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Basics of topological insulator

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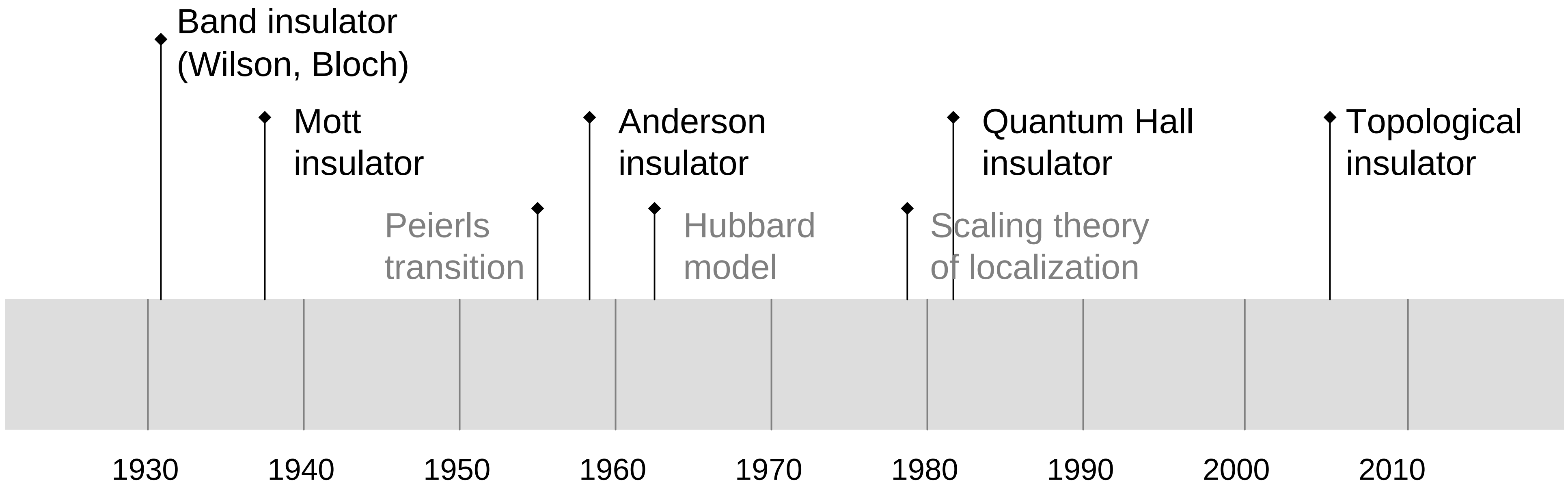
A brief history of insulators



Band insulator (Wilson, Bloch)

Mott insulator

Peierls transition

Anderson insulator

Hubbard model

Scaling theory of localization

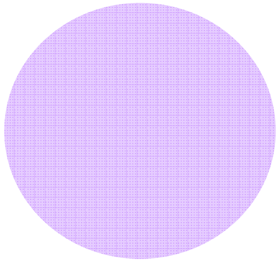
Quantum Hall insulator

Topological insulator

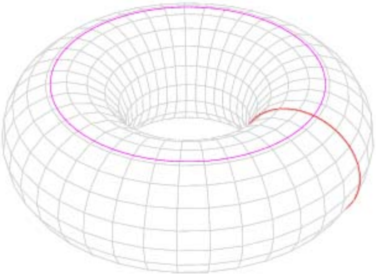
2D TI is also called QSHI

- Gauss-Bonnet theorem for a 2D surface with boundary

$$\int_M da\, G + \int_{\partial M} ds\, k_g = 2\pi\,\chi(M, \partial M)$$



$$\chi = 2$$



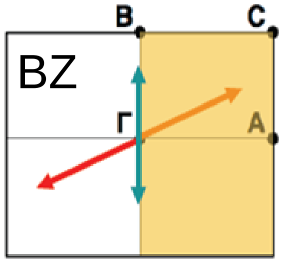
$$\chi = 0$$

$$\chi = 1$$

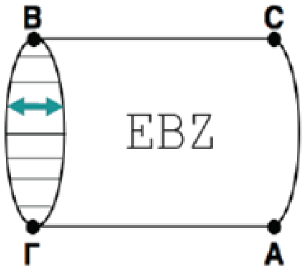
- Quantum Hall effect (2D lattice fermion in magnetic field)

$$\int_{2D\,BZ} d^2k\, \Omega_Z = 2\pi C_1 \qquad \sigma_H = C_1 \frac{e^2}{h}$$

2D Lattice fermion with time reversal symmetry (TRS)

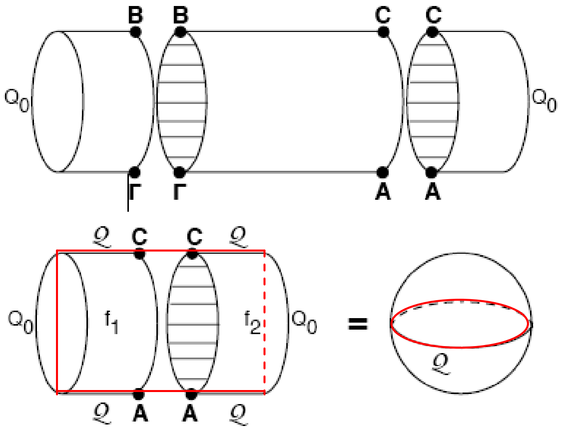


- Without B field, Chern number $C_1 = 0$
- Bloch states at *k*, -*k* are not independent



- EBZ is a cylinder, not a closed torus.

∴ No obvious quantization.



Moore and Balents PRB 07

- C_1 of closed surface may depend on caps
- C_1 of the EBZ (mod 2) is independent of caps

(topological insulator, TI)

→ 2 types of insulator, the “0-type”, and the “1-type”

- 2D TI characterized by a Z_2 number (Fu and Kane 2006)

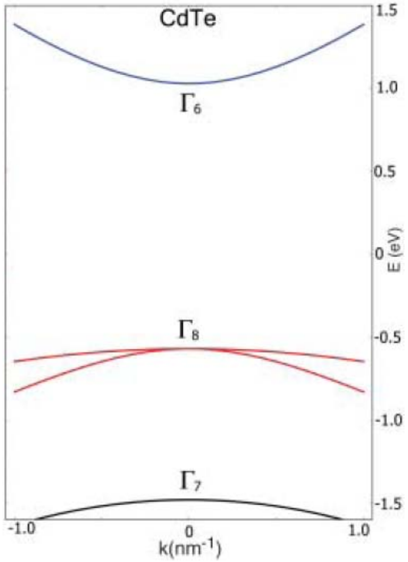
$$\nu = \frac{1}{2\pi}\left[\int_{EBZ} d^2k\,\Omega - \oint_{\partial(EBZ)} dk\,A\right] \text{mod } 2$$

~ Gauss-Bonnet theorem with edge

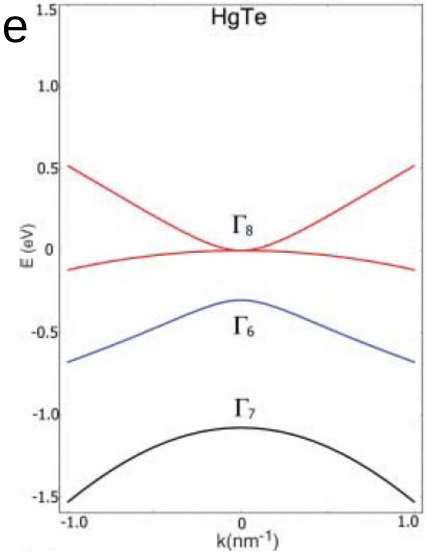
How can one get a TI?

: band inversion due to SO coupling

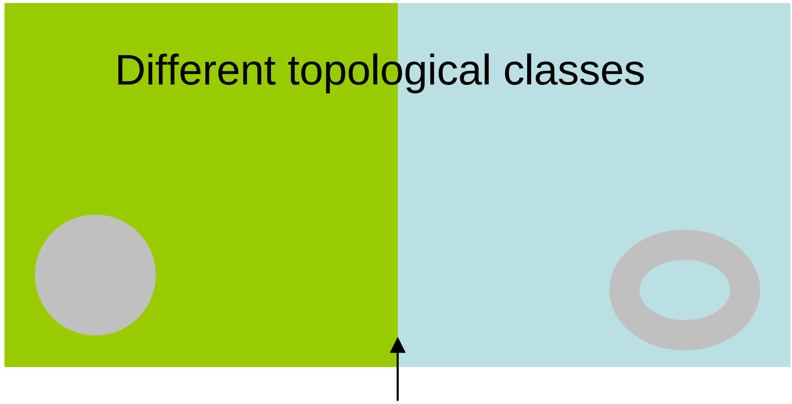
0-type



1-type

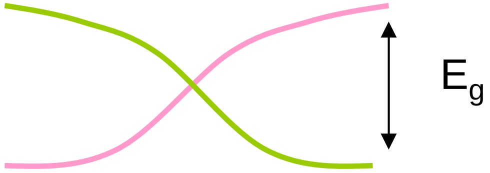


Bulk-edge correspondence



Different topological classes

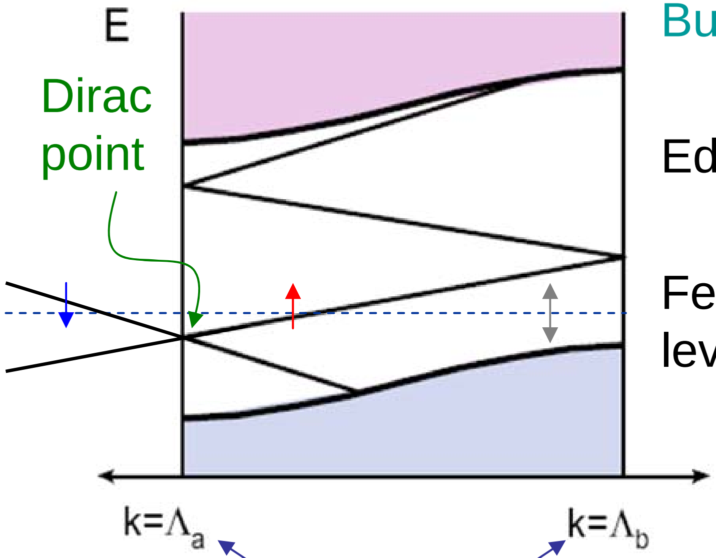
Semiclassical (adiabatic) picture:
energy levels must cross
(otherwise topology won’t change).



→ gapless states bound to the interface,
which are protected by topology.

Topological Goldstone theorem?

Bulk-edge correspondence in TI



Dirac point

Bulk states

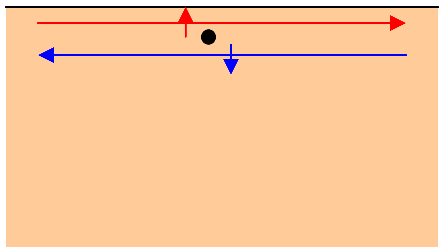
Edge states

Fermi level

TRIM

(2-fold degeneracy at TRIM due to Kramer's degeneracy)

helical edge states



robust

backscattering by non-magnetic impurity forbidden

Topological insulators in real life

SO coupling only
10^{-3} meV

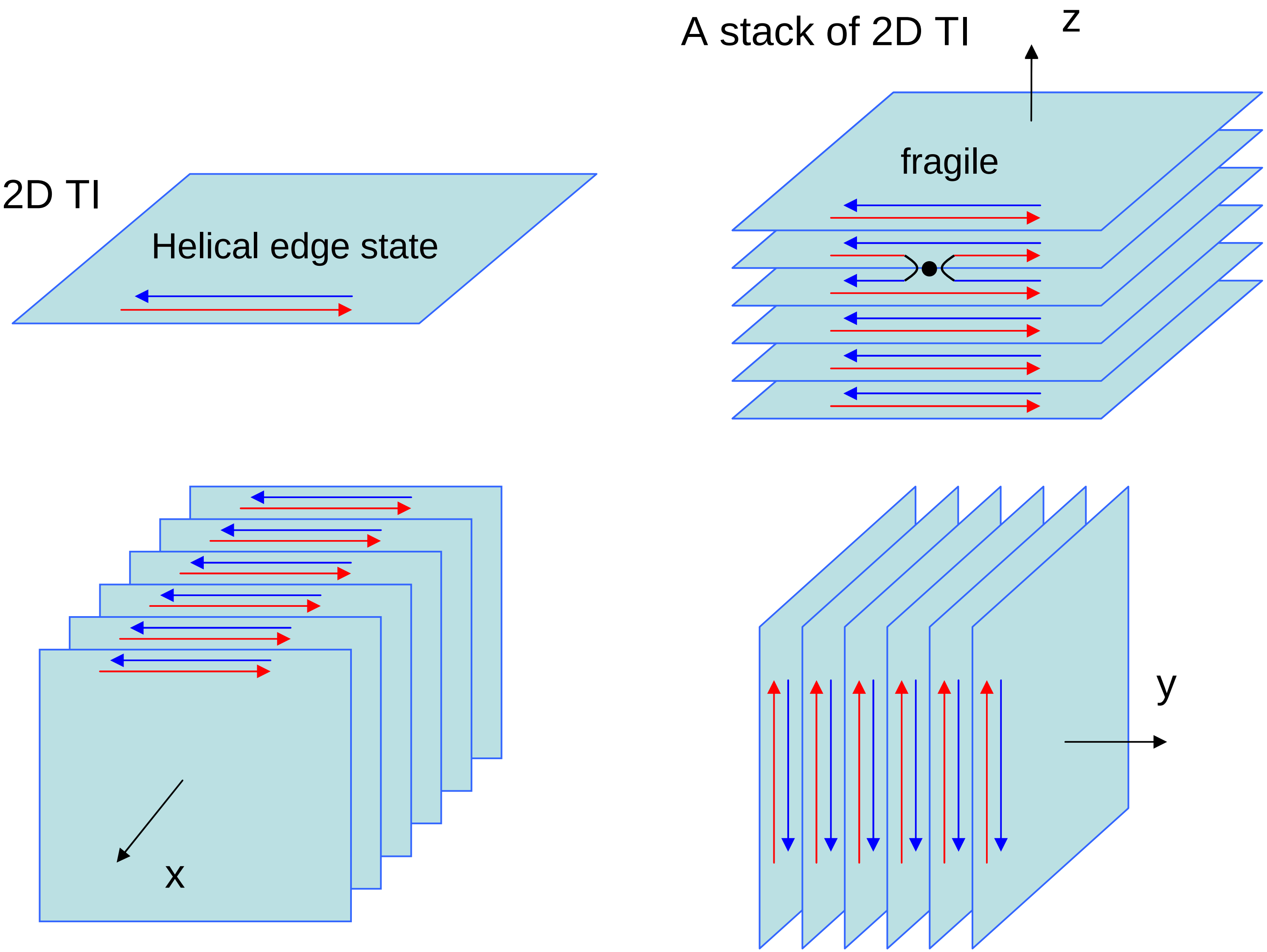
2D

- Graphene (Kane and Mele, PRLs 2005)
- HgTe/CdTe QW (Bernevig, Hughes, and Zhang, Science 2006)
- Bi bilayer (Murakami, PRL 2006)
- …

3D

- $Bi_{1-x}Sb_x$, α-Sn ... (Fu, Kane, Mele, PRL, PRB 2007)
- Bi_2Te_3 (0.165 eV), Bi_2Se_3 (0.3 eV) … (Zhang, Nature Phys 2009)
- The half Heusler compounds (LuPtBi, YPtBi …) (Lin, Nature Material 2010)
- thallium-based III-V-VI_2 chalcogenides ($TlBiSe_2$ …) (Lin, PRL 2010)
- $Ge_nBi_{2m}Te_{3m+n}$ family ($GeBi_2Te_4$ …)
- …

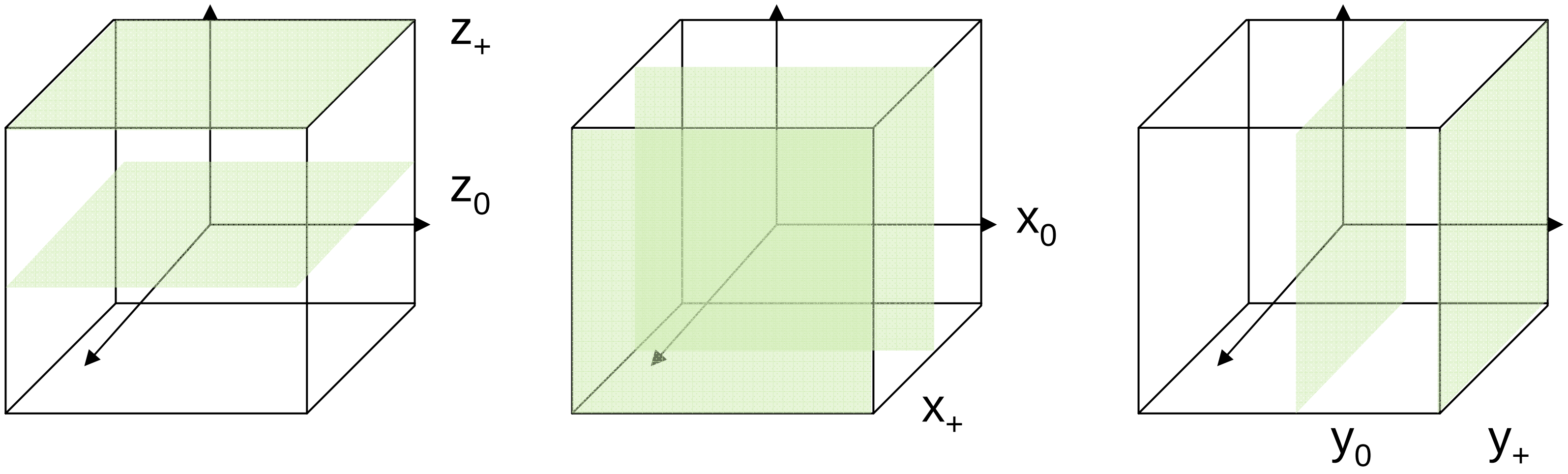
- strong spin-orbit coupling
- band inversion



3 TI indices

3D TI: 3 weak TI indices:

Eg., (x_0, y_0, z_0)

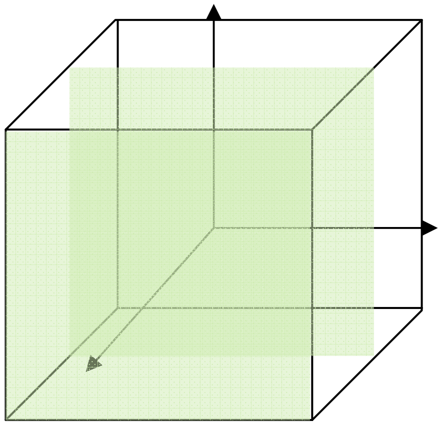


1 strong TI index: ν_0

$\nu_0 = z_+ - z_0$

(= $y_+ - y_0 = x_+ - x_0$)

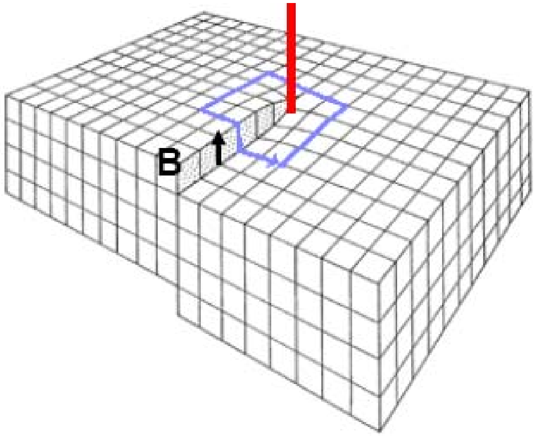
difference between
two 2D TI indices



Fu, Kane, and Mele PRL 07
Moore and Balents PRB 07
Roy, PRB 09

Weak TI index

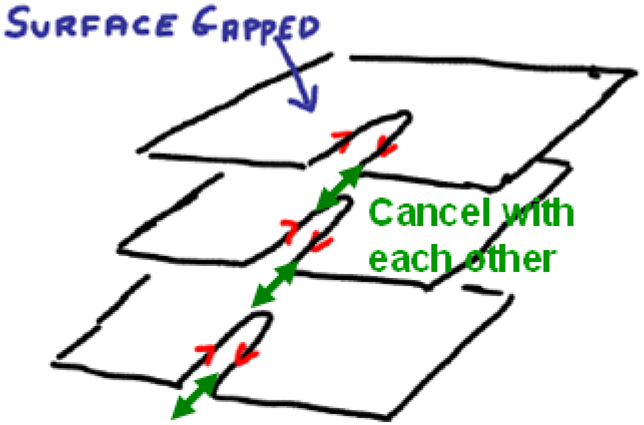
Screw dislocation of TI

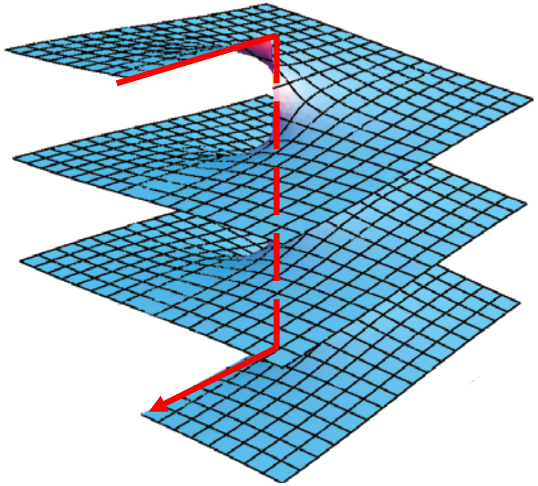


- not localized by disorder
- half of a regular quantum wire

Ran Y et al, Nature Phys 2009

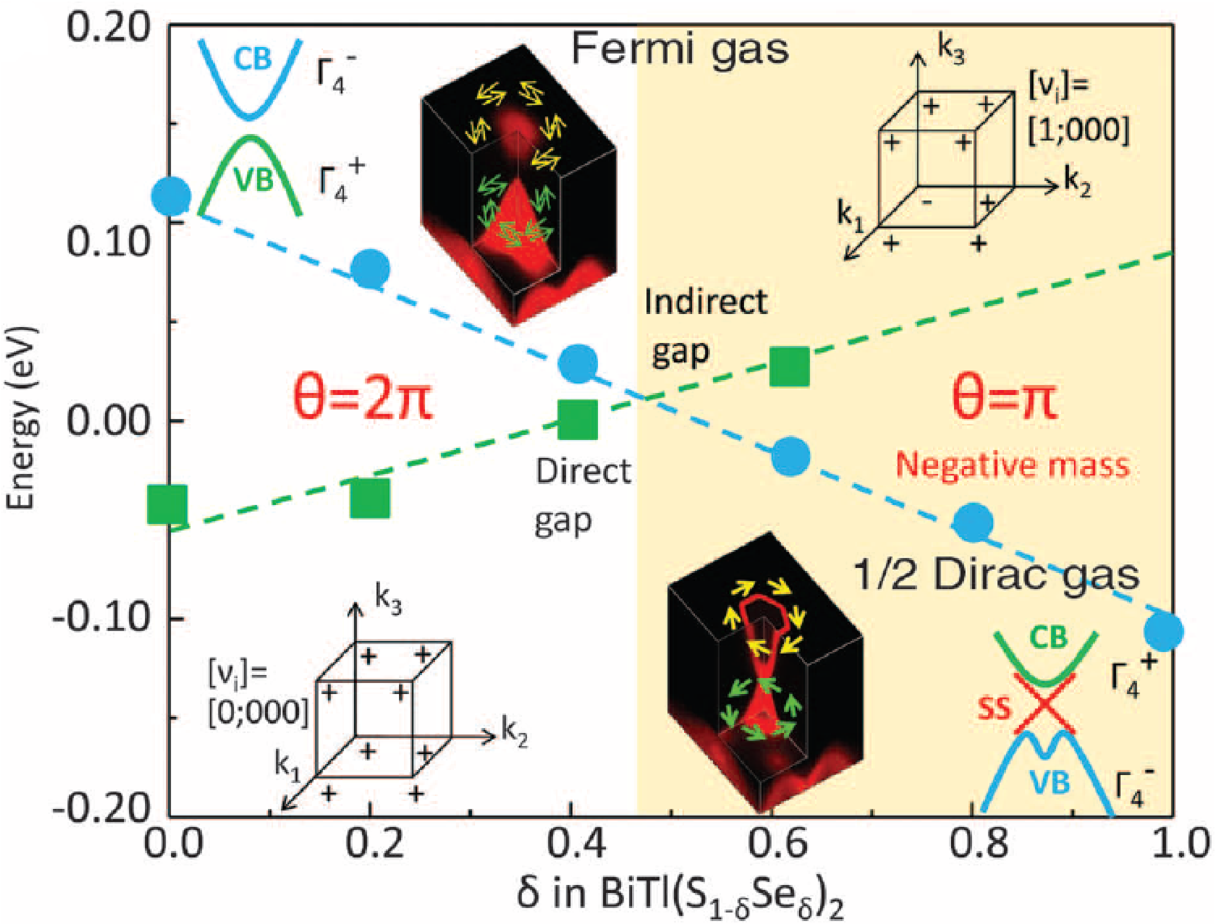
A stack of 2D TI





From Vishwanath's slides

Band inversion,
parity change,
spin-momentum locking (helical Dirac cone)

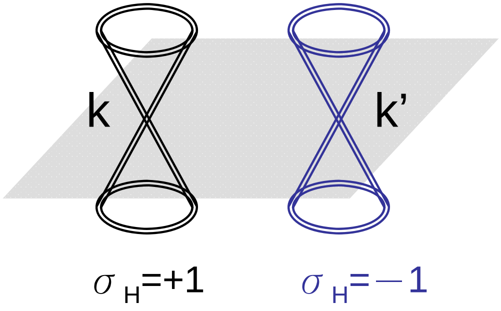


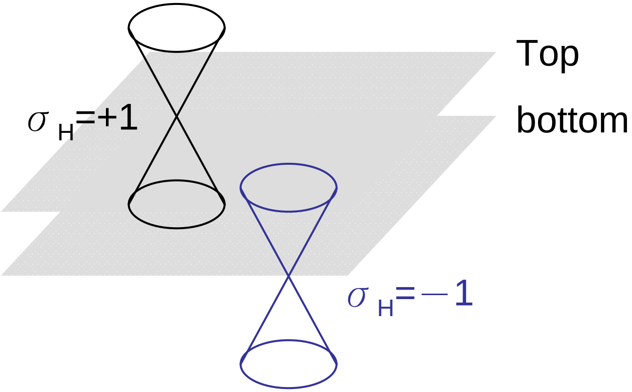
S.Y. Xu et al Science 2011

Dirac point:

Graphene vs. Topological insulator

Even number	Odd number (on one side)
located at Fermi energy	not located at E_F
half integer QHE (×4)	half integer QHE (if E_F is located at DP)
Spin is not locked with k	spin is locked with k
can be opened by substrate	cannot be opened

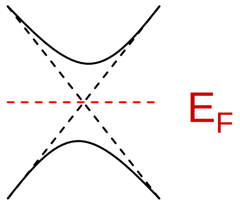




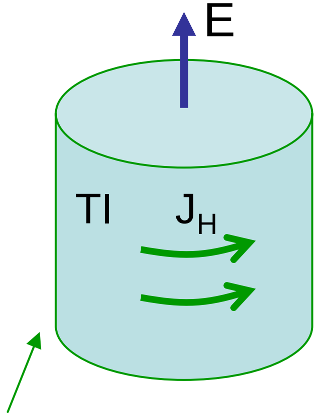
Electromagnetic response

Axion electrodynamics

First, a heuristic argument:



→ half-IQHE



Surface state ~2 DEG

- Hall current $J_H = \frac{e^2}{2h} E$
- Induced magnetization $M = \frac{e^2}{2h} E$

"magneto-electric" coupling

Effective Lagrangian for EM wave

$$L_{EM} = L_0 + L_{axion}$$

$$L_0 = \frac{1}{8\pi^2}\left(\frac{E^2}{c^2} - B^2\right)$$

"axion" coupling

$$L_{axion} = \frac{e^2}{2h}\frac{1}{c}\vec{E}\cdot\vec{B} = \alpha\frac{\Theta}{4\pi^2}\vec{E}\cdot\vec{B}$$

$$\text{note: } \alpha = \frac{4\pi}{c}\frac{e^2}{2h} = \frac{e^2}{\hbar c} \sim \frac{1}{137}$$

For systems with time-reversal symmetry,
Θ can only be 0 (usual insulator) or π (TI)

Cr_2O_3: $\theta \sim \pi/24$ (TRS is broken)

Maxwell eqs with axion coupling

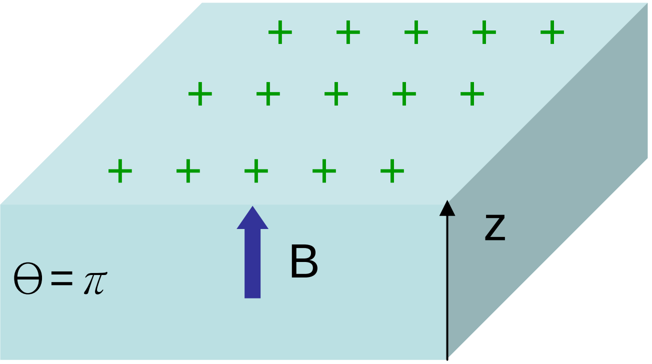
$$\nabla\cdot\left(\vec{E}+\alpha\frac{\Theta}{\pi}\vec{B}\right)=4\pi\rho$$

$$\nabla\times\left(\vec{B}-\alpha\frac{\Theta}{\pi}\vec{E}\right)=\frac{4\pi}{c}\vec{J}+\frac{\partial}{c\partial t}\left(\vec{E}+\alpha\frac{\Theta}{\pi}\vec{B}\right)$$

$$\nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{E}=-\frac{\partial}{c\partial t}\vec{B}$$

$$\rho_\Theta=\frac{\alpha}{4\pi}\delta(z)B_z$$



Effective charge and effective current

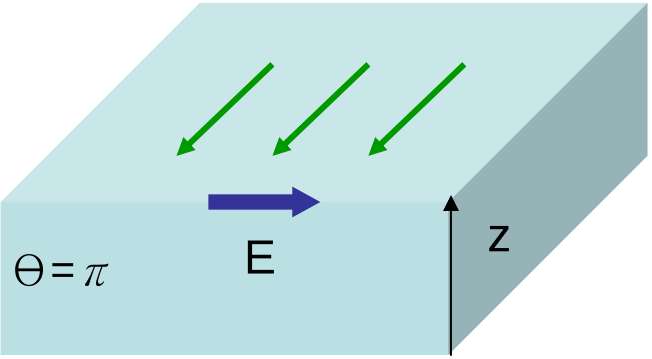
$$\nabla\cdot\vec{E}=4\pi\left(\rho+\rho_\Theta\right)$$

$$\rho_\Theta=-\frac{\alpha}{4\pi^2}\nabla\cdot\left(\Theta\vec{B}\right)$$

$$\nabla\times\vec{B}=\frac{4\pi}{c}\left(\vec{J}+\vec{J}_\Theta\right)+\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

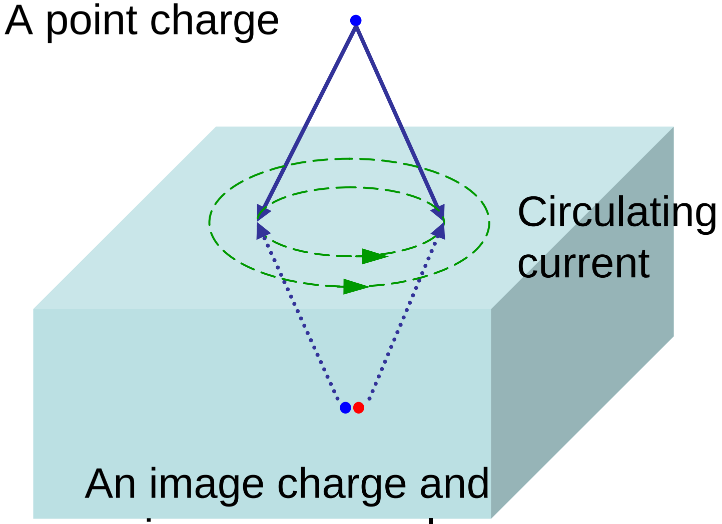
$$\vec{J}_\Theta=\frac{c\alpha}{4\pi^2}\nabla\times\left(\Theta\vec{E}\right)+\frac{\alpha}{4\pi^2}\frac{\partial}{\partial t}\left(\Theta\vec{B}\right)$$

$$\vec{J}_\Theta=-\frac{c\alpha}{4\pi}\delta(z)\hat{z}\times\vec{E}\rightarrow\sigma_{xy}=\frac{1}{2}\frac{e^2}{h}$$



Static:

Magnetic monopole in TI



A point charge

Circulating current

An image charge and an image monopole

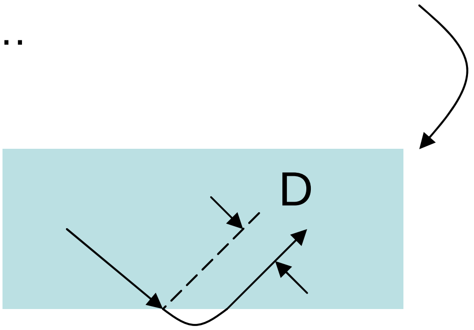
Qi, Hughes, and Zhang, Science 2009

Dynamic:

Optical signatures of TI?

axion effect on

- Snell's law
- Fresnel formulas
- Brewster angle
- Goos-Hänchen effect
- …



Longitudinal shift of reflected beam (total reflection)

Chang and Yang, PRB 2009

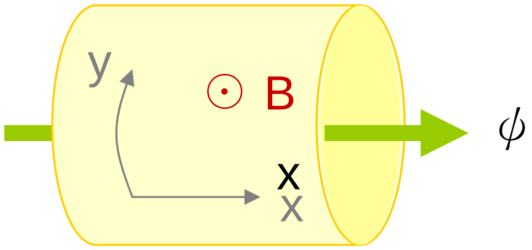
(Magnetic overlayer not included)

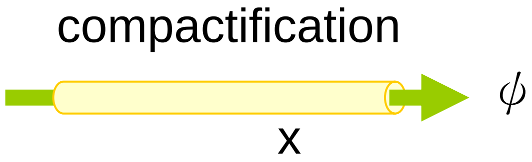
Dimensional reduction

Topological field theory

2D quantum Hall effect → 1D charge pump

Laughlin's argument (1981):





Berry curvature:

$f_{ij} = \partial_i a_j - \partial_j a_i$

Berry connection:

$a_k = i\langle u|\partial_k|u\rangle$

$A_y(x,t) \sim \theta(x,t)$, a parameter

polarization

$$C_1 = \frac{1}{2\pi}\int d^2k\, f_{xy}(\vec{k}) \longrightarrow P(\theta) = \frac{1}{2\pi}\int dk_x a_x$$

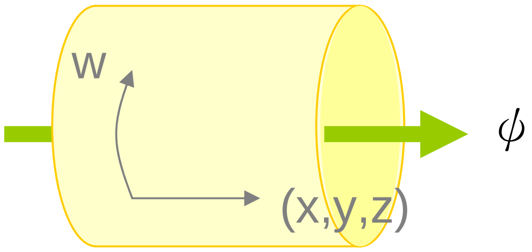
$$S_{CS} = \frac{C_1}{4\pi}\int d^2x dt\, \varepsilon^{\mu\nu\tau} A_\mu \partial_\nu A_\tau \longrightarrow \tilde{S} = \int dxdt\, P \varepsilon^{\alpha\beta}\partial_\alpha A_\beta$$

$$j^\mu = \frac{\delta S_{CS}}{\delta A_\mu} = \frac{C_1}{2\pi}\varepsilon^{\mu\nu\tau}\partial_\nu A_\tau$$

$$j_\alpha = -\varepsilon_{\alpha\beta}\partial_\beta P$$

Qi, Hughes, and Zhang PRB 2008

4D quantum Hall effect → 3D topological insulator



compactification

$$S=\frac{C_2}{24\pi^2}\int d^4x dt\,\varepsilon^{\mu\nu\rho\sigma\tau}A_\mu\partial_\nu A_\rho\partial_\sigma A_\tau$$

→

$$\tilde{S}=\frac{1}{8\pi^2}\int d^3x dt\,\Theta\varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha A_\beta\partial_\gamma A_\delta$$

$$j^\mu=\frac{C_2}{8\pi^2}\varepsilon^{\mu\nu\rho\sigma\tau}\partial_\nu A_\rho\partial_\sigma A_\tau$$

$$j^\alpha=\frac{1}{4\pi^2}\varepsilon^{\alpha\beta\gamma\delta}\partial_\beta\Theta\partial_\gamma A_\delta$$

nonlinear response.

$$C_2=\frac{1}{32\pi^2}\int d^4k\,\varepsilon^{ijkl}\mathrm{tr}\left(\underset{\sim}{f}_{ij}\underset{\sim}{f}_{kl}\right)$$

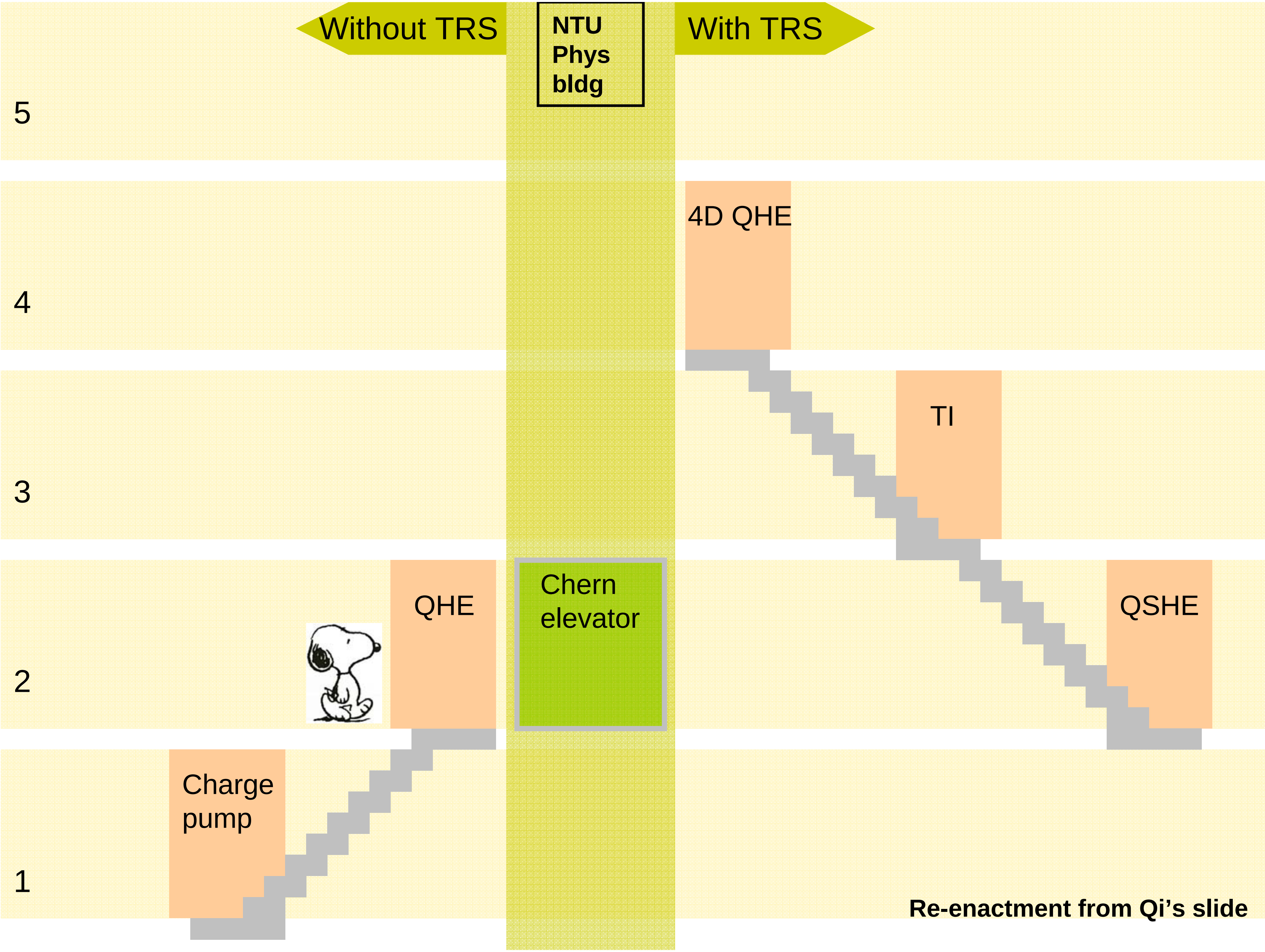
$$\underset{\sim}{f}_{ij}=\partial_i\underset{\sim}{a}_j-\partial_j\underset{\sim}{a}_i-i[\underset{\sim}{a}_i,\underset{\sim}{a}_j]$$

$$a_k^{mn}=i\langle u_m|\partial_k|u_n\rangle$$

→

$$\Theta=\frac{1}{8\pi}\int_{BZ}d^3k\,\varepsilon^{ijk}\mathrm{tr}\left(\underset{\sim}{a}_i\underset{\sim}{f}_{jk}+\frac{i}{3}\underset{\sim}{a}_i\left[\underset{\sim}{a}_j,\underset{\sim}{a}_k\right]\right)$$

Qi, Hughes, and Zhang PRB 2008



Without TRS

NTU Phys bldg

With TRS

5

4

4D QHE

3

TI

2

QHE

Chern elevator

QSHE

1

Charge pump

Re-enactment from Qi's slide

Alternative derivations of Θ

1. Semiclassical approach (Xiao Di et al, PRL 2009)

2. $\frac{\partial P_i}{\partial B_j}$ A. Essin et al, PRB 2010

3. $\frac{\partial M_j}{\partial E_i}$ A. Malashevich et al, New J. Phys. 2010

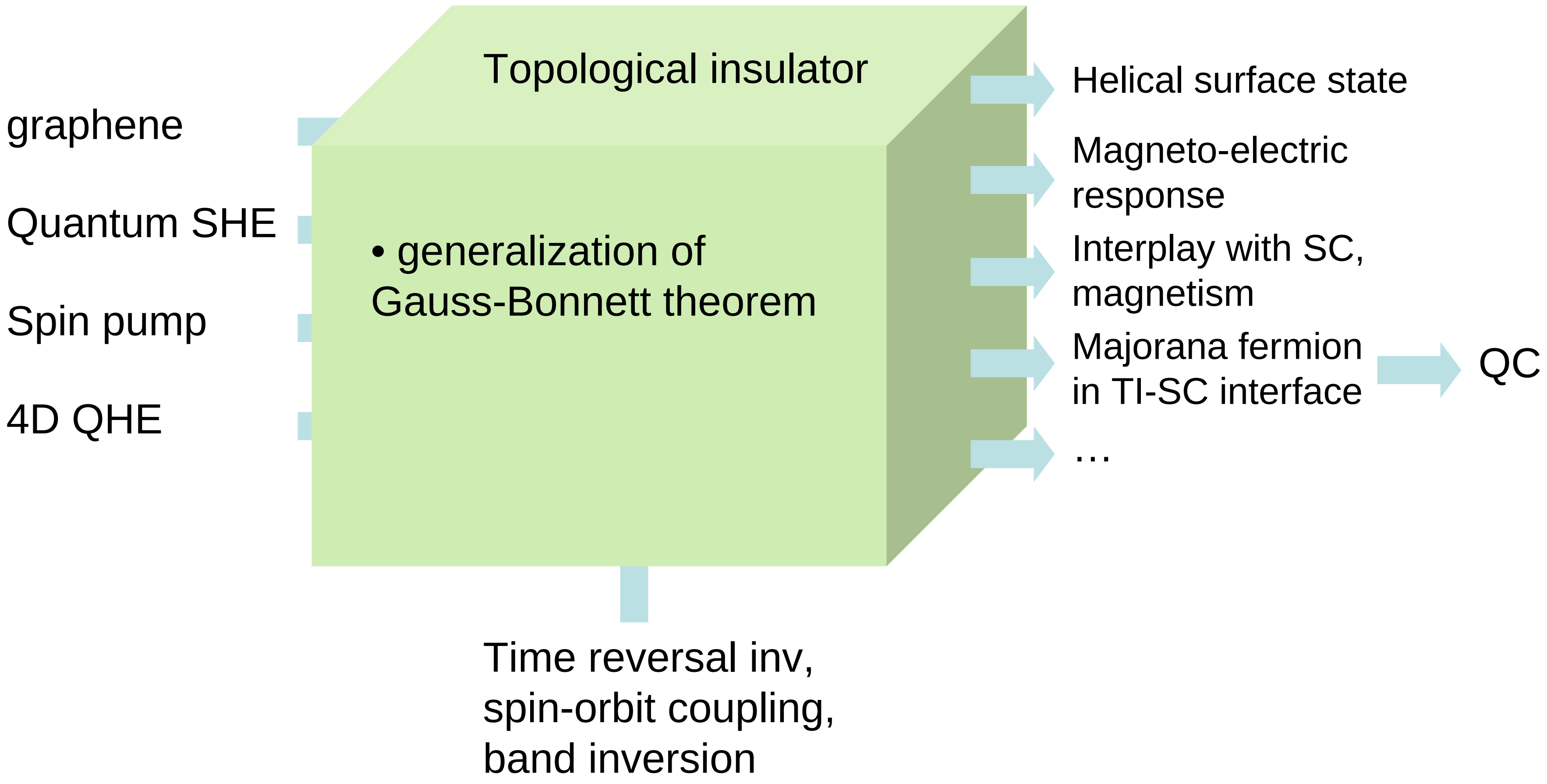
$$\frac{\partial P_i}{\partial B_j} = \frac{\partial M_j}{\partial E_i} = \alpha_{ij} = \tilde{\alpha}_{ij} + \alpha_\theta \delta_{ij}$$

$$\alpha_\theta = \frac{\Theta}{2\pi}\frac{e^2}{h}$$

Explicit proof of $\Theta = \pi$ for strong TI:

- Z. Wang et al, New J. Phys. 2010

Physics related to the Z_2 invariant



Topological insulator

- generalization of Gauss-Bonnett theorem

graphene

Quantum SHE

Spin pump

4D QHE

Helical surface state

Magneto-electric response

Interplay with SC, magnetism

Majorana fermion in TI-SC interface → QC

…

Time reversal inv,
spin-orbit coupling,
band inversion

Thank you!