Basics of topological insulator

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A brief history of insulators

Band insulator (Wilson, Bloch)

Mott insulator

Peierls transition

Anderson insulator

Hubbard model

Quantum Hall insulator

Scaling theory of localization

Topological insulator


2D TI is also called QSHI
• Gauss-Bonnet theorem for a 2D surface with boundary

\[ \int_M da \, G + \int_{\partial M} ds \, k_g = 2\pi \, \chi(M, \partial M) \]

\[ \chi = 2 \quad \chi = 0 \quad \chi = 1 \]

• Quantum Hall effect (2D lattice fermion in magnetic field)

\[ \int_{2D \, BZ} d^2k \, \Omega_Z = 2\pi C_1 \quad \sigma = C_1 \frac{e^2}{h} \]
2D Lattice fermion with time reversal symmetry (TRS)

- Without B field, Chern number $C_1 = 0$
- Bloch states at $k, -k$ are not independent

- EBZ is a cylinder, not a closed torus.
  \[ \therefore \text{No obvious quantization.} \]

Moore and Balents PRB 07

- $C_1$ of closed surface may depend on caps
- $C_1$ of the EBZ (mod 2) is independent of caps

(topological insulator, TI)

\[ \rightarrow \text{2 types of insulator, the “0-type”, and the “1-type”} \]
• 2D TI characterized by a $Z_2$ number (Fu and Kane 2006)

$$
\nu = \frac{1}{2\pi} \left[ \int_{EBZ} d^2k \Omega - \oint_{\partial(EBZ)} dk A \right] \mod 2
$$

~ Gauss-Bonnet theorem with edge

How can one get a TI?

: band inversion due to SO coupling

0-type

![CdTe](image1)

1-type

![HgTe](image2)
Bulk-edge correspondence

Different topological classes

Semiclassical (adiabatic) picture:
energy levels must cross
(otherwise topology won’t change).

\[ E_g \]

\[ \rightarrow \] gapless states bound to the interface, which are protected by topology.

Topological Goldstone theorem?
Bulk-edge correspondence in TI

Bulk states

Edge states

Fermi level

Dirac point

TRIM

$k = \Lambda_a$

$k = \Lambda_b$

(2-fold degeneracy at TRIM due to Kramer’s degeneracy)

helical edge states

robust backscattering by non-magnetic impurity forbidden
Topological insulators in real life

2D
• Graphene (Kane and Mele, PRLs 2005)
• HgTe/CdTe QW (Bernevig, Hughes, and Zhang, Science 2006)
• Bi bilayer (Murakami, PRL 2006)
• Bi$_{1-x}$Sb$_x$, α-Sn ... (Fu, Kane, Mele, PRL, PRB 2007)
• Bi$_2$Te$_3$ (0.165 eV), Bi$_2$Se$_3$ (0.3 eV) ... (Zhang, Nature Phys 2009)

3D
• The half Heusler compounds (LuPtBi, YPtBi ...) (Lin, Nature Material 2010)
• thallium-based III-V-VI$_2$ chalcogenides (TIBiSe$_2$ ...) (Lin, PRL 2010)
• Ge$_n$Bi$_{2m}$Te$_{3m+n}$ family (GeBi$_2$Te$_4$ ...)

• SO coupling only $10^{-3}$ meV
• strong spin-orbit coupling
• band inversion
A stack of 2D TI

Helical edge state

3 TI indices

fragile
3D TI: 3 weak TI indices:

\[ \nu_0 = z_+ - z_0 \]
\[ \nu_0 = y_+ - y_0 = x_+ - x_0 \]

Eg., \( (x_0, y_0, z_0) \)

1 strong TI index: \( \nu_0 \)

\[ \nu_0 = z_+ - z_0 \]
\[ \nu_0 = y_+ - y_0 = x_+ - x_0 \]

difference between two 2D TI indices

Fu, Kane, and Mele PRL 07
Moore and Balents PRB 07
Roy, PRB 09
Weak TI index

Screw dislocation of TI

• not localized by disorder
• half of a regular quantum wire

Ran Y et al, Nature Phys 2009

A stack of 2D TI

From Vishwanath's slides
Band inversion, parity change, spin-momentum locking (helical Dirac cone)
**Dirac point:**
Graphene vs. Topological insulator

<table>
<thead>
<tr>
<th>Even number</th>
<th>Odd number (on one side)</th>
</tr>
</thead>
<tbody>
<tr>
<td>located at Fermi energy</td>
<td>not located at $E_F$</td>
</tr>
<tr>
<td>half integer QHE ($\times 4$)</td>
<td>half integer QHE (if $E_F$ is located at DP)</td>
</tr>
<tr>
<td>Spin is not locked with $k$</td>
<td>spin is locked with $k$</td>
</tr>
<tr>
<td>can be opened by substrate</td>
<td>cannot be opened</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\sigma_H &= +1 \\
\sigma_H &= -1
\end{align*}
\]
Electromagnetic response
Axion electrodynamics
First, a heuristic argument:

\[ \Theta = \pi \]

**Effective Lagrangian for EM wave**

\[
L_{EM} = L_0 + L_{axion}
\]

\[
L_0 = \frac{1}{8\pi^2} \left( \frac{E^2}{c^2} - B^2 \right)
\]

\[
L_{axion} = \frac{e^2}{2\hbar c} \frac{1}{\Theta} \vec{E} \cdot \vec{B} = \alpha \frac{\Theta}{4\pi^2} \vec{E} \cdot \vec{B}
\]

Note: \[
\alpha = \frac{4\pi}{c} \frac{e^2}{2\hbar} = \frac{e^2}{\hbar c} \sim \frac{1}{137}
\]

For systems with time-reversal symmetry, \(\Theta\) can only be 0 (usual insulator) or \(\pi\) (TI)

\[ \text{Cr}_2\text{O}_3: \Theta \sim \pi/24 \text{ (TRS is broken)} \]

- Hall current \(J_H = \frac{e^2}{2\hbar} E\)
- Induced magnetization \(M = \frac{e^2}{2\hbar} E\)

“magneto-electric” coupling
Maxwell eqs with axion coupling

\[ \nabla \cdot \left( \vec{E} + \alpha \frac{\Theta}{\pi} \vec{B} \right) = 4\pi \rho \]

\[ \nabla \times \left( \vec{B} - \alpha \frac{\Theta}{\pi} \vec{E} \right) = \frac{4\pi}{c} \vec{J} + \frac{\partial}{c\partial t} \left( \vec{E} + \alpha \frac{\Theta}{\pi} \vec{B} \right) \]

\[ \nabla \cdot \vec{B} = 0 \]

\[ \nabla \times \vec{E} = -\frac{\partial}{c\partial t} \vec{B} \]

Effective charge and effective current

\[ \nabla \cdot \vec{E} = 4\pi (\rho + \rho_\Theta) \]

\[ \rho_\Theta = -\frac{\alpha}{4\pi^2} \nabla \cdot (\Theta \vec{B}) \]

\[ \nabla \times \vec{B} = \frac{4\pi}{c} \left( \vec{J} + \vec{J}_\Theta \right) + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \]

\[ \vec{J}_\Theta = -\frac{c\alpha}{4\pi} \delta(z) \hat{z} \times \vec{E} \rightarrow \sigma_{xy} = \frac{1}{2} \frac{e^2}{h} \]
Magnetic monopole in TI

A point charge

An image charge and an image monopole

Circulating current

Optical signatures of TI?

• Snell’s law
• Fresnel formulas
• Brewster angle
• Goos-Hänchen effect
• ...

axion effect on

Qi, Hughes, and Zhang, Science 2009

D

Longitudinal shift of reflected beam (total reflection)

Chang and Yang, PRB 2009

(Magnetic overlayer not included)
Dimensional reduction

Topological field theory
When the flux is changed by $\Phi_0$, integer charges are transported from edge to edge → IQHE

Laughlin’s argument (1981):

$\psi_{xx}$ $\psi_{xy}$ $\psi_{yy}$

Berry curvature:

$f_{ij} = \partial_i a_j - \partial_j a_i$

Berry connection:

$a_k = i \langle u | \partial_k | u \rangle$

Compactification

$C_1 = \frac{1}{2\pi} \int d^2 k f_{xy}(\vec{k})$

$S_{CS} = \frac{C_1}{4\pi} \int d^2 x d t \epsilon^{\mu \nu \tau} A_\mu \partial_\nu A_\tau$

$f^\mu = \frac{\delta S_{CS}}{\delta A_\mu} = \frac{C_1}{2\pi} \epsilon^{\mu \nu \tau} \partial_\nu A_\tau$

$A_y(x,t) \sim \theta(x,t)$, a parameter polarization

$P(\theta) = \frac{1}{2\pi} \int d k_x a_x$

$\tilde{S} = \int dx dt P \epsilon^{\alpha \beta} \partial_\alpha A_\beta$

$\dot{\alpha} = -\epsilon_{\alpha \beta} \partial_\beta P$

Qi, Hughes, and Zhang PRB 2008
4D quantum Hall effect

\[ S = \frac{C_2}{24\pi^2} \int d^4x dt \varepsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau \]

\[ j^\mu = \frac{C_2}{8\pi^2} \varepsilon^{\mu\nu\rho\sigma} \partial_\nu A_\rho \partial_\sigma A_\tau \]

nonlinear response.

\[ C_2 = \frac{1}{32\pi^2} \int d^4k \varepsilon^{ijkl} \text{tr} \left( f_{ij} f_{kl} \right) \]

\[ f_{ij} = \partial_i a_j - \partial_j a_i - i[a_i, a_j] \]

\[ a_k^{mn} = i \langle u_m | \partial_k | u_n \rangle \]

3D topological insulator

\[ \tilde{S} = \frac{1}{8\pi^2} \int d^3x dt \Theta \varepsilon^{\alpha\beta\gamma\delta} \partial_\alpha A_\beta \partial_\gamma A_\delta \]

\[ j^\alpha = \frac{1}{4\pi^2} \varepsilon^{\alpha\beta\gamma\delta} \partial_\beta \Theta \partial_\gamma A_\delta \]

\[ \Theta = \frac{1}{8\pi} \int_{BZ} d^3k \varepsilon^{ijk} \text{tr} \left( a_i f_{jk} + \frac{i}{3} a_i \left[ a_j, a_k \right] \right) \]

Qi, Hughes, and Zhang PRB 2008
Without TRS

NTU Phys bldg

With TRS

5

4

3

2

1

Charge pump

QHE

Chern elevator

4D QHE

TI

QSHE

Re-enactment from Qi’s slide
Alternative derivations of $\Theta$

1. Semiclassical approach (Xiao Di et al, PRL 2009)
   \[
   \frac{\partial P_i}{\partial B_j} = \frac{\partial M_j}{\partial E_i}
   \]

2. A. Essin et al, PRB 2010


\[
\frac{\partial P_i}{\partial B_j} = \frac{\partial M_j}{\partial E_i} = \alpha_{ij} = \tilde{\alpha}_{ij} + \alpha_\theta \delta_{ij}
\]

\[
\alpha_\theta = \frac{\Theta e^2}{2\pi h}
\]

Explicit proof of $\Theta = \pi$ for strong TI:

Physics related to the $\mathbb{Z}_2$ invariant

- Topological insulator
  - generalization of Gauss-Bonnet theorem
  - Time reversal inv, spin-orbit coupling, band inversion
  - Helical surface state
  - Magneto-electric response
  - Interplay with SC, magnetism
  - Majorana fermion in TI-SC interface
  - QC

- graphene
- Quantum SHE
- Spin pump
- 4D QHE
Thank you!