octupole.

tric quadrupole moment,

$$Q = \int dv' \rho(\mathbf{r}'), \tag{4.4}$$

$$\mathbf{p} = \int dv' \rho(\mathbf{r}')\mathbf{r}', \tag{4.5}$$

$$\Theta_{ij} = \frac{1}{2}\int dv' \rho(\mathbf{r}')\left(3r_i' r_j' - r'^2 \delta_{ij}\right). \tag{4.6}$$

Hence,

$$\phi(\mathbf{r}) \simeq \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \Theta_{ij}\frac{r_i r_j}{r^5}\right). \tag{4.7}$$

Note that Q, p_i and Θ_{ij} are simply sets of numbers, not functions of $\mathbf{r}$. Once these numbers are known for the charge distribution of interest, then its potential everywhere can be easily obtained from Eq. (4.7). The potentials of monopole, dipole, and quadrupole decrease with distance as $1/r$, $1/r^2$, and $1/r^3$. At large distance, higher multipoles can be neglected.

The quadrupole moment Θ_{ij} is a 3×3 matrix. It is not difficult to see from Eq. (4.6) that

$$\Theta_{ji} = \Theta_{ij} \tag{4.8}$$

$$\mathrm{tr}\ \Theta_{ij} \equiv \Theta_{ii} = 0. \tag{4.9}$$

That is, it is a *traceless*, *symmetric* matrix. Hence it has only 5 independent matrix elements. Thus, the multipole moments Q, p_i and Θ_{ij} have 1, 3, and 5 independent components respectively.

For a set of point charges $\{q_\alpha, \alpha = 1, \cdots, , N\}$, their charge density is (see Chap 2),

$$\rho(\mathbf{r}) = \sum_{\alpha=1}^{N} q_\alpha \delta(\mathbf{r} - \mathbf{r}_\alpha). \tag{4.10}$$

Substitute this to Eqs. (4.4), (4.5), and (4.6), we will get

$$Q = \sum_\alpha q_\alpha, \tag{4.11}$$

$$p_i = \sum_\alpha q_\alpha r_\alpha, \tag{4.12}$$

$$\Theta_{ij} = \frac{1}{2}\sum_\alpha q_\alpha \left(3r_{\alpha i} r_{\alpha j} - r_\alpha^2 \delta_{ij}\right) \tag{4.13}$$

See Fig. 2 for examples of multipoles with point charges.

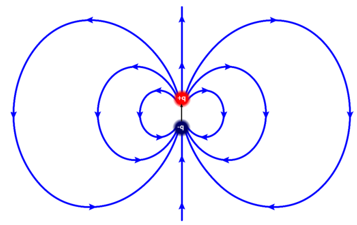

FIG. 30 The electric field of an electric dipole.

B. Electric dipole

From the dipole potential,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \tag{4.14}$$

we can derive its electric field,

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi = \frac{1}{4\pi\varepsilon_0}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}) - \mathbf{p}}{r^3}. \tag{4.15}$$

The field weakens as $1/r^3$ and has the distribution shown in Fig. 3.

Example:
Suppose there are charges $\rho(\mathbf{r})$ inside a ball V with volume V, show that the average of the electric field over the ball,

$$\langle \mathbf{E}(\mathbf{r})\rangle_V \equiv \frac{1}{V}\int_V dv \mathbf{E}(\mathbf{r}) = -\frac{1}{3\varepsilon_0}\frac{\mathbf{p}}{V}, \tag{4.16}$$

where $\mathbf{p}$ is the electric dipole moment due to the charges (see Fig. 4(a)). On the other hand, if the charges $\rho(\mathbf{r})$ are outside V, then the averaged field is equal to the field at the center of the sphere (Fig. 4(b)),

$$\langle \mathbf{E}(\mathbf{r})\rangle_V = \mathbf{E}(0). \tag{4.17}$$

The latter is analogous to the mean value theorem of electrostatic potential (Chap 3).
Pf: The Coulomb integral for electric field is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv' \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{4.18}$$

Thus,

$$\begin{aligned}\frac{1}{V}\int_V dv \mathbf{E}(\mathbf{r}) &= \frac{1}{V}\int_V dv \frac{1}{4\pi\varepsilon_0}\int_{\rho\neq 0} dv' \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \\ &= -\frac{1}{V}\int_{\rho\neq 0} dv' \rho(\mathbf{r}')\underbrace{\frac{1}{4\pi\varepsilon_0}\int_V dv \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}}_{=\tilde{\mathbf{E}}(\mathbf{r}')},\end{aligned}$$

where $\tilde{\mathbf{E}}(\mathbf{r}')$ is the electric field of a *fictitious* ball V with charge density $\tilde{\rho} = 1$.

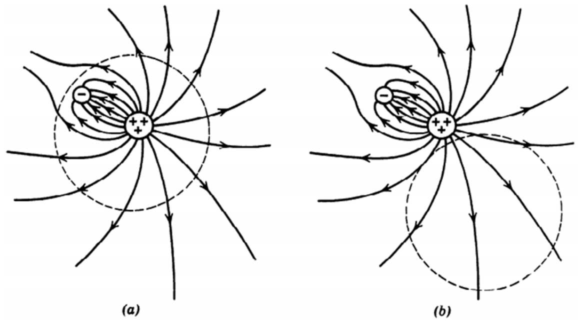

FIG. 31 Charges are inside a sphere (a), and outside a sphere (b). (Fig. from Jackson)

Note that $\mathbf{r}'$ is inside V if all of the charges are inside V. To get the electric field inside, choose a sphere S with radius r' and use

$$\int_S d\mathbf{s}\cdot\tilde{\mathbf{E}} = \frac{\tilde{Q}(r')}{\varepsilon_0}, \tag{4.19}$$

$$\to \tilde{E}4\pi r'^2 = \frac{1}{\varepsilon_0}\frac{4}{3}\pi r'^3, \tag{4.20}$$

$$\to \tilde{\mathbf{E}}(\mathbf{r}') = \frac{1}{3\varepsilon_0}\mathbf{r}'. \tag{4.21}$$

Thus,

$$\langle\mathbf{E}\rangle_V = -\frac{1}{V}\int dv'\rho(\mathbf{r}')\frac{1}{3\varepsilon_0}\mathbf{r}' \tag{4.22}$$

$$= -\frac{1}{3\varepsilon_0}\frac{\mathbf{p}}{V}. \tag{4.23}$$

Why there is a minus sign in front of $\mathbf{p}$? From Fig. 4(a), you can see that $\mathbf{p}$ points to the center, but most of the field lines inside the sphere point *away* from the center. This is why $\langle\mathbf{E}\rangle_V$ is anti-parallel to $\mathbf{p}$.

If $\rho(\mathbf{r})$ is on the outside of the sphere V, then the first two steps of the proof above are the same, but now $\tilde{\mathbf{E}}(\mathbf{r}')$ is the field *outside* the uniformly charged ball. It follows that

$$\mathbf{E}(\mathbf{r}') = \frac{V}{4\pi\varepsilon_0}\frac{\mathbf{r}'}{r'^3} \quad (\tilde{\rho}=1). \tag{4.24}$$

Thus,

$$\langle\mathbf{E}\rangle_V = -\frac{1}{V}\int dv'\rho(\mathbf{r}')\frac{V}{4\pi\varepsilon_0}\frac{\mathbf{r}'}{r'^3} \tag{4.25}$$

$$= \frac{1}{4\pi\varepsilon_0}\int dv'\rho(\mathbf{r}')\frac{0-\mathbf{r}'}{|0-\mathbf{r}'|^3} \tag{4.26}$$

$$= \mathbf{E}(0). \tag{4.27}$$

QED.

In general, when there are charges both inside and outside of the sphere, then

$$\langle\mathbf{E}(\mathbf{r})\rangle_V = -\frac{1}{3\varepsilon_0}\frac{\mathbf{p}_{in}}{V} + \mathbf{E}_{out}(0), \tag{4.28}$$

where $\mathbf{p}_{in}$ is due to the charges inside, and $\mathbf{E}_{out}(0)$ is due to the charges outside.

1. Point electric dipole

Consider the electric dipole shown in Fig. 3. The two charges $\pm q/s$ are separated by $s\mathbf{b}$ and has an electric dipole moment $\mathbf{p} = q\mathbf{b}$. In the limit $s \to 0$, it becomes a **point electric dipole**, but the dipole moment is not changed. Thus, the dipole field remains the same,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p})-\mathbf{p}}{r^3}. \tag{4.29}$$

As we have explained at the beginning of this chapter, the multipole expansion is valid when $r \gg r'$. For a point dipole, $r' \to 0$, thus the range of validity of the dipole field above extends down to the region close to the point $r \to 0$.

However, if you integrate the field in Eq. (4.29) over a ball V centered at $\mathbf{r} = 0$, then

$$\int_V dv\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V dv\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p})-\mathbf{p}}{r^3} = 0. \tag{4.30}$$

It is zero due to angular integration, no matter if the ball is large or small. This contradicts the result in Eq. (4.16).

Eq. (4.29) is valid almost everywhere, except at $\mathbf{r} = 0$, where the field diverges. In order to resolve the contradiction and ensure that

$$\frac{1}{V}\int_V dv\mathbf{E}(\mathbf{r}) = -\frac{1}{3\varepsilon_0}\frac{\mathbf{p}}{V}, \tag{4.31}$$

we can add a delta function to Eq. (4.29), so that

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p})-\mathbf{p}}{r^3} - \frac{\mathbf{p}}{3\varepsilon_0}\delta(\mathbf{r}). \tag{4.32}$$

The delta function can only be non-zero (in fact, infinite) when $\mathbf{r} = 0$. Now the equation is valid everywhere, *including* the origin.

C. Electric quadrupole

Recall that the quadrupole moment is a traceless, symmetric matrix. Like the **moment of inertia** in classical mechanics, we can always find a coordinate so that the matrix is diagonalized. Under this circumstance, the coordinate axes are called **principle axes**. If the charge distribution has certain symmetry, then the principle axes are along the symmetry axes.

For example, for the ellipsoids in Fig. 5, the principle axes are along the dotted lines. There is no distinction between x-axis and y-axis, so we expect $\Theta_{xx} = \Theta_{yy}$. Furthermore, since the quadrupole moment matrix is traceless,

$$\Theta_{xx} + \Theta_{yy} + \Theta_{zz} = 0, \tag{4.33}$$

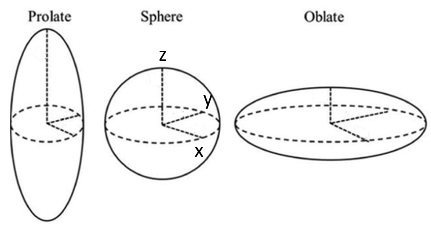

FIG. 32 From left to right, a prolate ellipsoid, a sphere, and an oblate ellipsoid.

so it must be of the form

$$\Theta = \begin{pmatrix} -\Theta_{zz}/2 & 0 & 0 \\ 0 & -\Theta_{zz}/2 & 0 \\ 0 & 0 & \Theta_{zz} \end{pmatrix}, \tag{4.34}$$

where

$$\begin{aligned} \Theta_{zz} &= \int dv \rho(\mathbf{r})(3z^2 - r^2) && (4.35) \\ &= \int dv \rho(\mathbf{r})(2z^2 - x^2 - y^2). && (4.36) \end{aligned}$$

If the prolate ellipsoid is uniformly charged, then it's not difficult to see that $\Theta_{zz} > 0$. On the other hand, the oblate ellipsoid has $\Theta_{zz} < 0$. A uniformly charged ball has no quadrupole moment, $\Theta_{zz} = 0$.

D. Potential energy and force

The potential energy of a charge distribution $\rho(\mathbf{r})$ in an external potential $\phi(\mathbf{r})$ is,

$$V_E = \int dv \rho(\mathbf{r}) \phi(\mathbf{r}). \tag{4.37}$$

Assume that the potential varies smoothly compared to the charge distribution, then we can expand it with respect to a point 0 near the charges,

$$\begin{aligned} \phi(\mathbf{r}) &\simeq \phi(0) + \mathbf{r} \cdot \nabla \phi(0) + \frac{1}{2} (\mathbf{r} \cdot \nabla)^2 \phi(0) && (4.38) \\ &= \phi(0) - \mathbf{r} \cdot \mathbf{E}(0) - \frac{1}{2} r_i r_j \frac{\partial E_j}{\partial r_i}(0). && (4.39) \end{aligned}$$

Since $\nabla \cdot \mathbf{E} = 0$ for the external field near the charges, we can add $\frac{1}{6} r^2 \nabla \cdot \mathbf{E}(0)$ to the last term and get

$$\phi(\mathbf{r}) = \phi(0) - \mathbf{r} \cdot \mathbf{E}(0) - \frac{1}{6} \left(3 r_i r_j - r^2 \delta_{ij} \right) \frac{\partial E_j}{\partial r_i}(0). \tag{4.40}$$

Thus, with Eqs. (4.4), (4.5), and (4.6), we have

$$V_E = q\phi(0) - \mathbf{p} \cdot \mathbf{E}(0) - \frac{1}{3} \Theta_{ij} \frac{\partial E_j}{\partial r_i}(0). \tag{4.41}$$

It is composed of monopole energy, dipole energy, and quadrupole energy (higher order terms are neglected). Note that the monopole energy depends on the potential, the dipole energy depends on the field, while the quadrupole energy depends on the field gradient. Hence, if the field is *uniform*, then there is *no* quadrupole energy.

In particular, a dipole $\mathbf{p}_1$ in the field of another dipole $\mathbf{p}_2$ (see Eq. (4.15)) has the dipole-dipole interaction energy,

$$\begin{aligned} V_{12} &= -\mathbf{p}_1 \cdot \mathbf{E}_2(\mathbf{r}_1) && (4.42) \\ &= \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p}_1 \cdot \mathbf{p}_2 - 3(\hat{\mathbf{R}} \cdot \mathbf{p}_1)(\hat{\mathbf{R}} \cdot \mathbf{p}_2)}{R^3}, && (4.43) \end{aligned}$$

which can either be repulsive or attractive, and decreases as $1/R^3$, $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$.

The forces on the multipoles are given by the gradient of potential energy. Thus,

$$\begin{aligned} \mathbf{F} &= -\nabla V_E && (4.44) \\ &= q\mathbf{E}(0) - \nabla(\mathbf{p} \cdot \mathbf{E}) - \frac{1}{3} \Theta_{ij} \frac{\partial^2 \mathbf{E}}{\partial r_i \partial r_j}. && (4.45) \end{aligned}$$

That is, you need a field gradient to have a dipole force, and a non-zero second order derivative of field to have a quadrupole force.

E. Macroscopic polarizable medium

Consider a polarizable medium that is composed of polarizable atoms or molecules. If the dipole moment of the i-th atom is $\mathbf{p}_i$, then we can define the **electric polarization** as,

$$\mathbf{P}(\mathbf{r}') = \frac{\sum_{i \text{ in } \Delta V} \mathbf{p}_i}{\Delta V}, \tag{4.46}$$

where ΔV is a volume element around $\mathbf{r}'$ (Fig. 6(a)). The volume element is microscopically large but macroscopically small (e.g., 1 μm in size), so that there are many atoms in ΔV, but $\mathbf{r}'$ remains a point from human's point of view.

Since the volume element ΔV has charge $q = \rho \Delta V$ and dipole $\mathbf{p} = \mathbf{P} \Delta V$, it produces a potential at $\mathbf{r}$ far away,

$$\begin{aligned} \Delta\phi(\mathbf{r}) &\simeq \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{R} + \frac{\mathbf{p} \cdot \mathbf{R}}{R^3} \right), \quad \mathbf{R} = \mathbf{r} - \mathbf{r}' && (4.47) \\ &= \frac{1}{4\pi\varepsilon_0} \left[\frac{\rho(\mathbf{r}')\Delta V}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathbf{P}(\mathbf{r}')\Delta V \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right]. \end{aligned}$$

After integration, we have the total potential,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left[\int dv' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \int dv' \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \right]. \tag{4.48}$$

Since (see Chap 1)

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|}, \tag{4.49}$$

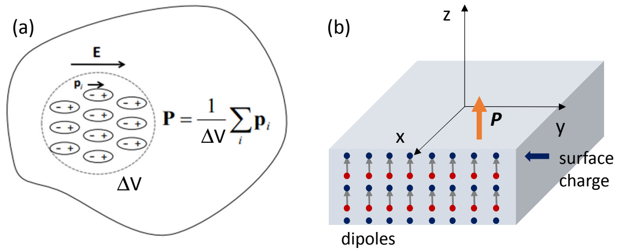

FIG. 33 (a) A volume element with many dipoles in a polarizable medium. (b) A semi-infinite dielectric below the $x-y$ plane.

the second term of Eq. (4.48), after integration by parts, can be written as

$$\int dv' \mathbf{P}(\mathbf{r}')\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\int dv' \nabla' \cdot \mathbf{P}(\mathbf{r}') \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{4.50}$$

It follows that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv' \frac{\rho(\mathbf{r}') - \nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{4.51}$$

The numerator can be considered as an effective charge density $\rho_{eff} = \rho + \rho_P$, where

$$\rho_P(\mathbf{r}) \equiv -\nabla\cdot\mathbf{P}(\mathbf{r}) \tag{4.52}$$

is the **polarization charge density**. Since the charge density in the integral above directly links with the one in Gauss's law (see Chap 2), so we have

$$\nabla\cdot\mathbf{E} = \frac{\rho_{eff}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(\rho - \nabla\cdot\mathbf{P}\right). \tag{4.53}$$

Define the **electric displacement field**,

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \tag{4.54}$$

$$\text{then}\quad \nabla\cdot\mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}). \tag{4.55}$$

This is Gauss's law in material (rather then in vacuum).

A side note: Maxwell coined the term *displacement*, which might be based on his (now out-of-date) mechanical model of *ether*. This field can be dispensed with, since we can just use **E** and **P** instead. According to Purcell, this quantity is sometimes treated "with more respect than it deserves" (Purcell, 2004).

If the polarization is proportional to the electric field,

$$\mathbf{P}(\propto \mathbf{E}) = \varepsilon_0\chi_e\mathbf{E}, \tag{4.56}$$

then

$$\mathbf{D} = \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon\mathbf{E}, \tag{4.57}$$

where χ_e is **electric susceptibility**, and $\varepsilon \equiv \varepsilon_0(1+\chi_e)$ is **electric permittivity**.

1. Polarization charge

Non-uniform polarization generates effective charge, $\rho_P = -\nabla\cdot\mathbf{P}$. We'll use a simple example to illustrate this: In Fig. 6(b) there is a semi-infinite dielectric with uniform polarization,

$$\mathbf{P} = P_0\theta(-z)\hat{z}, \tag{4.58}$$

in which θ is the step function. Its polarization charge density is,

$$\rho_P = -\nabla\cdot\mathbf{P} = P_0\delta(z)\hat{z}. \tag{4.59}$$

We can see from the figure that the bulk is charge-neutral, and only the outer-most electrons can be exposed. So its reasonable for the polarization charges to reside on the surface of the dielectric.

Note that the polarization charges are bounded to molecules. They cannot move away like free electrons in metals.

F. Electrostatic energy

The electrostatic energy of a charge distribution equals the total work required to assemble these charges, starting from the initial state when every bit of charges are far away from each other. Suppose we are in the middle of the process of building up the charges, when a charge distribution $\rho(\mathbf{r})$ that produces a potential $\phi(\mathbf{r})$ has been assembled. Then the work it takes to add $\delta\rho(\mathbf{r})$ to this system is

$$\delta W = \int dv\delta\rho(\mathbf{r})\phi(\mathbf{r}). \tag{4.60}$$

The extra charges result in a change of $\delta\mathbf{D}(\mathbf{r})$, and

$$\nabla\cdot\delta\mathbf{D}(\mathbf{r}) = \delta\rho(\mathbf{r}). \tag{4.61}$$

Thus,

$$\begin{aligned} \delta W &= \int dv\nabla\cdot[\delta\mathbf{D}(\mathbf{r})\phi(\mathbf{r})] && (4.62)\\ &= \int dv\nabla\cdot(\phi\delta\mathbf{D}) - \int dv\nabla\phi\cdot\delta\mathbf{D} && (4.63)\\ &= \int dv\mathbf{E}\cdot\delta\mathbf{D}. && (4.64)\end{aligned}$$

The first integral in the second line can be turned into a surface integral at infinity. For localized charges this surface integral vanishes.

Therefore, to build up the system from $\mathbf{D}=0$ to its final state $\mathbf{D}$, we need to do the work

$$W = \int dv \int_0^{\mathbf{D}} \mathbf{E}\cdot\delta\mathbf{D}, \tag{4.65}$$

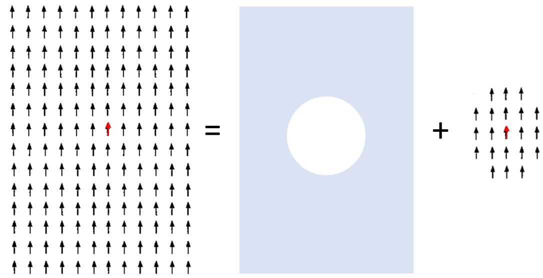

FIG. 34 The local field $\mathbf{E}_{loc}$ at the red arrow is approximately the sum of the field from the continuous medium with a hole, plus the field from dipoles inside the hole.

which is also the electrostatic energy U_E of this system. If the medium is linear, then

$$\mathbf{E}\cdot\delta\mathbf{D} = \frac{1}{2}\delta(\mathbf{E}\cdot\mathbf{D}). \tag{4.66}$$

Hence

$$U_E = \frac{1}{2}\int dv\mathbf{E}\cdot\mathbf{D}. \tag{4.67}$$

The integrand is the energy density

$$u_E(\mathbf{r}) = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}. \tag{4.68}$$

For charges in vacuum, $\mathbf{D} = \varepsilon_0\mathbf{E}$, and we are back to the result in Chap 3, $u_E = \frac{\varepsilon_0}{2}|\mathbf{E}|^2$.

G. Local field and electric permittivity

Apply an electric field $\mathbf{E}_{ex}$ to a polarizable medium, then the medium is polarized with $\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}_{loc}$. For a rarefied medium, $\mathbf{E}_{loc}$ is just the applied field $\mathbf{E}_{ex}$. However, for a dense medium, it is the applied field *plus* the induced field $\mathbf{E}_p$ due to polarization,

$$\mathbf{E}_{loc} = \mathbf{E}_{ex} + \mathbf{E}_p. \tag{4.69}$$

An atom or a molecule inside the material is polarized by the **local field** $\mathbf{E}_{loc}$. Instead of adding up the dipolar fields from other molecules, we use the following trick to calculate $\mathbf{E}_{loc}$: Divide the medium into two regions, a spherical region with radius R and a region *without* the sphere. The molecule (or atom) of interest is inside the sphere that is macroscopically small but microscopically large.

For the charge outside the sphere, we can adopt the coarse-grained, macroscopic electric field $\mathbf{E}$. Inside the sphere near the molecule, the material is not treated as a continuous spherical medium that produces $\mathbf{E}_{sph}$, but as a collection of dipoles that produces $\mathbf{E}_{near}$. Thus,

$$\mathbf{E}_{loc} = \mathbf{E} - \mathbf{E}_{sph} + \mathbf{E}_{near}. \tag{4.70}$$

That is, we remove the field $\mathbf{E}_{sph}$ from $\mathbf{E}$ and fill in the field $\mathbf{E}_{near}$ (see Fig. 7).

We can estimate $\mathbf{E}_{sph}$ with Eq. (4.16),

$$\mathbf{E}_{sph} \simeq \langle\mathbf{E}\rangle_V = -\frac{1}{3\varepsilon_0}\frac{\mathbf{p}_V}{V} = -\frac{1}{3\varepsilon_0}\mathbf{P}, \tag{4.71}$$

where $\mathbf{p}_V$ is the total dipole moments inside V, $\mathbf{p}_V = \mathbf{P}V$. The field $\mathbf{E}_{near}$ depends on crystal symmetry. For a regular lattice, or a random distribution of dipoles, $\mathbf{E}_{near} \simeq 0$ due to the cancellation from dipoles at symmetric positions. Thus,

$$\mathbf{E}_{loc} = \mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}. \tag{4.72}$$

This is the **Lorentz relation**.

The molecule is polarized by the local field,

$$\mathbf{p} = \varepsilon_0\gamma_m\mathbf{E}_{loc}, \tag{4.73}$$

where γ_m is the **molecular polarizability**. If the density of the number of dipoles is n, then

$$\mathbf{P} = n\mathbf{p} = n\varepsilon_0\gamma_m\left(\mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}\right), \tag{4.74}$$

which gives

$$\mathbf{P} = \varepsilon_0\frac{n\gamma_m}{1-\frac{1}{3}n\gamma_m}\mathbf{E} = \varepsilon_0\chi_e\mathbf{E}. \tag{4.75}$$

The **relativity permittivity** $\varepsilon_r \equiv \varepsilon/\varepsilon_0 = 1+\chi_e$. Thus,

$$\varepsilon_r = 1 + \frac{n\gamma_m}{1-\frac{1}{3}n\gamma_m}, \tag{4.76}$$

or

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{\gamma_m}{3}n. \tag{4.77}$$

This is the **Clausius-Mossotti relation**, which links the macroscopic quantity ε_r with the microscopic quantity γ_m. We also see that, for a given material, $(\varepsilon_r - 1)/(\varepsilon_r + 2)$ is proportional to the density of molecules.

For example, the molecular polarizability of methane (CH_4) is $\gamma_m = 4\pi \times 2.6 \times 10^{-30}$ m^3. At freezing point and 1 atm, there are 2.8×10^{25} molecules per cubic meter, hence

$$\varepsilon_r = 1.00091, \tag{4.78}$$

which is close to the dielectric constant measured $\varepsilon_r = 1.00088$ (Purcell, 2004).

Note that according to Eq. (4.72), the local field

$$\mathbf{E}_{loc} = \left(1+\frac{\chi_e}{3}\right)\mathbf{E} \tag{4.79}$$

$$= \frac{1}{1-\frac{1}{3}n\gamma_m}\mathbf{E} = \frac{\varepsilon_r + 2}{3}\mathbf{E}. \tag{4.80}$$

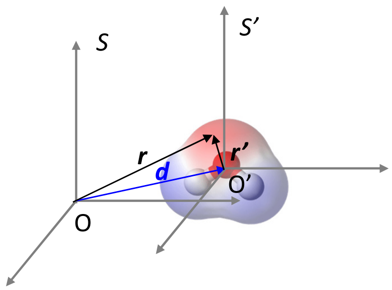

FIG. 35 The same charge distribution viewed from two different coordinates.

It is larger then the macroscopic field if $\varepsilon_r > 1$.

Problem:

1. From the third term of the binomial expansion in Eq. (4.2), we get the quadrupole potential in Eq. (4.3). Show that the quadrupole potential can also be written as

$$\phi_{quad}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} Q_{ij} \frac{3r_i r_j - \delta_{ij} r^2}{r^5}, \quad (4.81)$$

$$\text{where } Q_{ij} \equiv \frac{1}{2} \int dv' \rho(\mathbf{r}') r_i' r_j'. \quad (4.82)$$

2. The magnitude of an electric *charge* is independent of the choice of coordinate (Fig. 8). However, in general p_i and Θ_{ij} do. Show that
(a) if a system is neutral ($Q = 0$), then $\mathbf{p}$ is independent of the choice of coordinate.
(b) If both Q and $\mathbf{p}$ vanish, then Θ_{ij} is independent of the choice of coordinate.

V. MAGNETOSTATICS

A. Introduction

There are several ways to find out a magnetic field. Given a current distribution, we can always use the Biot-Savart law,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V dv' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (5.1)$$

where $\mu_0 = 4\pi \times 10^{-7}$ N/A^2. Alternatively, we can find out the vector potential using

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int dv' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \quad (5.2)$$

then take its curl to find the field, $\mathbf{B} = \nabla \times \mathbf{A}$.

Two of the Maxwell equations govern the magnetostatic field,

$$\int_S d\mathbf{s} \cdot \mathbf{B}(\mathbf{r}) = 0, \quad (5.3)$$

$$\oint_C d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = \mu_0 I, \quad (5.4)$$

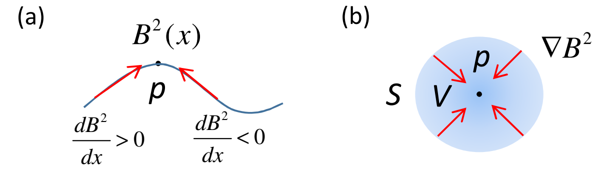

FIG. 36 (a) $B^2(x)$ and its slope in one dimension. (b) $B^2(\mathbf{r})$ and its gradient in three dimension.

where I is the current flowing through loop C. If the current distribution has certain symmetry, then it is convenient to find out $\mathbf{B}$ using the Ampère law in Eq. (5.4).

The differential form of the Maxwell equations are,

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0, \quad (5.5)$$

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}. \quad (5.6)$$

Since a field without divergence can be written as a curl, the first equation implies $\mathbf{B} = \nabla \times \mathbf{A}$. Substitute it to the second equation, and recall that $\nabla \cdot \mathbf{A} = 0$ for steady current (see Chap 2), we have the **vector Poission equation**,

$$\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r}). \quad (5.7)$$

It needs to be solved together with the boundary condition. Again, as in electrostatics, we will not use this approach in this course.

1. Magnetic force

A point charge q moving in a magnetic field $\mathbf{B}$ feels a magnetic force, called the **Lorentz force**,

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \quad (5.8)$$

For a thin wire carrying a current I, each line element $d\mathbf{r}$ feels a Lorentz force,

$$d\mathbf{F} = I d\mathbf{r} \times \mathbf{B}. \quad (5.9)$$

For the whole wire, just integrate to have the total force,

$$\mathbf{F} = I \int_C d\mathbf{r} \times \mathbf{B}(\mathbf{r}). \quad (5.10)$$

For a general current distribution $\mathbf{J}(\mathbf{r})$, just replace $Id\mathbf{r}$ with $\mathbf{J}(\mathbf{r})dv$, so that

$$\mathbf{F} = \int dv \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}). \quad (5.11)$$

Note that the field $\mathbf{B}$ in the equation is an *external* one, not including the field produced by $\mathbf{J}$ itself.

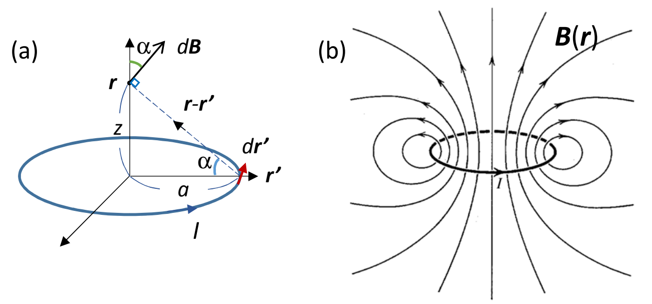

FIG. 37 (a) A ring with radius a and current I. (b) Distribution of magnetic field near a ring.

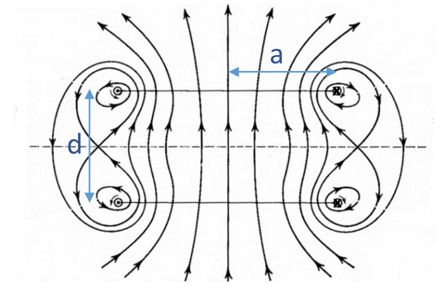

FIG. 38 The magnetic field at the center of a Helmholtz coil is nearly uniform. Fig. from Zangwill, 2013.

2. Thomson's theorem

In a region V without any current, a magnetic field $|\mathbf{B}(\mathbf{r})|$ can have local minimum, but *not* local maximum.
Pf: We'll prove this by contradiction. Suppose $|\mathbf{B}(\mathbf{r})|$, or $B^2(\mathbf{r})$, has local maximum at a point p, then near the point, ∇B^2 points toward p (Fig. 36). Therefore, if we integrate it over a spherical surface S surrounding p, then

$$\int_S d\mathbf{s} \cdot \nabla B^2 < 0. \tag{5.12}$$

Using the divergence theorem,

$$\int_S d\mathbf{s} \cdot \mathbf{V} = \int_V dv \nabla \cdot \mathbf{V}, \tag{5.13}$$

we have

$$\begin{aligned} \int_S d\mathbf{s} \cdot \nabla B^2 &= \int_S dv \nabla^2 B^2 & (5.14) \\ &= \int_V \nabla_i \nabla_i B_j B_j. & (5.15) \end{aligned}$$

The integrand

$$\begin{aligned} \nabla_i (\nabla_i B_j B_j) &= \nabla_i [2B_j(\nabla_i B_j)] & (5.16) \\ &= 2(\nabla_i B_j)^2 + 2B_j(\nabla^2 B_j). & (5.17) \end{aligned}$$

The integral of the second term is zero (proved later), thus

$$\int_S d\mathbf{s} \cdot \nabla^2 B^2 = \int_V dv 2(\nabla_i B_j)^2 > 0. \tag{5.18}$$

This contradicts with Eq. (5.12). Thus the premise that $|\mathbf{B}(\mathbf{r})|$ has local maximum can't be valid. QED.

We now prove that the integral of the second term in Eq. (5.17) is zero. First, since there is no current inside V, $\nabla \times \mathbf{B} = 0$, thus

$$\nabla_i B_j = \nabla_j B_i. \tag{5.19}$$

It follows that

$$\begin{aligned} \int_V dv B_j (\nabla^2 B_j) &= \int_V dv B_j \nabla_i \underbrace{\nabla_i B_j}_{=\nabla_j B_i} & (5.20) \\ &= \int_V dv B_j \nabla_j (\nabla \cdot \mathbf{B}) & (5.21) \\ &= 0. & (5.22) \end{aligned}$$

B. Biot-Savart law

Let's start with a classic example:
Find out the magnetic field along the central axis of a circular wire with radius a and current I.
Sol'n:
According to the Biot-Savart law, a line element $Id\mathbf{r}'$ generates

$$\begin{aligned} d\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} I d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} & (5.23) \\ &= \frac{\mu_0}{4\pi} I dr' \frac{1}{|\mathbf{r}-\mathbf{r}'|^2} \text{ along } d\mathbf{B}. & (5.24) \end{aligned}$$

Note that $d\mathbf{r}' \perp \mathbf{r} - \mathbf{r}'$, and $d\mathbf{B}$ is shown in Fig. 37(a).
When one integrates over the circle, the horizontal component of $d\mathbf{B}$ vanishes, but $dB_z = dB \cos\alpha$ survives. Therefore,

$$\begin{aligned} B_z(z) &= \frac{\mu_0}{4\pi} \oint_C I dr' \frac{\cos\alpha}{z^2+a^2}, \quad \cos\alpha = \frac{a}{\sqrt{z^2+a^2}} \\ &= \frac{\mu_0 I}{2} \frac{a^2}{(z^2+a^2)^{3/2}}, & (5.25) \end{aligned}$$

and $\mathbf{B}(z) = B_z(z)\hat{\mathbf{z}}$. The field decreases as $1/z^3$ at large distance. The distribution of field lines is shown in Fig. 37(b).
A **Helmholtz coil** consists of two rings with the same radius, and the same magnitude and direction of current (Fig. 38). Along the central axis,

$$B_z(z) = \frac{\mu_0}{2} \left\{ \frac{Ia^2}{[(z-d/2)^2+a^2]^{3/2}} + \frac{Ia^2}{[(z+d/2)^2+a^2]^{3/2}} \right\}. \tag{5.26}$$

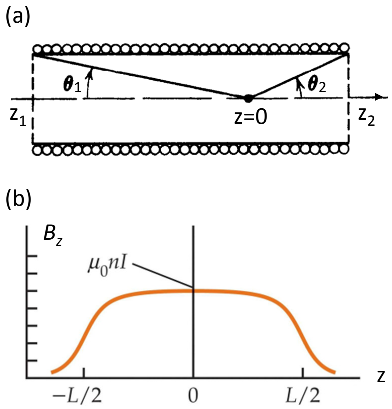

FIG. 39 (a) A solenoid with finite length. (b) The magnetic field along the central axis of the solenoid.

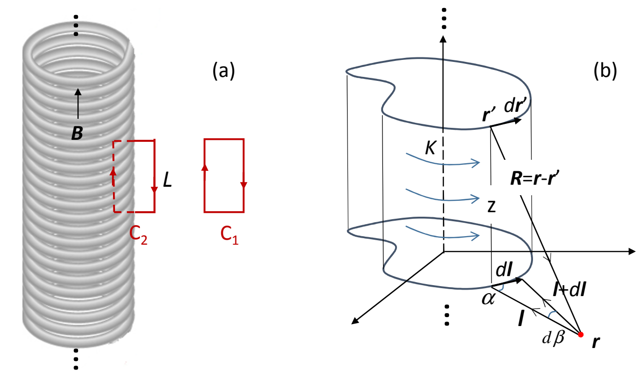

FIG. 40 (a) A solenoid with infinite length. (b) An infinite solenoid with a non-circular cross section that is uniform along its length.

It can be shown that $dB_z(z)/dz = 0$ at the center ($z = 0$). Actually, from the symmetry of the Helmholtz coil, one can argue that the derivatives of odd orders at $z = 0$ should be zero. It is left as an exercise to show that when the separation between rings $d = a$, then $d^2B_z(z)/dz^2|_{z=0} = 0$. Thus, the first non-zero derivative is of the fourth order, $d^4B_z(z)/dz^4$. As a result, the magnetic field is nearly uniform at the center of the Helmholtz coil with $d = a$.

1. Solenoid

Consider a solenoid with finite length L (Fig. 39(a)). It has a uniform **surface current density** (current per unit length) $K = nI$, where n is the number of coils per unit length. Let's find out the magnetic field $\mathbf{B}(z)$ along the central axis inside the solenoid. The observation point is set as the origin of the coordinate. A slice of the solenoid with width dz has current $dI = Kdz$, which produces a magnetic field at the origin (see Eq. (5.25)),

$$dB_z = \frac{\mu_0}{2} K dz \frac{a^2}{(z^2 + a^2)^{3/2}}. \tag{5.27}$$

Let $z = a \cot\theta$, then $dz = -a \csc^2\theta d\theta$. Integrate over the whole solenoid to get,

$$\begin{aligned} B_z(z=0) &= \frac{\mu_0}{2} K a^2 \int_{z_1}^{z_2} \frac{dz}{(z^2 + a^2)^{3/2}} &(5.28)\\ &= -\frac{\mu_0}{2} K \int_{\pi-\theta_1}^{\theta_2} \sin\theta d\theta &(5.29)\\ &= \frac{\mu_0}{2} K (\cos\theta_1 + \cos\theta_2). &(5.30) \end{aligned}$$

The dependence of B_z on z is shown in Fig. 39(b). If the solenoid has infinite length, then $\theta_1, \theta_2 \to 0$, and

$$B_z(z) = \mu_0 K = \mu_0 nI, \tag{5.31}$$

where n is the density of coils per unit length.

For the infinite solenoid, due to the translation symmetry along the z-axis, we expect the magnetic field to be uniform along z and directs along the z-direction, $\mathbf{B}(\mathbf{r}) = B_z(\rho)\hat{\mathbf{z}}$, both inside and outside the solenoid. We can find out the magnetic field easily using Ampère's law. First, choose the loop C_1 in Fig. 40(a). Since there is no current flowing through C_1, hence

$$\oint_{C_1} d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = 0. \tag{5.32}$$

Because the choice of C_1 is arbitrary (as long as it is outside), the magnetic field outside must be a constant and can only be zero.

Next choose the loop C_2, then

$$\oint_{C_2} d\mathbf{r} \cdot \mathbf{B}(\mathbf{r}) = B_z(\rho) L = \mu_0 I. \tag{5.33}$$

Thus, $B_z(\rho) = \mu_0 K$ is independent of ρ inside the solenoid. Note that the derivation that leads to Eq. (5.31) applies only to the magnetic field along the central axis, while the derivation here applies to any location inside the solenoid.

2. Solenoid with non-circular cross section

Consider a solenoid with infinite length but arbitrary cross section, as shown in Fig. 40(b). The cross section is uniform along its length. The surface current density $\mathbf{K}(\mathbf{r})$ is uniform and flows horizontally. Consider a point $\mathbf{r}$ outside or inside the solenoid. From the Biot-Savart law,

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} K dz d\mathbf{r}' \times \frac{\mathbf{R}}{R^3}, \quad \mathbf{R} = \mathbf{r} - \mathbf{r}' \tag{5.34}$$

in which we have replaced $Id\mathbf{r}'$ with $(Kdz)d\mathbf{r}'$. From the geometry in Fig. 40(b), we can see that

$$\begin{aligned} d\mathbf{r}' &= d\boldsymbol{\ell}, & (5.35)\\ \mathbf{R}+\boldsymbol{\ell} &= -z\hat{\mathbf{z}}, & (5.36)\\ \text{and}\quad R^2 &= z^2+\ell^2. & (5.37)\end{aligned}$$

Thus,

$$d\mathbf{r}'\times\frac{\mathbf{R}}{R^3} = -\frac{d\boldsymbol{\ell}\times\boldsymbol{\ell}}{R^3} - \frac{d\boldsymbol{\ell}\times z\hat{\mathbf{z}}}{R^3}. \tag{5.38}$$

After integration,

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= -\frac{\mu_0}{4\pi}K\int_{-\infty}^{\infty}dz\oint\left(\frac{\boldsymbol{\ell}\times d\boldsymbol{\ell}}{R^3}+\frac{z\hat{\mathbf{z}}\times d\boldsymbol{\ell}}{R^3}\right) \qquad (5.39)\\ &= -\frac{\mu_0}{4\pi}K\oint\left(\boldsymbol{\ell}\times d\boldsymbol{\ell}\int_{-\infty}^{\infty}\frac{dz}{R^3}+\hat{\mathbf{z}}\times d\boldsymbol{\ell}\int_{-\infty}^{\infty}dz\frac{z}{R^3}\right).\end{aligned}$$

The second integral is zero; the first integral equals $2/\ell^2$. Thus,

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{2\pi}K\oint\frac{\boldsymbol{\ell}\times d\boldsymbol{\ell}}{\ell^2}. \tag{5.40}$$

Note that

$$|\boldsymbol{\ell}\times d\boldsymbol{\ell}| = \ell d\ell\sin\alpha, \tag{5.41}$$

and

$$d\ell\sin\alpha = |\boldsymbol{\ell}+d\boldsymbol{\ell}|\sin d\beta \simeq \ell d\beta. \tag{5.42}$$

Hence,

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{2\pi}K\oint d\beta\ \hat{\mathbf{z}} & (5.43)\\ &= \begin{cases}\mu_0 K\hat{\mathbf{z}} & \text{inside the solenoid}\\ 0 & \text{outside the solenoid}\end{cases} & (5.44)\end{aligned}$$

Note that in a real solenoid, the current cannot be purely azimuthal since as a whole it needs to flow forward along the central axis. When we take this into account, the magnetic field would have certain azimuthal component B_ϕ.

C. Ampère's law

If the distribution of current has a simple symmetry, then we can use the integral form of the Ampère's law to find out the magnetic field.

Example:
Suppose there is a straight wire with infinite length lying along the z-axis. It has a cylindrical shape with radius a and carries a uniform current I. Find out the magnetic field generated by this wire.
Sol'n:

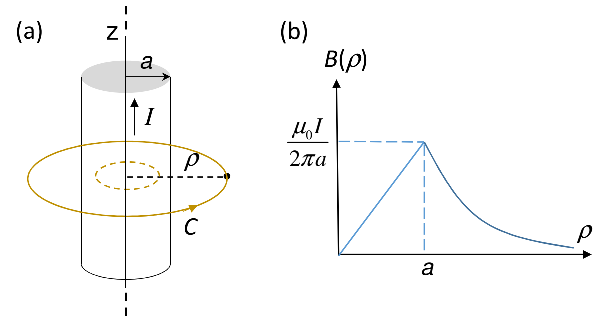

FIG. 41 (a) A cylindrical wire along the z-axis. (b) The magnetic field inside and outside the wire.

Let's choose the cylindrical coordinate. The system is invariant if you rotate around z-axis, or translate along z-axis, so the magnetic field cannot depend on ϕ, z. It follows that,

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(\rho) = B_\rho(\rho)\hat{\boldsymbol{\rho}} + B_\phi(\rho)\hat{\boldsymbol{\phi}} + B_z(\rho)\hat{\mathbf{z}}. \tag{5.45}$$

From Ampère's right-hand rule, we expect the magnetic field to be along $\hat{\boldsymbol{\phi}}$, so

$$\mathbf{B}(\mathbf{r}) = B_\phi(\rho)\hat{\boldsymbol{\phi}}. \tag{5.46}$$

You may reach the same conclusion with a more detailed analysis of the Biot-Savart integral.

Choose a loop C with radius ρ around the wire (Fig. 41(a)), then

$$\begin{aligned}\oint_C d\mathbf{r}\cdot\mathbf{B}(\mathbf{r}) &= \mu_0 I(\rho), & (5.47)\\ \to 2\pi\rho B_\phi(\rho) &= \mu_0 I(\rho), & (5.48)\end{aligned}$$

where $I(\rho)$ is the current passing through the circle C,

$$I(\rho) = \begin{cases} I\frac{\rho^2}{a^2} & \text{if } \rho<a\\ I & \text{if } \rho>a\end{cases}. \tag{5.49}$$

Thus (Fig. 41(b)),

$$\mathbf{B}(\mathbf{r}) = \begin{cases}\frac{\mu_0 I}{2\pi a}\frac{\rho}{a}\ \hat{\boldsymbol{\phi}} & \text{if } \rho<a\\ \frac{\mu_0 I}{2\pi\rho}\ \hat{\boldsymbol{\phi}} & \text{if } \rho>a\end{cases} \tag{5.50}$$

Example:
In Fig. 42(a), a hollow cylindrical can with radius R and height L has a wire at its center. A current I flows up the wire, spreads out, flows down, converges at the bottom of the wire and flows up again.
(a) Using cylindrical coordinate, argue that the magnetic field has the following form everywhere, both inside and outside the can,

$$\mathbf{B}(\mathbf{r}) = B(\rho,z)\hat{\boldsymbol{\phi}}. \tag{5.51}$$

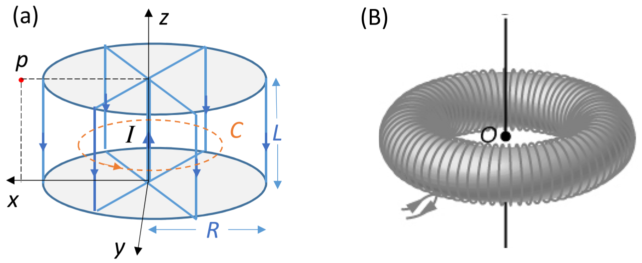

FIG. 42 (a) A hollow can with a wire inside along its central axis. (b) A toroidal solenoid.

(b) Find out $B(\rho, z)$.
Sol'n:
(a) Since the system is invariant with respect to the rotation around the wire (z-axis), so the magnetic field cannot depend on ϕ,

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \mathbf{B}(\rho, z) \qquad (5.52) \\ &= B_\rho(\rho, z)\hat{\boldsymbol{\rho}} + B_\phi(\rho, z)\hat{\boldsymbol{\phi}} + B_z(\rho, z)\hat{\mathbf{z}}. \qquad (5.53) \end{aligned}$$

There is no obvious reason to rule out certain component of $\mathbf{B}$. But from a detailed analysis of the Biot-Savart law, we can show that the field $\mathbf{B}$ is circular and has only the $\hat{\boldsymbol{\phi}}$ component:
First, align the x-axis with the direction of observation point p, which can be inside or outside the can (Fig. 42(a)). In general,

$$\begin{aligned} \mathbf{J}(\mathbf{r}') &= J_x\hat{\mathbf{x}} + J_y\hat{\mathbf{y}} + J_z\hat{\mathbf{z}}, \qquad (5.54) \\ \mathbf{r} - \mathbf{r}' &= (x - x')\hat{\mathbf{x}} + (0 - y')\hat{\mathbf{y}} + (z - z')\hat{\mathbf{z}}. \qquad (5.55) \end{aligned}$$

The distribution of current has a mirror symmetry with respect to the x-z plane. So for a current element $\mathbf{J}(\mathbf{r}')dv'$, there is a mirror counterpart $\tilde{\mathbf{J}}(\tilde{\mathbf{r}}')dv'$, with

$$\tilde{\mathbf{J}} = (J_x, -J_y, J_z), \text{ and } \tilde{\mathbf{r}}' = (x', -y', z'). \qquad (5.56)$$

The magnetic field produced by this pair of current elements is

$$\begin{aligned} d\mathbf{B} &\sim \mathbf{J} \times (\mathbf{r} - \mathbf{r}') + \tilde{\mathbf{J}} \times (\mathbf{r} - \tilde{\mathbf{r}}') \qquad (5.57) \\ &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ J_x & J_y & J_z \\ x - x' & -y' & z - z' \end{vmatrix} + \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ J_x & -J_y & J_z \\ x - x' & +y' & z - z' \end{vmatrix} \\ &= \begin{vmatrix} \hat{\mathbf{x}} & 2\hat{\mathbf{y}} & \hat{\mathbf{z}} \\ J_x & 0 & J_z \\ x - x' & 0 & z - z' \end{vmatrix} \sim \hat{\mathbf{y}}. \qquad (5.58) \end{aligned}$$

Thus, after integration, $\mathbf{B} \sim \hat{\mathbf{y}} = \hat{\boldsymbol{\phi}}$.
(b) After the form of $\mathbf{B}(\mathbf{r})$ has been narrowed down, it's easy to evaluate the Ampère integral. Choose the path C to be a horizontal circle with radius ρ, then

$$\oint_C d\mathbf{r} \cdot \mathbf{B} = 2\pi\rho B(\rho, z) = \begin{cases} \mu_0 I \text{ if } C \text{ is inside the can} \\ 0 \text{ if } C \text{ is outside the can} \end{cases} \qquad (5.59)$$

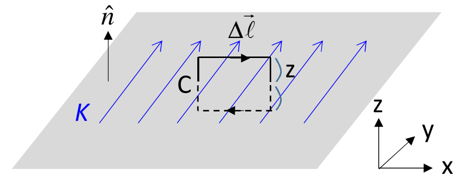

FIG. 43 An infinite plate on the x-y plane with a uniform sheet of current $\mathbf{K} = K\hat{\mathbf{y}}$.

Thus, inside the can,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}}. \qquad (5.60)$$

There is no magnetic field outside the can.
Note that the same argument applies to other systems with azimuthal symmetry and radial current flow, and their magnetic fields must be circular. For example, for the toroidal solenoid in Fig. 42(b), the magnetic field inside is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 N I}{2\pi\rho}\hat{\boldsymbol{\phi}}, \qquad (5.61)$$

where N is the number of coils. There is no magnetic field outside the solenoid.
Example:
There is a thin plate on the x-y plane with a uniform current. Its current per unit length, or surface current density $\mathbf{K} = K\hat{\mathbf{y}}$ (Fig. 43). Find out the magnetic field on both sides of the plate.
Sol'n:
Since the plate can be considered as a collection of wires along y direction, according to Ampère's right-hand rule, we expect the magnetic field to be along the x-axis (see Zangwill, 2013 for a detailed analysis based on symmetry),

$$\mathbf{B}(\mathbf{r}) = B(z)\hat{\mathbf{x}}. \qquad (5.62)$$

Choose a small rectangular loop C with a surface normal parallel to $\mathbf{K}$, as shown in Fig. 43. The current passing through C is $I_\square = K\Delta\ell$. Thus, the circulation

$$\oint_C d\mathbf{r} \cdot \mathbf{B} = \mathbf{B}_+ \cdot \Delta\boldsymbol{\ell} + \mathbf{B}_- \cdot (-\Delta\boldsymbol{\ell}) = \mu_0 I_\square, \qquad (5.63)$$

where $\mathbf{B}_+$ ($\mathbf{B}_-$) is the field above (below) the plane. We expect $\mathbf{B}_- = -\mathbf{B}_+$, thus

$$\begin{aligned} \mathbf{B}_+ &= +\frac{\mu_0}{2}K\hat{\mathbf{x}} \text{ or } \frac{\mu_0}{2}\mathbf{K} \times \hat{\mathbf{n}}, \; K = \frac{I_\square}{\Delta\ell} \qquad (5.64) \\ \mathbf{B}_- &= -\frac{\mu_0}{2}K\hat{\mathbf{x}} \text{ or } -\frac{\mu_0}{2}\mathbf{K} \times \hat{\mathbf{n}}, \qquad (5.65) \end{aligned}$$

in which $\hat{\mathbf{n}}$ points up. The magnetic field is uniform and does not decrease with distance z.

D. Boundary condition for B

In general, the magnetic fields on opposite sides of a current sheet are not the same. Their difference is caused by the current on the surface. Suppose a surface has surface current density $\mathbf{K}(\mathbf{r})$. At a point $\mathbf{r}$ on the surface, the magnetic fields on opposite sides are $\mathbf{B}_1(\mathbf{r})$ and $\mathbf{B}_2(\mathbf{r})$ (Fig. 44). What's the relation between this two magnetic fields?

First, divide the surface S into a small rectangle $\square$ and a surface S' (S with $\square$ removed),

$$S = \square + S'. \tag{5.66}$$

The rectangle is microscopically large, but macroscopically small (say, with a size of 1 μm). It can be considered as flat since it is just a small part of the smooth surface S. The field, $\mathbf{B}_1(\mathbf{r})$ or $\mathbf{B}_2(\mathbf{r})$, is the superposition of the fields produced by $\square$ and S'.

When one infinitesimally approaches the center of the rectangle, the field is close to the field of an infinite plane, $\mathbf{B}(\mathbf{r}) = \pm\frac{\mu_0}{2}\mathbf{K}\times\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points from region 1 to region 2. Suppose the field produced by S' is $\mathbf{B}_S$, then

$$\mathbf{B}_1 = \mathbf{B}_S - \frac{\mu_0}{2}\mathbf{K}\times\hat{\mathbf{n}}, \tag{5.67}$$

$$\mathbf{B}_2 = \mathbf{B}_S + \frac{\mu_0}{2}\mathbf{K}\times\hat{\mathbf{n}}. \tag{5.68}$$

Even though $\mathbf{B}_S$ remains unknown, we can substrate the field to get

$$\mathbf{B}_2(\mathbf{r}) - \mathbf{B}_1(\mathbf{r}) = \mu_0\mathbf{K}(\mathbf{r})\times\hat{\mathbf{n}}. \tag{5.69}$$

This is the BC for fields near a current sheet. Sometimes it is written as,

$$\hat{\mathbf{n}}\cdot(\mathbf{B}_2 - \mathbf{B}_1) = 0, \tag{5.70}$$

$$\hat{\mathbf{n}}\times(\mathbf{B}_2 - \mathbf{B}_1) = \mu_0\mathbf{K}. \tag{5.71}$$

1. Force on current sheet

To find out the magnetic force on a current sheet, divide the surface S into $\square$ and S', as in previous section. The rectangle exerts no force on itself. So the force is due to the magnetic field produce by S',

$$\mathbf{F}_\square = I_\square\Delta\mathbf{L}\times\mathbf{B}_S \tag{5.72}$$

$$= (\mathbf{K}\Delta\ell)\Delta L\times\mathbf{B}_S, \tag{5.73}$$

where $I_\square$ is the current passing through $\square$ (Fig. 44). Since

$$\mathbf{B}_S = \frac{1}{2}(\mathbf{B}_1 + \mathbf{B}_2), \tag{5.74}$$

so the force density (or pressure)

$$\mathbf{f}_\square \equiv \frac{\mathbf{F}_\square}{\Delta\ell\Delta L} = \frac{1}{2}\mathbf{K}\times(\mathbf{B}_1 + \mathbf{B}_2). \tag{5.75}$$

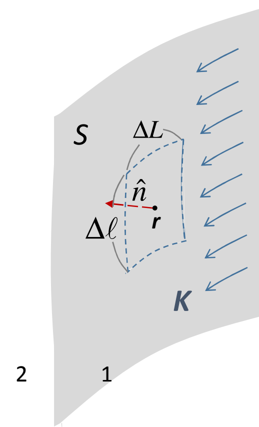

FIG. 44 A surface S is divided into a small rectangle and a surface without the rectangle, S'.

For example, for a long solenoid, the magnetic field inside and outside are (see Eq. (5.44))

$$\mathbf{B}_1 = \mu_0 K\hat{\mathbf{z}}, \quad \mathbf{B}_2 = 0. \tag{5.76}$$

So the magnetic pressure on the wall of the solenoid is

$$\mathbf{f} = \mu_0\frac{K^2}{2}\hat{\mathbf{n}}, \tag{5.77}$$

where $\hat{\mathbf{n}}$ points outward.

For example, for a solenoid with $I = 0.1$ A and coil density $a = 10^3$/m, its surface current density $K = 100$ A/m. The magnetic pressure on the wall $\mathbf{f} = 2\pi\times10^{-3}$ N/m^2.

E. Vector potential

Assume that two vector potentials differ by a gradient $\nabla\chi(\mathbf{r})$,

$$\mathbf{A}' = \mathbf{A} + \nabla\chi. \tag{5.78}$$

Since $\nabla\times\nabla\chi(\mathbf{r}) = 0$ for any scalar function $\chi(\mathbf{r})$ without singularity, so $\mathbf{A}'$ and $\mathbf{A}$ yield the same magnetic field $\mathbf{B}$. That is, one magnetic field can have different vector potentials. This is called **gauge degree of freedom**.

For example, given $\mathbf{B}(\mathbf{r}) = B_0\hat{\mathbf{z}}$, then its vector potential can be

$$\mathbf{A}(\mathbf{r}) = B_0(0, x, 0), \tag{5.79}$$

$$\text{or}\quad \mathbf{A}(\mathbf{r}) = \frac{B_0}{2}(-y, x, 0). \tag{5.80}$$

They differ by the gradient in Eq. (5.78) with $\chi(\mathbf{r}) = -B_0xy/2$.

With the help of χ, one can demand the vector potential to satisfy the **Coulomb gauge**,

$$\nabla\cdot\mathbf{A}(\mathbf{r}) = 0. \tag{5.81}$$

Pf: Suppose $\nabla\cdot\mathbf{A}\neq 0$, then we can choose a $\chi(\mathbf{r})$ such that

$$\nabla\cdot\mathbf{A}' = \nabla\cdot\mathbf{A} + \nabla^2\chi = 0. \tag{5.82}$$

What we need is a χ that satisfies

$$\nabla^2\chi(\mathbf{r}) = -\nabla\cdot\mathbf{A}(\mathbf{r}). \tag{5.83}$$

The RHS is like the source term of the Poisson equation in electrostatics, and in principle a solution $\chi(\mathbf{r})$ always exists. Thus, we can always have $\nabla\cdot\mathbf{A}'=0$. QED.
Usually, all we need to know is that χ exists. It is not necessary to actually find out $\chi(\mathbf{r})$.

Note: In Chap 2, we have shown that Eq. (5.81) is always valid for steady current. But its validity extends to dynamic field, as we will show in a later chapter.

We can write Ampère's law in terms of the vector potential,

$$\begin{aligned}\nabla\times\mathbf{B} &= \mu_0\mathbf{J},\quad \mathbf{B}=\nabla\times\mathbf{A}, &(5.84)\\ \rightarrow\quad \nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A} &= \mu_0\mathbf{J}. &(5.85)\end{aligned}$$

With the Coulomb gauge $\nabla\cdot\mathbf{A}(\mathbf{r})=0$, we have the vector Poisson equation,

$$\nabla^2\mathbf{A}(\mathbf{r}) = -\mu_0\mathbf{J}(\mathbf{r}). \tag{5.86}$$

Each component of Eq. (5.86) is a scalar Poisson equation. Thus, it has the formal solution,

$$\begin{aligned}A_i &= \frac{\mu_0}{4\pi}\int dv'\frac{J_i(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|},\quad i=x,y,z &(5.87)\\ \text{or}\quad \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int dv'\frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, &(5.88)\end{aligned}$$

which is consistent with Eq. (5.2). For a thin wire, just replace $dv'\mathbf{J}$ with $Id\mathbf{r}'$, and

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}. \tag{5.89}$$

<u>Example:</u>
Find out the vector potential of a wire that is straight, infinite, and carries a current I.
Sol'n:

Adopt the cylindrical coordinate, and lay the wire along z-axis (Fig. 45(a)). In the integral,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{5.90}$$

$d\mathbf{r}'=dz\hat{\mathbf{z}}$, thus $\mathbf{A}(\mathbf{r})\parallel\hat{\mathbf{z}}$.

With the help of

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \ln\left(2x+2\sqrt{x^2+a^2}\right), \tag{5.91}$$

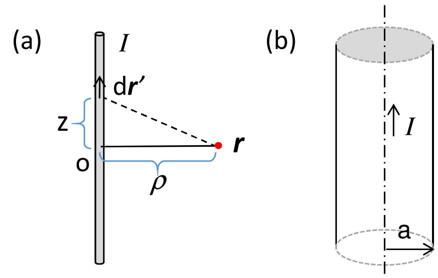

FIG. 45 (a) A thin wire with a current. (b) A cylindrical wire with a uniform current inside.

for a wire with length $2L$, we have

$$\begin{aligned}A_z(\rho) &= \frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{dz}{\sqrt{z^2+\rho^2}} &(5.92)\\ &= \frac{\mu_0 I}{4\pi}\ln\frac{\sqrt{1+(\rho/L)^2}+1}{\sqrt{1+(\rho/L)^2}-1}. &(5.93)\end{aligned}$$

If $L\gg\rho$, then

$$\ln\frac{\sqrt{1+(\rho/L)^2}+1}{\sqrt{1+(\rho/L)^2}-1}\simeq \ln 4 - 2\ln\frac{\rho}{L}, \tag{5.94}$$

hence

$$A_z(\rho)\simeq -\frac{\mu_0 I}{2\pi}\ln\rho + \text{const.} \tag{5.95}$$

It diverges when $\rho\to 0$ (and at infinity). Finally, it's not difficult to show that,

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0 I}{2\pi}\frac{1}{\rho}\hat{\phi}. \tag{5.96}$$

<u>Example:</u>
Find out the vector potential of a straight, infinite cylindrical wire with radius a (Fig. 45(b)). The wire carries a uniform current I.
Sol'n:

When the wire has a finite radius, the divergence of $\mathbf{A}$ as $\rho\to 0$ can be avoided. However, it's no longer convenient to use the integral formula in Eq. (5.88). Thus we will use the vector Poisson equation instead.

First, since $\mathbf{J}(\mathbf{r})=J_0\hat{\mathbf{z}}$, where $J_0=I/\pi a^2$ is a constant, Eq. (5.88) tells us that $\mathbf{A}(\mathbf{r})=A_z(\mathbf{r})\hat{\mathbf{z}}$. Furthermore, we expect $A_z(\mathbf{r})=A_z(\rho)$, thus $\nabla\cdot\mathbf{A}=0$ is automatically satisfied, and

$$\nabla^2\mathbf{A}(\mathbf{r}) = -\mu_0\mathbf{J}(\mathbf{r}). \tag{5.97}$$

Since

$$\nabla^2 A_z(\mathbf{r}) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_z}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial\phi^2}+\frac{\partial^2 A_z}{\partial z^2}, \tag{5.98}$$

it follows that

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dA_z}{d\rho}\right) = -\mu_0 J_0. \tag{5.99}$$

Direct integration gives

$$A_z(\rho) = -\frac{\mu_0}{4}J_0\rho^2 + C\ln\rho + \text{constant} \tag{5.100}$$

$$= \begin{cases} -\frac{\mu_0}{4}J_0\rho^2 + D & \text{for } \rho \le a, \\ +C\ln\rho + D' & \text{for } \rho \ge a. \end{cases} \tag{5.101}$$

Some terms have been dropped to avoid unphysical divergence.

The vector potential needs be continuous at $\rho = a$ (otherwise the magnetic field would diverge there). This gives

$$A_z(\rho) = \begin{cases} -\frac{\mu_0}{4}J_0\rho^2 + D & \text{for } \rho \le a, \\ -\frac{\mu_0}{4}J_0a^2 + C\ln\frac{\rho}{a} + D & \text{for } \rho \ge a. \end{cases} \tag{5.102}$$

We can ignore the constant D, but C is still unknown.

To find C, we require that the curl of $\mathbf{A}$ be continuous across the boundary. That is,

$$\mathbf{B}_{out} - \mathbf{B}_{in} = 0. \tag{5.103}$$

This so because the surface current density is zero, $\mathbf{K} = 0$, for a boundary layer that is infinitely thin. Now

$$\mathbf{B}(\mathbf{r}) = \nabla\times\mathbf{A}(\mathbf{r}) = -\frac{dA_z}{d\rho}\hat{\boldsymbol{\phi}}. \tag{5.104}$$

Match $\mathbf{B}_{in}$ and $\mathbf{B}_{out}$ at the boundary to get $C = -\frac{\mu_0}{2}J_0a^2$.

Finally, drop D to get

$$A_z(\rho) = \begin{cases} -\frac{\mu_0}{4}J_0\rho^2 & \text{for } \rho \le a, \\ -\frac{\mu_0}{4}J_0a^2\left[1 + 2\ln(\rho/a)\right] & \text{for } \rho \ge a. \end{cases} \tag{5.105}$$

It's not difficult to see that

$$B_\phi(\rho) = \begin{cases} \frac{\mu_0 I}{2\pi}\frac{\rho}{a^2} & \text{for } \rho \le a, \\ \frac{\mu_0 I}{2\pi\rho} & \text{for } \rho \ge a. \end{cases} \tag{5.106}$$

This agrees with the result in Eq. (5.50), which was obtained by a simpler approach.

F. Magnetic scalar potential

Since a vector field with zero curl can be written as a gradient, for a static magnetic field *in vacuum* with $\nabla\times\mathbf{B} = 0$, one can write

$$\mathbf{B}(\mathbf{r}) = -\nabla\psi(\mathbf{r}), \tag{5.107}$$

where ψ is the magnetic scalar potential. Combined with the equation $\nabla\cdot\mathbf{B} = 0$, we have

$$\nabla^2\psi(\mathbf{r}) = 0. \tag{5.108}$$

Unlike the vector potential, the magnetic scalar potential is not applicable to dynamic magnetic field.

1. Potential of a current loop

Suppose a magnetic field is generated from a loop of thin wire C with current I. From Biot-Savart law,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}I\oint_C d\mathbf{r}'\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{5.109}$$

With the help of the identity,

$$\oint_C d\mathbf{r}'\times\mathbf{V} = \int_S ds_k\nabla V_k - \int_S d\mathbf{s}\nabla\cdot\mathbf{V}, \tag{5.110}$$

where C is the boundary of S, we can write

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_S ds_k\nabla'\frac{(\mathbf{r}-\mathbf{r}')_k}{|\mathbf{r}-\mathbf{r}'|^3} - \frac{\mu_0 I}{4\pi}\int_S d\mathbf{s}\nabla'\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{5.111}$$

The integrand of the second term,

$$\nabla'\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \nabla'\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{5.112}$$

$$= -4\pi\delta(\mathbf{r}-\mathbf{r}'). \tag{5.113}$$

Thus the second term is zero as long as the observation point $\mathbf{r}$ is not *on* the surface S. For the first term, switch ∇' to ∇ (getting a minus sign), then

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\nabla\int_S \underbrace{d\mathbf{s}\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}}_{=-d\Omega_S} \tag{5.114}$$

$$= \frac{\mu_0 I}{4\pi}\nabla\Omega_S(\mathbf{r}), \tag{5.115}$$

where $\Omega_S(\mathbf{r})$ is the solid angle of S with respect to the observation point $\mathbf{r}$. Therefore, the magnetic scalar potential

$$\psi(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\Omega_S(\mathbf{r}). \tag{5.116}$$

Take the ring in Fig. 37(a) as an example. Note that the current flows counter-clockwise, hence the normal vector of S points up, instead of pointing down, away from the observation point. As a result, there is an extra minus sign in Ω_S, and

$$\Omega_S = -2\pi\left[1 - \cos\left(\frac{\pi}{2} - \alpha\right)\right], \quad \sin\alpha = \frac{z}{\sqrt{z^2+a^2}}$$

$$= -2\pi\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right) \tag{5.117}$$

Taking the gradient of Ω_S to obtain

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{d}{dz}\Omega_S(z)\hat{\mathbf{z}} = \frac{\mu_0 I}{2}\frac{a^2}{(z^2+a^2)^{3/2}}\hat{\mathbf{z}}. \tag{5.118}$$

This agrees with the result in Eq. (5.25).

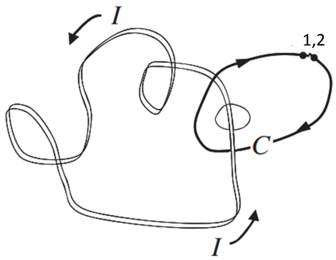

FIG. 46 A current pierces through a surface S bounded by C. Fig. from Zangwill, 2013

2. Multi-valuedness of ψ

It is known that for static electric field,

$$\oint_C d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}) = 0. \tag{5.119}$$

This implies that the potential difference,

$$\psi_{(}\mathbf{r}_2) - \psi(\mathbf{r}_1) = -\int_1^2 d\mathbf{r}\cdot\mathbf{E}(\mathbf{r}), \tag{5.120}$$

is independent of the path of the integral from point-1 to point-2.

For a static magnetic field, however, the loop integral of $\mathbf{B}$ may not be zero. Thus, if one moves from $\mathbf{r}_1$ to $\mathbf{r}_2 = \mathbf{r}_1$ around a loop C that encloses a current I (Fig. 46), then

$$\psi(\mathbf{r}_2) - \psi(\mathbf{r}_1) = -\oint_C d\mathbf{r}\cdot\mathbf{B}(\mathbf{r}) = \pm\mu_0 I. \tag{5.121}$$

That is, ψ is not single-valued. To prevent it from having multiple values at the same location, we can refrain the path C from crossing the surface bounded by the current loop (Zangwill, 2013).

<u>Problem</u>:

1. Along the central axis of a Helmholtz coils (Fig. 38),

$$B_z(z) = \frac{\mu_0}{2}\left\{\frac{Ia^2}{[(z-d/2)^2+a^2]^{3/2}} + \frac{Ia^2}{[(z+d/2)^2+a^2]^{3/2}}\right\}.$$

(a) Show that $dB_z(z)/dz = 0$ at the center ($z = 0$).
(b) Argue that the derivatives of odd orders at $z = 0$ should be zero.
(c) Show that when the separation between rings $d = a$, $d^2B_z(z)/dz^2|_{z=0} = 0$.

VI. MAGNETIC MULTIPOLES

A. Multipole expansion

Recall that in Chap 4, given an electric potential,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int dv' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{6.1}$$

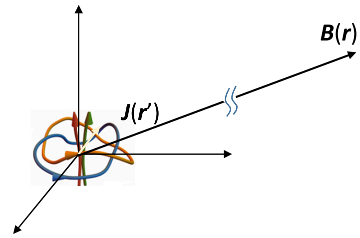

FIG. 47 An observation point is far away from a localized current distribution.

if $r \gg r'$, then we can expand

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \simeq \frac{1}{r} + \frac{\hat{\mathbf{r}}}{r^2}\cdot\mathbf{r}' + \frac{1}{2r^3}\left[3(\hat{\mathbf{r}}\cdot\mathbf{r}')^2 - |\mathbf{r}'|^2\right]. \tag{6.2}$$

Each term contributes to the potential of a certain electric multipole.

Similar approximation can be applied to the vector potential,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int dv' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{6.3}$$

If $r \gg r'$, that is, the current source is localized and the observer is far away (Fig. 1), then we can use the expansion in Eq. (6.2) and keep terms to the first order to get

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &\simeq \frac{\mu_0}{4\pi r}\int dv'\mathbf{J}(\mathbf{r}') + \frac{\mu_0}{4\pi r^3}\mathbf{r}\cdot\mathbf{r}'\int dv'\mathbf{J}(\mathbf{r}') && (6.4)\\ &\simeq 0 + \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}, && (6.5)\end{aligned}$$

where $\mathbf{m}$ is the magnetic dipole moment. The magnetic quadrupole potential from the second-order term is not considered here.

We now explain how Eq. (6.5) is obtained. First, two identities are required. For a steady, localized current distribution,

$$\int dv' J_i(\mathbf{r}') = 0, \quad i = x, y, z \tag{6.6}$$

$$\int dv' \left[r_i' J_j(\mathbf{r}') + r_j' J_i(\mathbf{r}')\right] = 0. \tag{6.7}$$

Pf: From the equation of continuity, for a steady current,

$$\begin{aligned}\nabla\cdot\mathbf{J} &= 0, && (6.8)\\ \nabla\cdot(r_i\mathbf{J}) &= J_i + r_i\cancel{\nabla\cdot\mathbf{J}}, && (6.9)\\ \nabla\cdot(r_ir_j\mathbf{J}) &= r_iJ_j + r_jJ_i + r_ir_j\cancel{\nabla\cdot\mathbf{J}}. && (6.10)\end{aligned}$$

The integration of Eq. (6.9) over the whole space gives,

$$\begin{aligned}\int dv' J_i &= \int dv'\nabla'\cdot(r_i'\mathbf{J}) && (6.11)\\ &= \int d\mathbf{s}'\cdot(r_i'\mathbf{J}) = 0. && (6.12)\end{aligned}$$

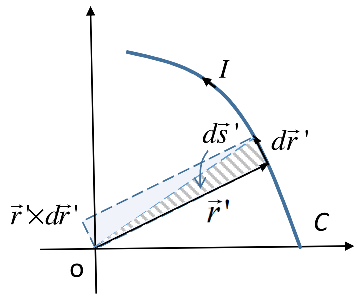

FIG. 48 A planar loop with current I.

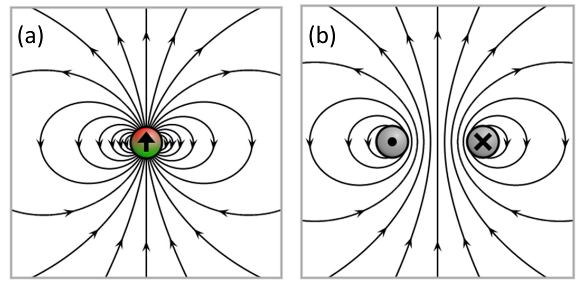

FIG. 49 The fields from (a) an electric dipole and (b) a current loop.

The integral is zero since the current is localized while the surface of integration is at infinity. Thus, the monopole term in Eq. (6.4) vanishes.

The integration of Eq. (6.10) over the whole space gives,

$$\int dv'(r'_i J_j + r'_j J_i) = \int d\mathbf{s}' \cdot (r'_i r'_j \mathbf{J}) = 0. \qquad (6.13)$$

Thus, we can write the integral of the dipole term in Eq. (6.4) as,

$$r_i \int dv' r'_i J_j = \frac{r_i}{2} \int dv' \underbrace{(r'_i J_j - r'_j J_i)}_{=\epsilon_{ijk}(\mathbf{r}'\times\mathbf{J})_k} \qquad (6.14)$$

$$= \frac{1}{2} \int dv' \left[(\mathbf{r}' \times \mathbf{J}) \times \mathbf{r}\right]_j \qquad (6.15)$$

$$= (\mathbf{m} \times \mathbf{r})_j, \qquad (6.16)$$

where

$$\mathbf{m} \equiv \frac{1}{2} \int dv' \mathbf{r}' \times \mathbf{J}(\mathbf{r}'). \qquad (6.17)$$

Hence, up to the first order,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \qquad (6.18)$$

Eq. (6.17) is the most general form of the **magnetic dipole moment**. It reduces to other forms under special circumstances:

1. Thin wire:

For the current carried by a thin wire of loop C, just replace $dv'\mathbf{J}$ with $Id\mathbf{r}'$ to get

$$\mathbf{m} \equiv \frac{I}{2} \oint_C \mathbf{r}' \times d\mathbf{r}'. \qquad (6.19)$$

If furthermore, C is a *planar loop*, then (see Fig. 2),

$$\frac{1}{2} \mathbf{r}' \times d\mathbf{r}' = d\mathbf{s}'. \qquad (6.20)$$

Hence, after integration,

$$\mathbf{m} = I \oint d\mathbf{s}' = I\mathbf{S}. \qquad (6.21)$$

The magnetic moment is proportional to the surface area of the loop. The direction of $\mathbf{S}$ is determined by the right-hand rule.

2. Point charges:

A set of moving charges has the current density,

$$\mathbf{J}(\mathbf{r}) = \sum_{k=1}^{N} q_k \mathbf{v}_k \delta(\mathbf{r} - \mathbf{r}_k). \qquad (6.22)$$

Substitute it to Eq. (6.17) and get

$$\mathbf{m} = \sum_{k=1}^{N} \frac{q_k}{2} \int dv' \mathbf{r}' \times \mathbf{v}_k \delta(\mathbf{r}' - \mathbf{r}_k) \qquad (6.23)$$

$$= \frac{1}{2} \sum_k q_k (\mathbf{r}_k \times \mathbf{v}_k) \qquad (6.24)$$

$$= \sum_k \frac{q_k}{2m_k} \mathbf{L}_k, \quad \mathbf{L}_k \equiv m_k \mathbf{r}_k \times \mathbf{v}_k. \qquad (6.25)$$

If q_k/m_k is a constant, then the **orbital magnetic moment**

$$\mathbf{m} = \frac{q}{2m} \mathbf{L}, \qquad (6.26)$$

where $\mathbf{L}$ is the total angular momentum of these charges.

B. Magnetic dipole

From the vector potential of a magnetic dipole (valid for $r \gg r'$),

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}, \qquad (6.27)$$

we can calculate its magnetic field,

$$\mathbf{B}(\mathbf{r}) = \nabla \times \left(\frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3} \right) \qquad (6.28)$$

$$\cdots = \frac{\mu_0}{4\pi} \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m}) - \mathbf{m}}{r^3}. \qquad (6.29)$$

The field decreases as $1/r^3$ and has the distribution shown in Fig. 3, which is similar to the electric dipole field (Chap 4) when $r \gg r'$.

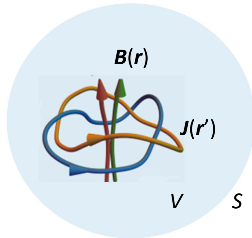

FIG. 50 The current is confined within a sphere.

Example:
Suppose current distribution $\mathbf{J}(\mathbf{r})$ flows inside a ball V with volume V, show that the average of the magnetic field over the ball,

$$\langle \mathbf{B}(\mathbf{r})\rangle_V \equiv \frac{1}{V}\int_V dv \mathbf{B}(\mathbf{r}) = \frac{2\mu_0}{3}\frac{\mathbf{m}}{V}, \tag{6.30}$$

where $\mathbf{m}$ is the magnetic dipole moment due to the current (see Fig. 4).
Pf: Start from the Biot-Savart law,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{J\neq 0} dv'\mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, \tag{6.31}$$

then

$$\begin{aligned}\int_V dv\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V dv \int_{J\neq 0} dv'\mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \\ &= -\frac{\mu_0}{4\pi}\int_{J\neq 0} dv'\mathbf{J}(\mathbf{r}') \times \underbrace{\int_V dv \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}}_{=\tilde{\mathbf{E}}(\mathbf{r}')},\end{aligned}$$

where $\tilde{\mathbf{E}}(\mathbf{r}')$ is the *fictitious* "electric" field of a ball V with charge density $\tilde{\rho} = 4\pi\varepsilon_0$. According to the analysis in Chap 4,

$$\tilde{\mathbf{E}}(\mathbf{r}') = \frac{4\pi}{3}\mathbf{r}'. \tag{6.32}$$

Thus,

$$\langle \mathbf{B}\rangle_V = -\frac{\mu_0}{V}\int dv'\mathbf{J}(\mathbf{r}') \times \frac{1}{3}\mathbf{r}' \tag{6.33}$$

$$= +\frac{2\mu_0}{3}\frac{\mathbf{m}}{V}. \tag{6.34}$$

Similar to the case of the electric dipole, if the current is outside of the sphere, then

$$\langle \mathbf{B}(\mathbf{r})\rangle_V = \mathbf{B}(0). \tag{6.35}$$

Its proof is similar to the case of electric dipole and will not be repeated here.

1. Point magnetic dipole

When a magnetic dipole is produced by the current in a tiny region (say a nucleus), we have a **point magnetic dipole**. The formula in Eq. (6.29) remains valid as long as $r \neq 0$.

However, if you integrate the field in Eq. (6.29) over a ball V centered at $\mathbf{r} = 0$, then

$$\int_V dv\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V dv\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m})-\mathbf{m}}{r^3} = 0. \tag{6.36}$$

It is zero due to angular integration, no matter if the ball is large or small. This contradicts the result in Eq. (6.34). To fix this discrepancy, we can add a delta function to Eq. (6.29), so that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m})-\mathbf{m}}{r^3} + \frac{2\mu_0}{3}\mathbf{m}\delta(\mathbf{r}). \tag{6.37}$$

The added term is important in the calculation of hyperfine structure (more later).

2. Magnetic dipole layer

In Fig. 5(a), there is a continuous distribution of magnetic dipoles on surface S. Suppose these dipole moments are from orbital motion of charges (not from electron spins), and are *perpendicular* to the surface. If S is an open surface, then the magnetic field from these dipoles is equal to the $\mathbf{B}$ field produced by a current flowing around the boundary C of S. This **Ampère's theorem**.
Pf: Each magnetic dipole is produced by a small current loop,

$$d\mathbf{m} = Id\mathbf{s}, \tag{6.38}$$

where $d\mathbf{s}$ is an area element. The dipole at $\mathbf{r}'$ on the surface generates a vector potential,

$$d\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}d\mathbf{m} \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{6.39}$$

Using the identity,

$$\int_S d\mathbf{s} \times \nabla f(\mathbf{r}) = \oint_C d\mathbf{r} f(\mathbf{r}), \tag{6.40}$$

where C is the boundary of surface S, one then has

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d\mathbf{m} \times \underbrace{\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}}_{=\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}} \tag{6.41}$$

$$= \frac{\mu_0}{4\pi}I\int_S d\mathbf{s}' \times \nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} \tag{6.42}$$

$$= \frac{\mu_0}{4\pi}I\oint_C d\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{6.43}$$

The line integral above equals the vector potential produced by a loop C carrying current I. QED.

If you're familiar with Stoke's theorem, then Ampère's theorem is simply a variant of Stokes theorem: The sum

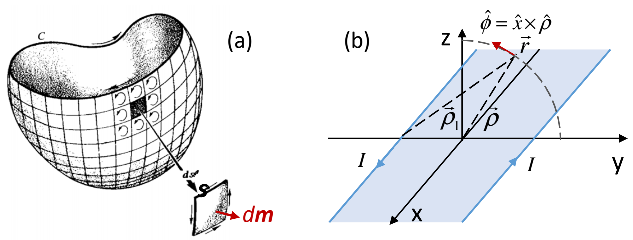

FIG. 51 (a) Magnetic dipole moments are standing on an open surface. (b) A magnetic tape on the x-y plane.

of the circulation of current loops packed together equals the circulation around the outer boundary of these loops (Fig. 5(a)).

Example:

A long magnetic tape with width d is lying along the x-axis, as shown in Fig. 5(b). The magnetic dipoles on the tape stand straight up, and the magnetic moment per unit area is M. Find out the magnetic field around this magnetic tape.

Sol'n: According to Ampère's theorem, we only need to calculate the $\mathbf{B}$ field produced by the current flowing along the boundary of the tape. Since $d\mathbf{m} = Id\mathbf{s}$, so

$$I = \frac{dm}{ds} = M. \tag{6.44}$$

We need to calculate the magnetic field of two long straight wires with current I.

If the wire is lying on the x-axis, then for a point $\mathbf{r}$ on y-z plane,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{2\pi}\frac{I}{\rho}\hat{\boldsymbol{\phi}}, \tag{6.45}$$

where $\boldsymbol{\rho}$ and $\hat{\boldsymbol{\phi}} = \hat{\mathbf{x}} \times \hat{\boldsymbol{\rho}}$ are shown in Fig. 5(b).

For a wire lying along $y = -d/2$,

$$\mathbf{B}_1 = \frac{\mu_0 I}{2\pi}\frac{\hat{\mathbf{x}} \times \hat{\boldsymbol{\rho}}_1}{\rho_1}, \tag{6.46}$$

where (Fig. 5(b))

$$\boldsymbol{\rho}_1 = \boldsymbol{\rho} + \frac{d}{2}\hat{\mathbf{y}} = \left(y + \frac{d}{2}\right)\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{6.47}$$

Similarly, for the other wire with current flowing along the opposite direction,

$$\mathbf{B}_2 = -\frac{\mu_0 I}{2\pi}\frac{\hat{\mathbf{x}} \times \hat{\boldsymbol{\rho}}_2}{\rho_2}, \tag{6.48}$$

and

$$\boldsymbol{\rho}_2 = \boldsymbol{\rho} - \frac{d}{2}\hat{\mathbf{y}} = \left(y - \frac{d}{2}\right)\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{6.49}$$

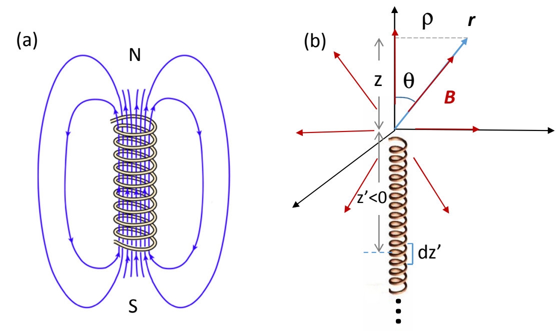

FIG. 52 (a) The magnetic field of a solenoid. (b) A semi-infinite solenoid along the negative z-axis.

Finally, the total magnetic field

$$\begin{aligned}\mathbf{B} &= \frac{\mu_0 I}{2\pi}\left(\frac{\hat{\mathbf{x}} \times \boldsymbol{\rho}_1}{\rho_1^2} - \frac{\hat{\mathbf{x}} \times \boldsymbol{\rho}_2}{\rho_2^2}\right) \qquad (6.50)\\ \text{or} \quad &= \frac{\mu_0 I}{2\pi}\left[\frac{\left(y + \frac{d}{2}\right)\hat{\mathbf{z}} - z\hat{\mathbf{y}}}{\left(y + \frac{d}{2}\right)^2 + z^2} - \frac{\left(y - \frac{d}{2}\right)\hat{\mathbf{z}} - z\hat{\mathbf{y}}}{\left(y - \frac{d}{2}\right)^2 + z^2}\right].\end{aligned}$$

C. Magnetic monopole

We have shown in Sec. A that the monopole potential of a localized current distribution is zero. Also, no magnetic monopole has been observed so far. Nevertheless, theory itself does not forbid the existence of magnetic monopole, as we'll show now.

The magnetic field produced by a finite solenoid is similar to that of a bar of magnetic (Fig. 6(a)). If a solenoid is very long, then its N-pole and S-pole are far away from each other. For a semi-infinite solenoid that extends from the origin to $z = -\infty$ (Fig. 6(b)), its S-pole is pushed to infinity and all of the magnetic field emanates from the N-pole — the opening at the origin. We can use it to simulate a magnetic monopole.

Example:

A semi-infinite solenoid along negative z-axis carries a current I. The cross section area is s, and the number of coils per unit length is n. Find out its vector potential and magnetic field.

Sol'n:

First, the vector potential of a current loop on the x-y plane with magnetic moment $\mathbf{m} = Is\hat{\mathbf{z}}$ is,

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}, \quad \mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}} \qquad (6.51)\\ &= \frac{\mu_0}{4\pi}m\frac{\rho}{(\rho^2 + z^2)^{3/2}}\hat{\boldsymbol{\phi}}. \qquad (6.52)\end{aligned}$$

Now, the number of loops within dz' at position z' (< 0) is ndz'. Its vector potential,

$$d\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}m(ndz')\frac{\rho}{[\rho^2 + (z - z')^2]^{3/2}}\hat{\boldsymbol{\phi}}. \tag{6.53}$$

After integration,

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} mn \int_{-\infty}^{0} dz' \frac{\rho}{[\rho^2 + (z-z')^2]^{3/2}} \hat{\boldsymbol{\phi}} \\ &= \frac{\mu_0}{4\pi} g \underbrace{\int_{-\infty}^{-z} dz' \frac{\rho}{(\rho^2 + z'^2)^{3/2}}}_{\equiv I(z)} \hat{\boldsymbol{\phi}}, \;\; g \equiv mn. \end{aligned} \quad (6.54)$$

Let $z' = -\rho \tan\varphi$, then $dz' = -\rho \sec^2\varphi d\varphi$, the integral becomes

$$\begin{aligned} I(z) &= \int_{\tan^{-1}\frac{z}{\rho}}^{\frac{\pi}{2}} d\varphi \frac{1}{\rho \sec\varphi} && (6.55) \\ &= \frac{1}{\rho} \sin\varphi \Big|_{\tan^{-1}\frac{z}{\rho}}^{\frac{\pi}{2}} && (6.56) \\ &= \frac{1}{\rho}\left(1 - \frac{z}{\sqrt{z^2+\rho^2}}\right). && (6.57) \end{aligned}$$

If we choose spherical coordinate ($\rho = r\sin\theta$), then

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{g}{r}\frac{1-\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}}. \quad (6.58)$$

Its magnetic field,

$$\begin{aligned} \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{\mathbf{r}} + \cdots \\ &= \frac{\mu_0 g}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}. \end{aligned} \quad (6.59)$$

This is valid as long as $\mathbf{r}$ is away from the solenoid. Finally, let $s \to 0$, and $n \to \infty$, such that $g = Isn$ remains fixed. Then Eqs. (6.58) and (6.59) are valid everywhere, except along the negative z-axis.

The monopole field $\mathbf{B}(\mathbf{r})$ is the same as the Coulomb field for a point charge and decreases as $1/r^2$. If magnetic monopole exists, then the divergence of $\mathbf{B}$ is no longer zero, but (for the example above)

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \mu_0 g \delta(\mathbf{r}). \quad (6.60)$$

Recall that the divergence of curl is always zero, so how can $\nabla\cdot\nabla\times\mathbf{A}(\mathbf{r})$ be non-zero here? In fact, $\nabla\cdot\nabla\times\mathbf{V}(\mathbf{r}) = 0$ is valid *only if* $\mathbf{V}(\mathbf{r})$ has no singularity, which is not the case for the $\mathbf{A}(\mathbf{r})$ here. The vector potential is singular along the negative z-axis, when $\theta = \pi$.

This string of singularity, called **Dirac string**, is an artifact of theory and cannot be detected in experiment if the monopole charge is quantized (Jackson, 1998). It's possible to simulate a monopole using a semi-infinite solenoid along the *positive* z-axis (or other places), then

$$\mathbf{A}'(\mathbf{r}) = -\frac{\mu_0}{4\pi}\frac{g}{r}\frac{1+\cos\theta}{\sin\theta}\hat{\boldsymbol{\phi}}, \quad (6.61)$$

which produces the same monopole field $\mathbf{B}(\mathbf{r})$. In this case, the Dirac string is along the positive z-axis. You may check that $\mathbf{A}$ and $\mathbf{A}'$ differ by a gauge transformation. That is, the position of the Dirac string is gauge dependent.

D. Force and energy

Consider a distribution of current in an external magnetic field $\mathbf{B}(\mathbf{r})$. Suppose the current is "rigid". That is, the external magnetic field cannot alter the distribution of current, then it feels a force,

$$\mathbf{F} = \int dv \mathbf{J}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}). \quad (6.62)$$

Assume the magnetic field varies slowly across the current, then we can expand it with respect to a point 0 near the current,

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}(0) + (\mathbf{r}\cdot\nabla)\mathbf{B}(0) + \cdots. \quad (6.63)$$

Thus,

$$\mathbf{F} \simeq \underbrace{\left(\int dv \mathbf{J}(\mathbf{r})\right)}_{=0} \times \mathbf{B}(0) + \int dv \mathbf{J}(\mathbf{r}) \times (\mathbf{r}\cdot\nabla)\mathbf{B}(0). \quad (6.64)$$

The first integral is zero, as has been shown in Eq. (6.12). When written in components, one has

$$F_i = \epsilon_{ijk} \int dv J_j r_\ell \nabla_\ell B_k. \quad (6.65)$$

Before moving on, recall that (Chap 1)

$$\begin{aligned} (\mathbf{u}\times\mathbf{v})_j &= \epsilon_{jkl} u_k v_l, && (6.66) \\ \epsilon_{klj}\epsilon_{kmn} &= \begin{vmatrix} \delta_{lm} & \delta_{ln} \\ \delta_{jm} & \delta_{jn} \end{vmatrix}. && (6.67) \end{aligned}$$

Also, for an arbitrary vector $\mathbf{w}$,

$$w_\ell \int dv r_\ell J_j = \frac{1}{2}\int dv \left[(\mathbf{r}\times\mathbf{J})\times\mathbf{w}\right]_j. \quad (6.68)$$

Pf:

$$\begin{aligned} \int dv \left[(\mathbf{r}\times\mathbf{J})\times\mathbf{w}\right]_j &= \int dv \epsilon_{jkl}(\mathbf{r}\times\mathbf{J})_k w_l && (6.69) \\ &= \int dv \underbrace{\epsilon_{jkl}\epsilon_{kmn}}_{=\delta_{lm}\delta_{jn}-\delta_{ln}\delta_{jm}} r_m J_n w_l \\ &= \int dv (r_l J_j w_l - r_j J_l w_l) && (6.70) \\ &= 2\int dv w_l r_l J_j, && (6.71) \end{aligned}$$

where we have switched the subscripts of $r_j J_l$ in the second term and used Eq. (6.13). Hence Eq. (6.68) follows. QED.

Replace w_l by $\nabla_l B_k$ (with a fixed k), then

$$\begin{aligned} \nabla_\ell B_k \int dv r_\ell J_j &= \frac{1}{2}\int dv \left[(\mathbf{r}\times\mathbf{J})\times\nabla B_k\right]_j \\ &= (\mathbf{m}\times\nabla)_j B_k. \end{aligned} \quad (6.72)$$

Thus Eq. (6.65) becomes

$$\mathbf{F} = (\mathbf{m} \times \nabla) \times \mathbf{B}. \tag{6.73}$$

With the help of

$$\begin{aligned}\nabla(\mathbf{a}\cdot\mathbf{b}) &= \mathbf{a}\nabla\cdot\mathbf{b} + \mathbf{b}\nabla\cdot\mathbf{a} \\ &+ (\mathbf{a}\times\nabla)\times\mathbf{b} + (\mathbf{b}\times\nabla)\times\mathbf{a},\end{aligned} \tag{6.74}$$

we have

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) \tag{6.75}$$
$$= -\nabla U, \tag{6.76}$$

where

$$U \equiv -\mathbf{m}\cdot\mathbf{B} \tag{6.77}$$

is the **magnetic dipole energy**.

1. Hyperfine structure

In an atom, such as the hydrogen atom, from the point of view of an orbiting electron, the nucleus is nearly a point since it is about 10^5 times smaller than the radius of the electron orbital. The magnetic field produced by the nucleus magnetic dipole moment $\mathbf{m}_N$ is (Eq. (6.37)),

$$\mathbf{B}_N(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}_N) - \mathbf{m}_N}{r^3} + \frac{2\mu_0}{3}\mathbf{m}_N\delta(\mathbf{r}). \tag{6.78}$$

An electron with dipole moment $\mathbf{m}_e$ would interact with $\mathbf{B}_N$. The Hamiltonian of the interaction is,

$$H_{HFS} = -\mathbf{m}_e\cdot\mathbf{B}_N(\mathbf{r}) \tag{6.79}$$
$$\begin{aligned}&= -\frac{\mu_0}{4\pi}\frac{3(\hat{\mathbf{r}}\cdot\mathbf{m}_e)(\hat{\mathbf{r}}\cdot\mathbf{m}_N) - \mathbf{m}_e\cdot\mathbf{m}_N}{r^3} \\ &\quad -\frac{2\mu_0}{3}\mathbf{m}_e\cdot\mathbf{m}_N\delta(\mathbf{r}).\end{aligned} \tag{6.80}$$

The first term is the typical *dipole-dipole interaction*, and the second term is a *contact interaction*.

For an s-orbital $\psi(\mathbf{r})$, which is non-zero at the origin (not so for a p-orbital or other non-s-orbitals, which vanishes at the origin), the second term causes an energy shift,

$$\Delta E_{HFS} = \langle\psi|H_{HFS}|\psi\rangle \tag{6.81}$$
$$= -\frac{2\mu_0}{3}\mathbf{m}_e\cdot\mathbf{m}_N|\psi(0)|^2. \tag{6.82}$$

The expectation of the first term in H_{HFS} is zero since s-orbital is spherical. As a result of this contact interaction, spin-up and spin-down electrons have slightly different energy levels (Fig. 7(a)). This is the **hyperfine structure** in atomic spectroscopy.

For an electron in the $1s$ orbital of H atom, $\Delta E_{HFS} \simeq 5.89\times10^{-6}$ eV. The electron transition between this two

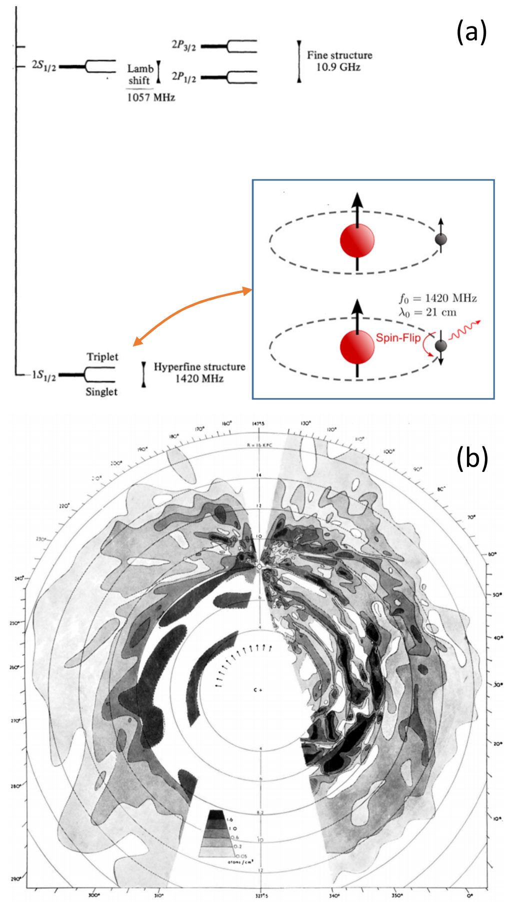

FIG. 53 (a) The hyperfine structure in hydrogen spectrum. (b) The global structure of the Galaxy determined by the 21 cm line. Fig. from Unknown, 1958.

energy levels emits a radio wave with wavelength 21 cm. This is the famous **21-centimeter line** in astrophysics that can help scientists mapping out the structure of the Galaxy (Fig. 7(b)).

In early times, some scientists thought that the magnetic moments in magnetic materials could be due to point magnetic charges, instead of tiny current loops (see Fig. 8). If so, then instead of magnetization current around the side surface, we should have magnetic charges on top and bottom surfaces, as in electric polarization.

However, from the calculation of hyperfine structure, we know that if the magnetic dipole is due to point charges, then the contact term should be $-\frac{\mu_0}{3}\mathbf{m}_N\delta(\mathbf{r})$, as in the case of electric dipole, instead of $+\frac{2\mu_0}{3}\mathbf{m}_N\delta(\mathbf{r})$. This would leads to a hyperfine splitting *half* of the present value, and in turn produces 42-cm hydrogen line (which is not observed). Thus, there are no magnetic monopoles hidden inside tiny magnetic dipoles.

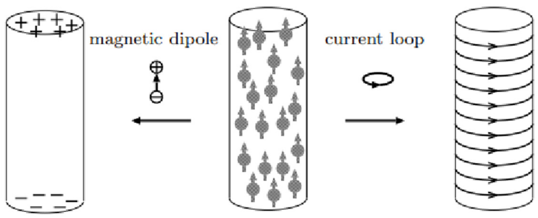

FIG. 54 Two possible scenarios of magnetic dipoles. Fig. from Kitano, 2006.

E. Macroscopic magnetizable medium

Consider a magnetic medium that is composed of small current loops. If the magnetic moment of the i-th element is $\mathbf{m}_i$, then we can define the **magnetization** as,

$$\mathbf{M}(\mathbf{r}') = \frac{\sum\limits_{i \text{ in } \Delta V} \mathbf{m}_i}{\Delta V}, \tag{6.83}$$

where ΔV is a volume element around $\mathbf{r}'$. The volume element is microscopically large but macroscopically small, so that there are many elements in ΔV, but it remains a point from human's point of view.

A volume element ΔV has magnetic moment $\mathbf{m} = \mathbf{M}\Delta V$ and produces a vector potential,

$$\Delta \mathbf{A}(\mathbf{r}) \simeq \frac{\mu_0}{4\pi}\left[\frac{\mathbf{J}(\mathbf{r}')\Delta V}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mathbf{M}(\mathbf{r}')\Delta V \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right].$$

After integration, we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int_V dv' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \int_V dv' \frac{\mathbf{M}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right], \tag{6.84}$$

where V is the volume of the material. Write

$$\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}, \tag{6.85}$$

use

$$\nabla \times (f\mathbf{v}) = \nabla f \times \mathbf{v} + f \nabla \times \mathbf{v}, \tag{6.86}$$

and integrate by parts, the second integral can be written as

$$\begin{aligned}\int_V dv' \mathbf{M}(\mathbf{r}') \times \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \int_V dv' \frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &- \int_V dv' \nabla' \times \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right).\end{aligned}$$

We then use

$$\int_V dv \nabla \times \mathbf{v} = \int_S d\mathbf{s} \times \mathbf{v}, \tag{6.87}$$

where S is the boundary of V, and write

$$\int_V dv' \nabla' \times \left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right) = \int_S d\mathbf{s}' \times \frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{6.88}$$

If follows that,

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int_V dv' \frac{\mathbf{J}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &+ \frac{\mu_0}{4\pi}\int_S ds' \frac{\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{n}}}{|\mathbf{r}-\mathbf{r}'|}.\end{aligned} \tag{6.89}$$

The numerator of the first integral can be considered as an effective current density $\mathbf{J}_{eff} = \mathbf{J} + \mathbf{J}_m$, where

$$\mathbf{J}_m(\mathbf{r}) \equiv \nabla \times \mathbf{M}(\mathbf{r}) \tag{6.90}$$

is the **magnetization current density**. The numerator of the second integral is the **magnetization surface current density**,

$$\mathbf{K}_m(\mathbf{r}) \equiv \mathbf{M}(\mathbf{r}) \times \hat{\mathbf{n}}, \tag{6.91}$$

where $\mathbf{r}$ is located on the surface.

Note that instead of integrating over the material body V, we can also integrate over the whole space, then S is the surface at infinity, and the surface integral vanishes since $M = 0$ at infinity. These two choices of V give the same result of $\mathbf{A}$, since in the second choice, $\mathbf{J}_m$ would automatically pick up the surface current on boundary (see next subsection). In what follows, we prefer to integrate over the whole space, so that the surface integral in Eq. (6.89) can be dispensed with.

Now, since the current density in the volume integral above directly links with the one in Ampère's law (see Chap 2), we have

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{eff} = \mu_0 \left(\mathbf{J} + \nabla \times \mathbf{M}\right). \tag{6.92}$$

Introduce an H field,

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}, \tag{6.93}$$

$$\text{then} \quad \nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}). \tag{6.94}$$

This is Ampère's law in material. The official term for $\mathbf{B}$ is **magnetic flux density**, which is in units of Tesla (N s/C m). The H field is called **magnetic field strength**, which is in units of A/m. For simplicity, we'll call them B field and H field, or simply magnetic field (for both) if the context is clear.

Note that

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \tag{6.95}$$

We call a magnet *simple* if it is isotropic and linear. That is, its $\mathbf{M}$ is proportional to and aligns with $\mathbf{H}$,

$$\mathbf{M} = \chi_m \mathbf{H}, \tag{6.96}$$

and

$$\mathbf{B} = \mu_0(1 + \chi_m)\mathbf{H} = \mu\mathbf{H}, \tag{6.97}$$

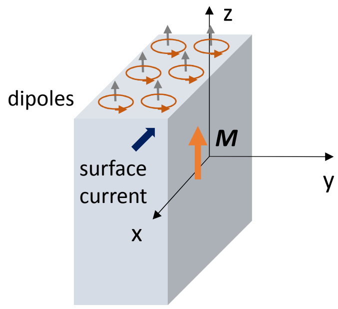

FIG. 55 Distribution of magnetic dipoles in the half-space $y < 0$.

where χ_m is the **magnetic susceptibility**, and μ the **magnetic permeability** of material.

For **paramagnetic material** ,

$$\mathbf{M} \parallel \mathbf{H}, \quad \text{and } \chi_m > 0. \tag{6.98}$$

For **diamagnetic material**,

$$\mathbf{M} \parallel -\mathbf{H}, \quad \text{and } \chi_m < 0. \tag{6.99}$$

The magnitude of χ_m is typically of the order of 10^{-5}. A simple magnet such as soft iron can have $\chi_m \sim 10^4$. The χ_m of hard ferromagnet materials can be as large as 10^6, but they are not simple magnets.

1. Magnetization current

Non-uniform magnetization generates effective current, $\mathbf{J}_m = \nabla \times \mathbf{M}$. We'll use a simple example to illustrate this: In Fig. 9 there is a semi-infinite magnet with uniform magnetization,

$$\mathbf{M} = M_0 \theta(-y) \hat{z}. \tag{6.100}$$

Its magnetization current density is,

$$\mathbf{J}_m = \nabla \times \mathbf{M} = -M_0 \delta(y) \hat{x}. \tag{6.101}$$

That is, magnetization current flows only on the surface of the magnet. In the figure, molecular currents generate magnetic dipoles. Near the interface between neighboring current loops, the currents flow along opposite directions. Thus, there is no current inside the bulk, and only the outer-most current exposed.

Note that the magnetization currents in the example are bounded to molecules. They cannot flow away like conduction current in metals.

Since the magnetization current flows on a surface, we can describe it with surface current density $\mathbf{K}_m$,

$$\begin{aligned} \mathbf{K}_m = \int dy \mathbf{J}_m &= -\int dy M_0 \delta(y) \hat{\mathbf{x}} && (6.102) \\ &= -M_0 \hat{\mathbf{x}} = \mathbf{M}_s \times \hat{\mathbf{n}}, && (6.103) \end{aligned}$$

where $\mathbf{M}_s$ is the magnetization on the surface.

2. Boundary condition

In previous chapter, we have learned about the boundary condition for magnetic field,

$$\begin{aligned} \hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) &= 0, && (6.104) \\ \hat{\mathbf{n}} \times (\mathbf{B}_2 - \mathbf{B}_1) &= \mu_0 \mathbf{K}. && (6.105) \end{aligned}$$

In the presence of magnetic materials, the boundary condition would depend on magnetization and needs be rederived. Let's start from the integral form of the Maxwell equations,

$$\begin{aligned} \int_S d\mathbf{s} \cdot \mathbf{B} &= 0, && (6.106) \\ \oint_C d\mathbf{r} \cdot \mathbf{H} &= I, && (6.107) \end{aligned}$$

where I is the current flowing through C, *not including the magnetization current.*

As shown in Fig. 10(a), near the boundary surface, we can choose the S in Eq. (6.106) to be a small pillar box with area ds and *nearly zero thickness*, then

$$\int_S d\mathbf{s} \cdot \mathbf{B} \simeq \mathbf{B}_1 \cdot ds(-\hat{\mathbf{n}}) + \mathbf{B}_2 \cdot ds\hat{\mathbf{n}} = 0, \tag{6.108}$$

where $\hat{\mathbf{n}}$ points from region 1 to region 2. Hence, the normal components

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0, \tag{6.109}$$

which is the same as Eq. (6.104).

Choose the C in Eq. (6.107) to be a small rectangular loop perpendicular to the current flow. Suppose the loop has width d and *nearly zero height*, then

$$\oint_C d\mathbf{r} \cdot \mathbf{H} \simeq \mathbf{H}_1 \cdot (-\mathbf{d}) + \mathbf{H}_2 \cdot \mathbf{d} = I, \tag{6.110}$$

where $\mathbf{d} = d\hat{\mathbf{d}}$ and $\hat{\mathbf{d}} = \hat{\mathbf{J}} \times \hat{\mathbf{n}}$, as shown in figure. Hence

$$\begin{aligned} (\mathbf{H}_2 - \mathbf{H}_1) \cdot \hat{\mathbf{d}} &= K, && (6.111) \\ \text{or } \hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) &= \mathbf{K}, && (6.112) \end{aligned}$$

which replaces Eq. (6.105).

F. Magnetostatic energy

The magnetostatic energy of a current distribution equals the total work required to assemble the current, starting from the state when there is no current. The increase of current leads to the increase of magnetic field, which induces an electric field $\mathbf{E}$ that interacts with the current.

For the reason above, even though we are discussing magneto*static* energy, Faraday's law needs be used,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{6.113}$$

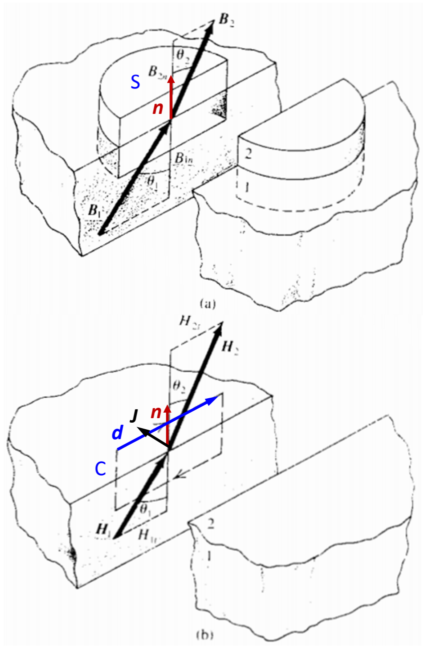

FIG. 56 (a) Gaussian surface for the flux of $\mathbf{B}$. (b) Ampère loop for the circulation of $\mathbf{H}$. Fig. from Lorrain and Corson, 1970.

It is assumed that the current builds up slowly so the process is *quasi-static*. The electromagnetic energy in a *dynamic* system will be discussed in Chap 15.

Suppose charge $\rho\Delta V$ is displaced by $\Delta\mathbf{r}$ due to fields, the mechanical work done *by electromagnetic field* on charged particles is,

$$\Delta w_m = \rho\Delta V(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\Delta\mathbf{r}. \tag{6.114}$$

The rate of total work done is

$$\begin{aligned} \frac{dW_m}{dt} &= \int dv\rho(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\mathbf{v} && (6.115)\\ &= \int dv\mathbf{J}\cdot\mathbf{E}, \quad \mathbf{J}=\rho\mathbf{v}. && (6.116) \end{aligned}$$

Note that *the magnetic force does no work.*

According to **Lenz's law**, the induced field $\mathbf{E}$ *opposes* the increase of current. To build up the current, an external agent must provide W_{ext} to work *against* W_m. It is *this* external work that increases the energy U_B of the system.

For simplicity, consider a thin wire of loop C. Replace $dv\mathbf{J}$ with $Id\mathbf{r}$, then

$$\frac{dW_m}{dt} = I\oint_C d\mathbf{r}\cdot\mathbf{E} = I\mathcal{E}, \tag{6.117}$$

where $\mathcal{E}$ is the **electromotive force** around C. The rate of external work is,

$$\begin{aligned} \frac{dW_{ext}}{dt} &= -I\int_C d\mathbf{r}\cdot\mathbf{E} && (6.118)\\ &= -I\int_S d\mathbf{s}\cdot\nabla\times\mathbf{E} && (6.119)\\ &= I\int_S d\mathbf{s}\cdot\frac{\partial\mathbf{B}}{\partial t} && (6.120)\\ &= I\frac{d\Phi}{dt}, && (6.121) \end{aligned}$$

where S is a surface (not moving) bounded by C, and Φ is the magnetic flux through S. Finally, in a time δt,

$$\delta W_{ext} = \frac{dW_{ext}}{dt}\delta t = I\delta\Phi, \tag{6.122}$$

hence

$$\delta U_B = \delta W_{ext} = I\delta\Phi. \tag{6.123}$$

This is the first form of δU_B.

With Stoke's theorem, the magnetic flux can be written as,

$$\Phi = \int_S d\mathbf{s}\cdot\nabla\times\mathbf{A} = \oint_C d\mathbf{r}\cdot\mathbf{A}. \tag{6.124}$$

The current is held fixed in δt, hence

$$\delta U_B = I\oint_C d\mathbf{r}\cdot\delta\mathbf{A}. \tag{6.125}$$

For a general current distribution, replace $Id\mathbf{r}$ with $dv\mathbf{J}$, then

$$\delta U_B = \int dv\mathbf{J}\cdot\delta\mathbf{A}. \tag{6.126}$$

This is the second form of δU_B.

We can also write U_B in terms of magnetic field. Recall that $\nabla\times\mathbf{H}$ equals J (not including the magnetization current), thus

$$\delta U_B = \int dv(\nabla\times\mathbf{H})\cdot\delta\mathbf{A}. \tag{6.127}$$

Using

$$\nabla\cdot(\mathbf{u}\times\mathbf{v}) = (\nabla\times\mathbf{u})\cdot\mathbf{v} - (\nabla\times\mathbf{v})\cdot\mathbf{u}, \tag{6.128}$$

then

$$\begin{aligned} \delta U_B &= \int dv(\nabla\times\delta\mathbf{A})\cdot\mathbf{H} + \int dv\nabla\cdot(\mathbf{H}\times\delta\mathbf{A})\\ &= \int dv\ \delta\mathbf{B}\cdot\mathbf{H}. && (6.129) \end{aligned}$$

The second integral can be converted to an integral over a boundary surface at infinity and vanishes when the field distribution is localized. This is the third form of δU_B.

Back to the first form of δU_B. Suppose the current (flux) increases from 0 to a final value I (Φ). In an intermediate state,

$$I(\lambda) = \lambda I, \text{ and } \delta\Phi(\lambda) = \delta\lambda\Phi \quad (0 \le \lambda \le 1), \tag{6.130}$$

then

$$\begin{aligned} U_B &= \int I(\lambda)\delta\Phi(\lambda) && (6.131)\\ &= \int_0^1 d\lambda\lambda\, I\Phi && (6.132)\\ &= \frac{1}{2}I\Phi. && (6.133) \end{aligned}$$

Similarly, the second form also has the factor 1/2,

$$U_B = \frac{1}{2}\int dv\mathbf{J}\cdot\mathbf{A}. \tag{6.134}$$

For the third form, in a simple magnet (such as a paramagnetic or a diamagnetic material), $\mathbf{B}$ is proportional to $\mathbf{H}$, and we can also have

$$U_B = \frac{1}{2}\int dv\mathbf{B}\cdot\mathbf{H}. \tag{6.135}$$

The integrand is the **energy density** for magnetic field,

$$u_B = \frac{1}{2}\mathbf{B}\cdot\mathbf{H} = \frac{\mu}{2}H^2. \tag{6.136}$$

However, for non-simple magnet (such as a ferromagnet), the original form in Eq. (6.129) needs be used to compute the change of energy step by step.

Problem:

1. Suppose a current distribution *outside* a sphere with volume V produces a magnetic field $\mathbf{B}(\mathbf{r})$. Show that the magnetic field averaged over the sphere (which has no current inside) equals the field at the center of the sphere,

$$\langle\mathbf{B}(\mathbf{r})\rangle_V \equiv \frac{1}{V}\int_V dv\mathbf{B}(\mathbf{r}) = \mathbf{B}(0).$$

2. A point magnetic dipole at the origin produces a magnetic field,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}}{r^3}, \quad r > 0.$$

Suppose $\mathbf{m} = m\hat{\mathbf{z}}$. Show that the field averaged over a sphere centered at the origin is zero,

$$\int_V dv\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V dv\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}}{r^3} = 0.$$

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