Frustrated Heisenberg model

and

Hofstadter spectrum

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A typical example of antiferromagnet : MnO

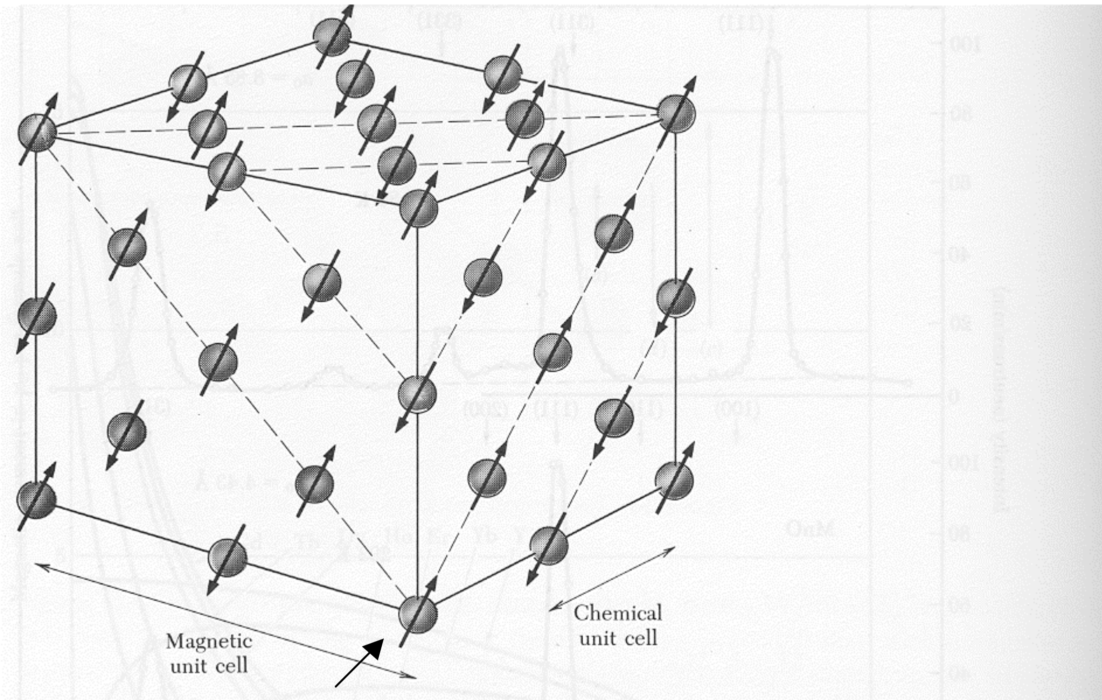


Figure 21 Ordered arrangements of spins of the Mn^{2+} ions in manganese oxide, MnO, as determined by neutron diffraction. The O^{2-} ions are not shown.

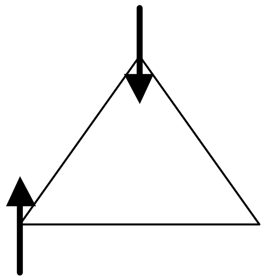
AF Heisenberg model

$$H = \sum_{<\vec{R},\vec{R}'>} J(\vec{R}-\vec{R}')\vec{S}(\vec{R})\cdot\vec{S}(\vec{R}'), \quad J(\vec{R}-\vec{R}')>0$$

Because of the positive exchange coupling, nearby spins prefer $\uparrow\downarrow$ to lower the energy

“Frustrated” antiferromagnet

- Triangular lattice ($LaCuO_2$)



? “><” frustrated

- Square lattice with NN *and NNN interactions*

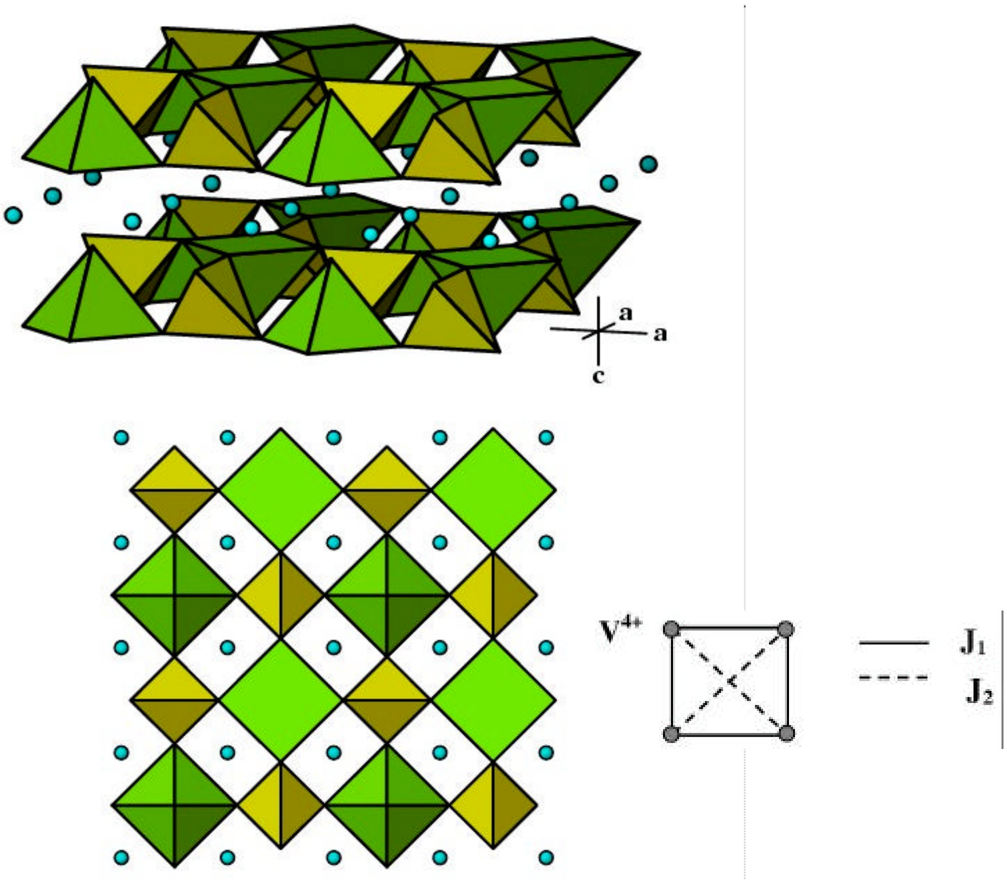
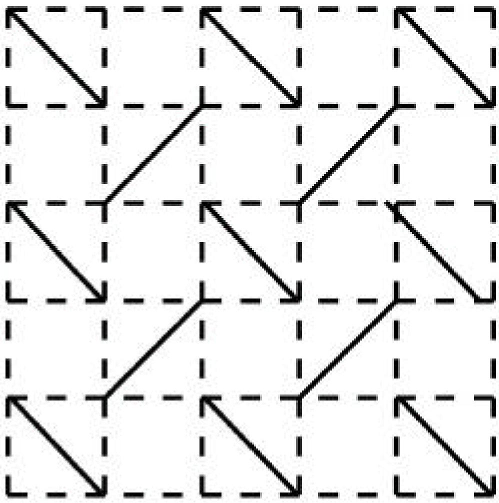


FIG. 1. Perspective view (upper panel) of the crystal structure of Li_2VOSiO_4 and projection along [001] (lower panel). The VO_5 pyramids (large diamonds) share the corners of the basal planes with SiO_4 tetrahedra (small diamonds). The Li^+ ions are indicated by circles.

R. Melzi *et al*, Phys. Rev. Lett. **85**, 1318 (2000);
R. Melzi *et al*, Phys. Rev. B **64**, 024409 (2001).

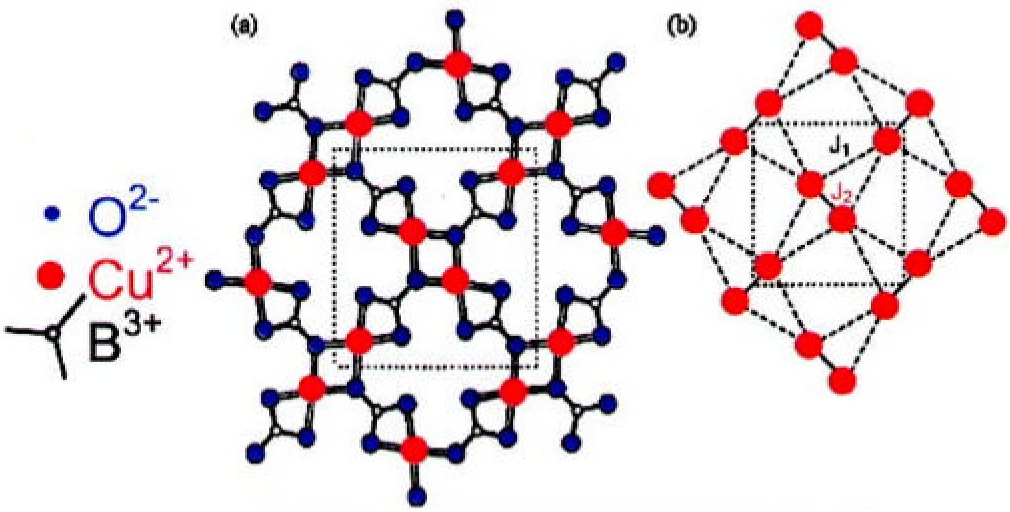
➢ Shastry-Sutherland lattice (Physica, 1981)



j < 0.7j' dimerized state (Bose's review 0107399)

j > 0.7j' square Heisenberg with Neel-order

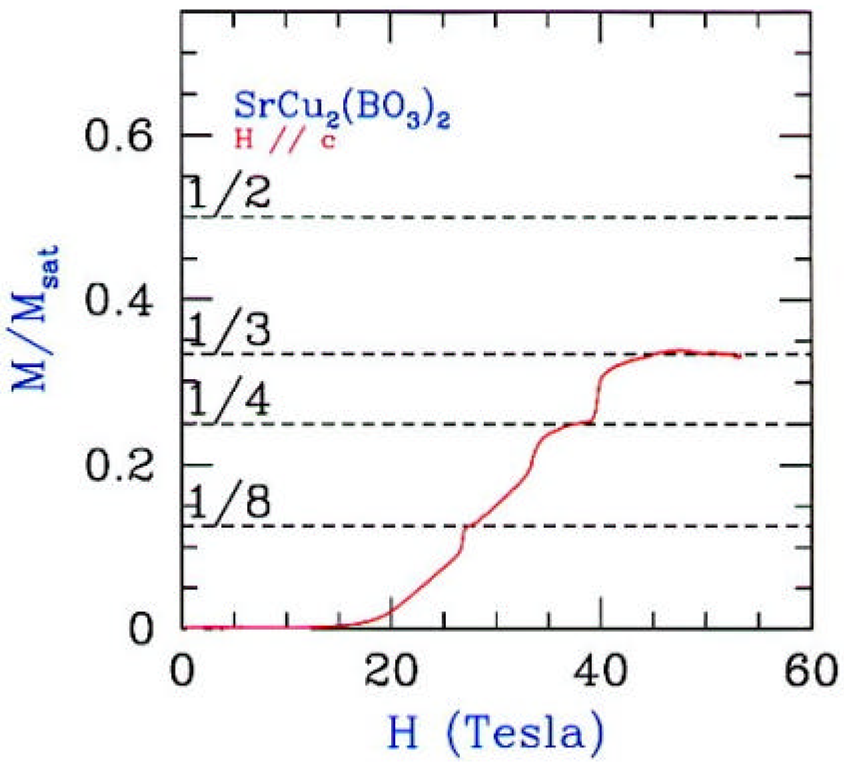
$SrCu_2(BO_3)_2$



Magnetization plateaus in $SrCu_2(BO_3)_2$

H.Kageyama et al., Phys. Rev. Lett. 82, 3168 (1999)

K.Onizuka, H.Kageyama et al., J. Phys. Soc. Jpn (2000)



Other examples of the magnetization plateaus

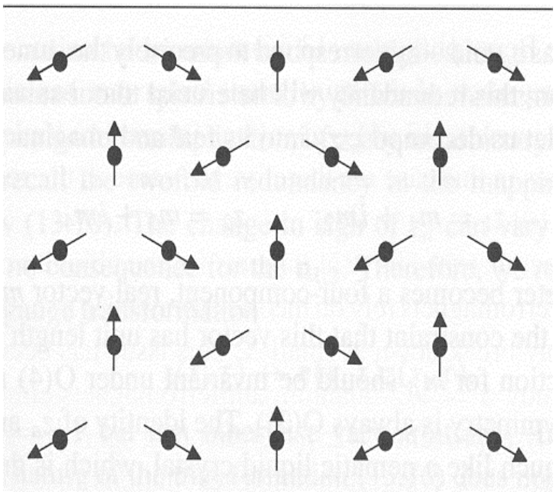
$CsCuCl_3$ (a stack of triangular lattices), M/Ms=1/3

NH_4CuCl_3 (coupled zigzag spin ladders), M/Ms=1/4,3/4

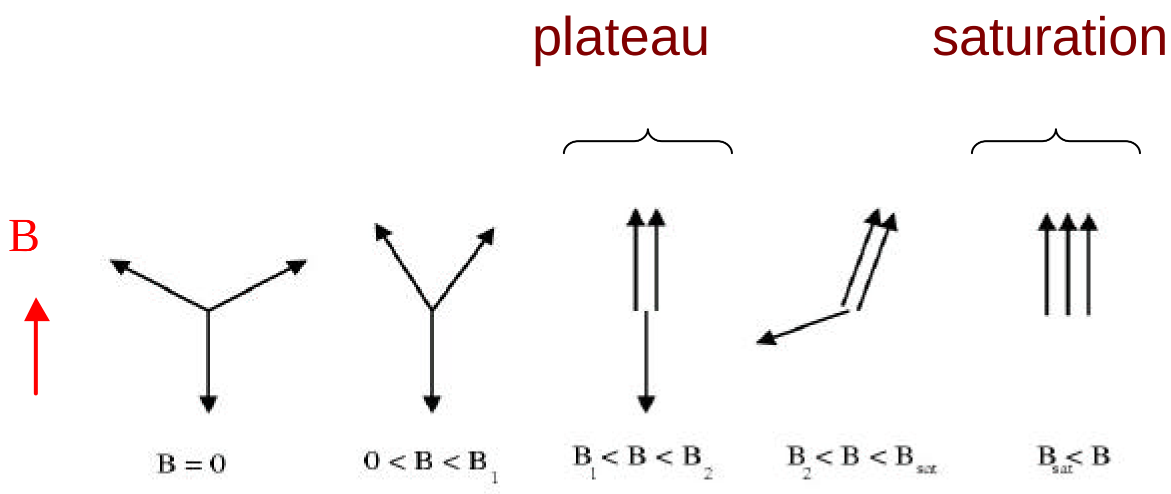
Plateaus for a classical magnet

(C. Lhullier and G. Misguich, cond/mat 0109146)

Classical ground state for a triangular lattice



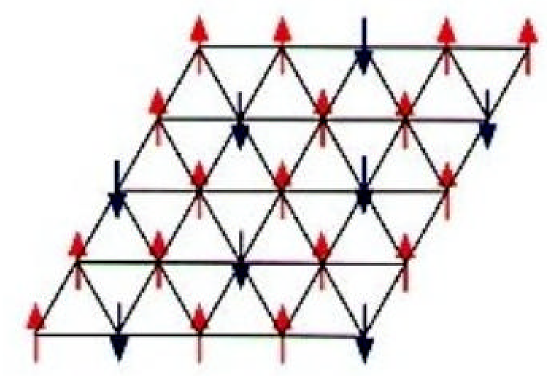
Collinearity criterion



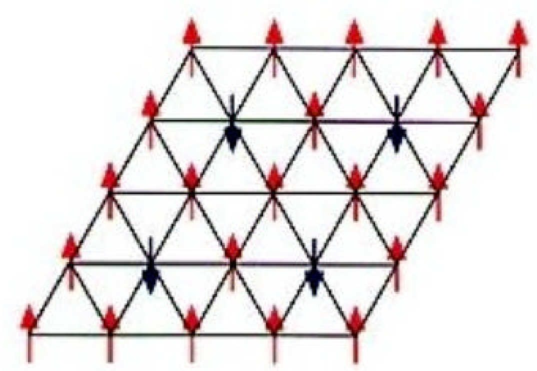
The three vectors represent the three sublattice magnetizations. A plateau at $M/M_{sat}=1/3$ is present for magnetic field between B_1 and B_2

Spin configuration within a plateau

uud structure (M/Ms=1/3)



uuud structure (M/Ms=1/2)



Plateaus for a quantum magnet

1D: Oshikawa, Yamanaka, and Affleck ('97)

$$\boxed{n(S-S_z) \in Z}$$

n: period of the unit cell

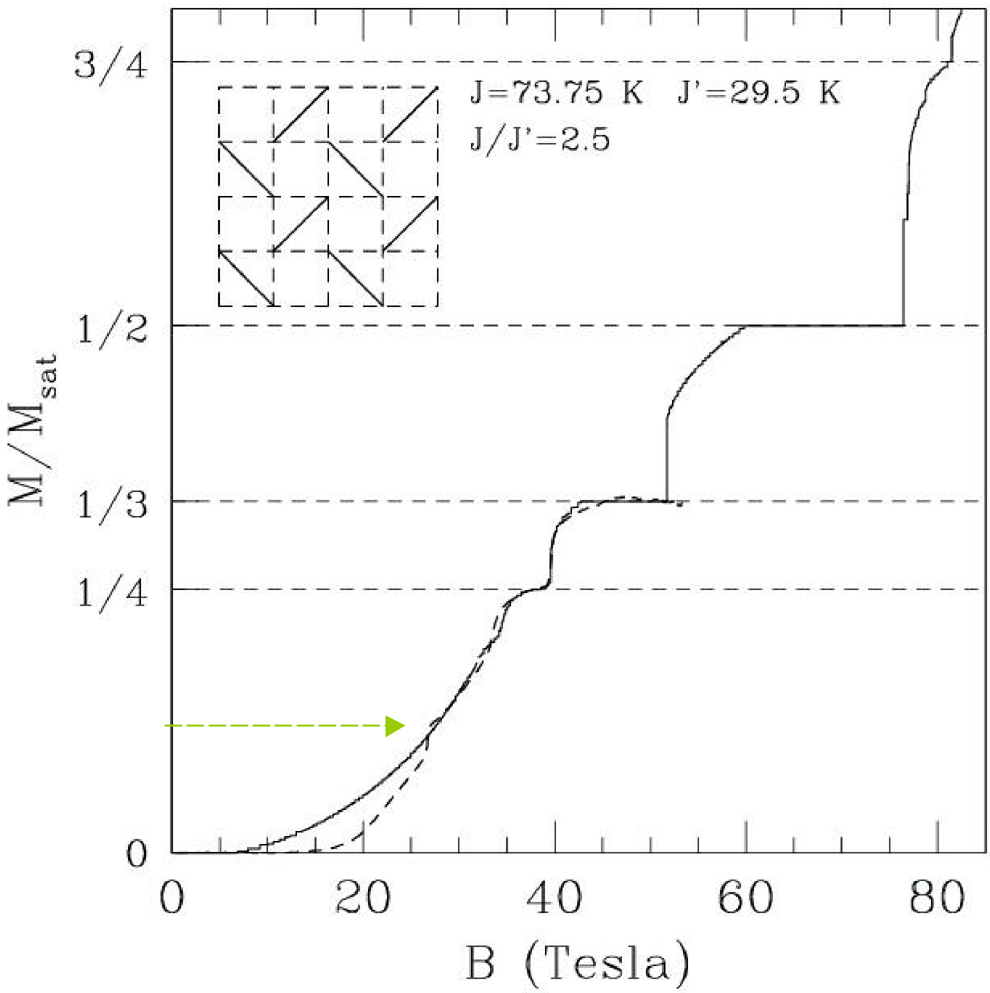
S: spin per site

→ Probably true in 2D as well

Magnetization of the Shastry-Sutherland lattice

G. Misguich, Th. Jolicoeur, and S. M. Girvin,

Phys. Rev. Lett. **87**, 097203 (2001)



Comparison between the magnetization curve measured

by Onizuka *et al*. (dashed line) and

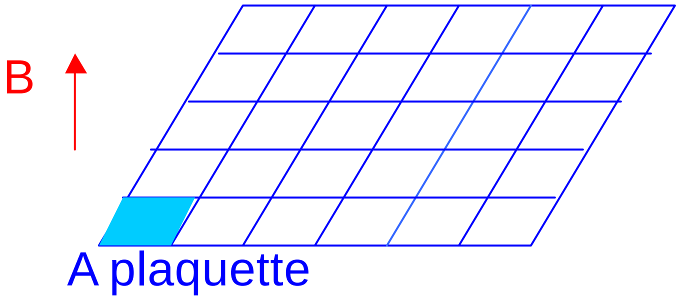
the mean-field result (solid line).

$\Rightarrow$ close theoretical connection with

the Hofstadter spectrum!

Hofstadter spectrum: (D.R. Hofstadter, PRB 1976)

The band structure of an electron subjects to both a lattice potential V(x,y) and a magnetic field B



- Can be studied using either the nearly free electron model or the tight-binding model
- Surprisingly complex spectrum!

  Split of energy band depends on flux/plaquette

  If Φ_{plaq}/Φ_0= p/q, where p, q are co-prime integers, then a Bloch band splits to q subbands (for TBM)
- The tricky part:

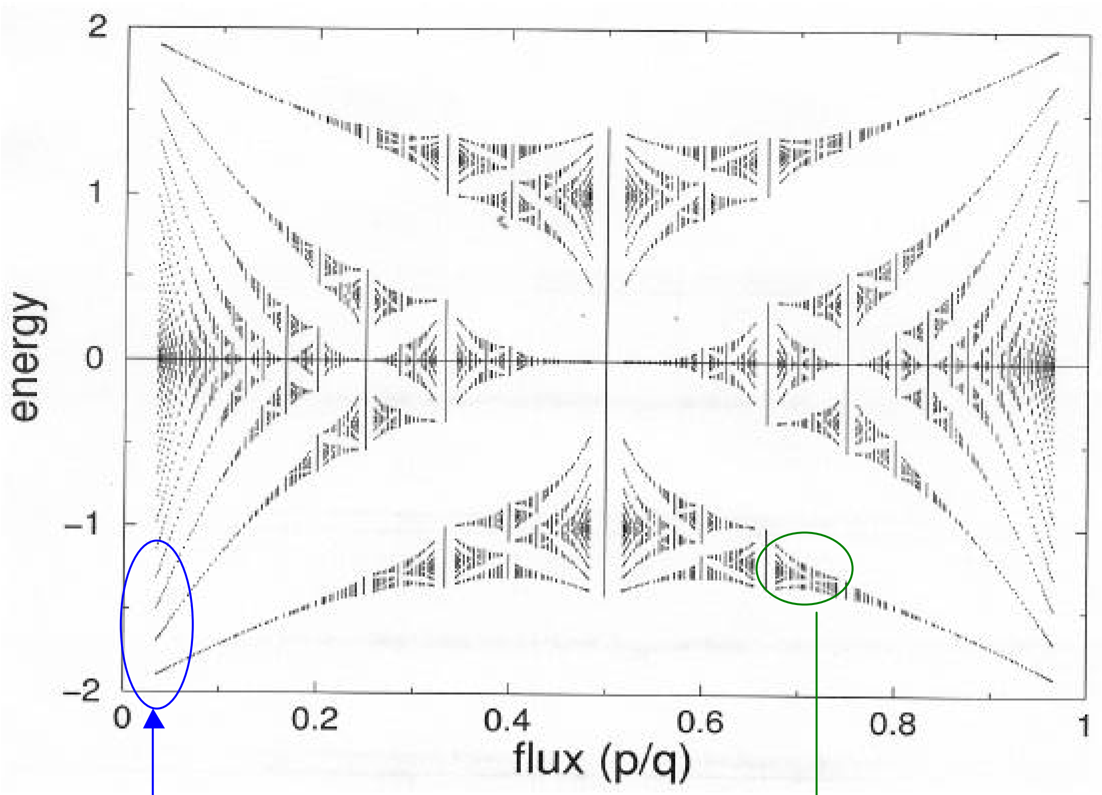
$$\frac{1}{3-\frac{1}{10}}=\frac{10}{\boxed{29}}=\frac{1}{\boxed{3}}+\frac{1}{87}$$

q=3 → q=29 upon a small change of B!

Also, when B → 0, q can be very large!

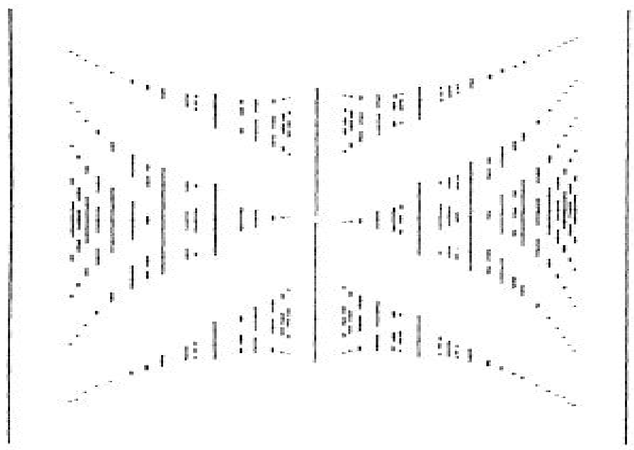
Hofstadter's butterfly

A fractal spectrum with self-similarity structure



B→0 near band button, evenly-spaced LLs

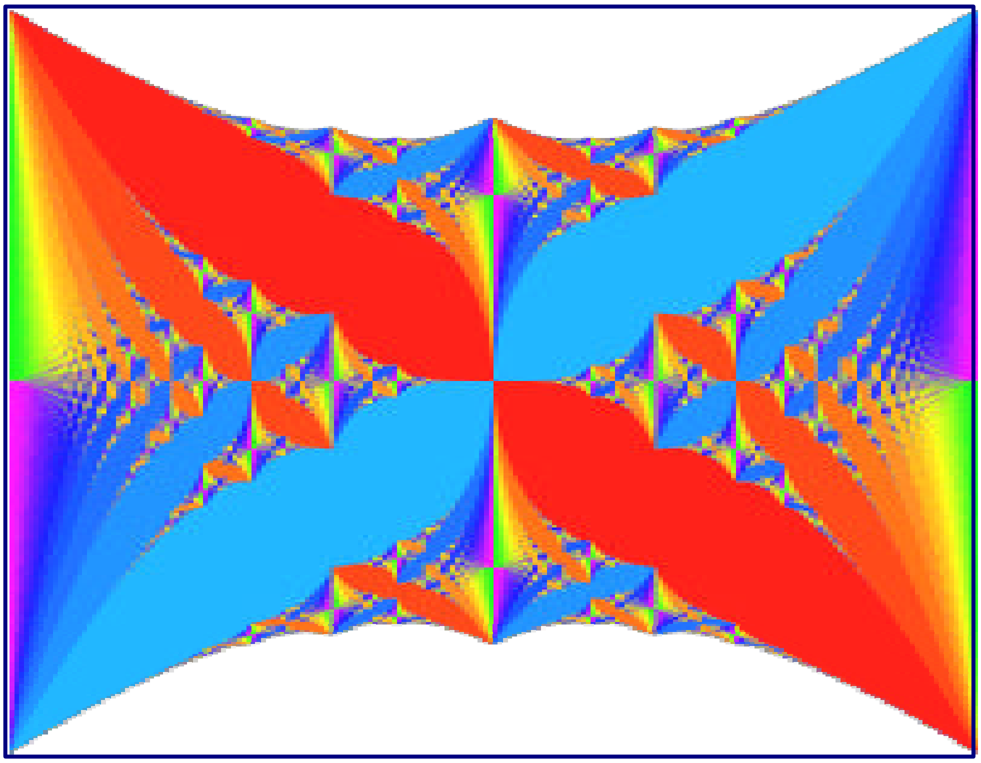
Self-similarity



✧ The total band width for an irrational q is of measure zero (as in a Cantor set)

Distribution of the Hall conductances

Each filled band carries an integer Hall conductance σ_{xy} (TKNdN paper, 1982)



D.Osadchy and J.Avron, J. Math. Phys. 46, 5225 (2001)

The figure shows the gaps, color coded according to the Hall conductance (determined by the Diophantine eq.). The warm colors represent positive values of Hall conductance, and the cold colors represent negative values.

GS of J_1-J_2 Heisenberg model on a square lattice

Neel order vs collinear order: semiclassical picture

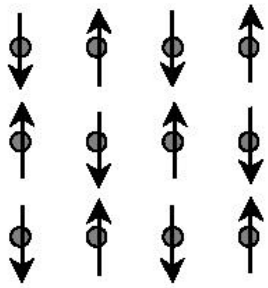
A. Honecker, Can. J. Phys. 79, 1557 (2001)

Magnon dispersion for the FM state

$$E(k_x, k_y) = -2(J_1 + J_2) + J_1(\cos k_x + \cos k_y) + 2J_2 \cos k_x \cos k_y + B$$

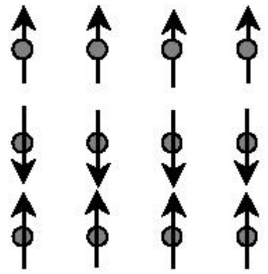
$J_2 < J_1/2$: energy minimum at (π, π)

Neel order with SU(2) symmetry



$J_2 > J_1/2$: minima at $(\pi, 0)$ or $(0, \pi)$

Collinear order (superlattice structure)

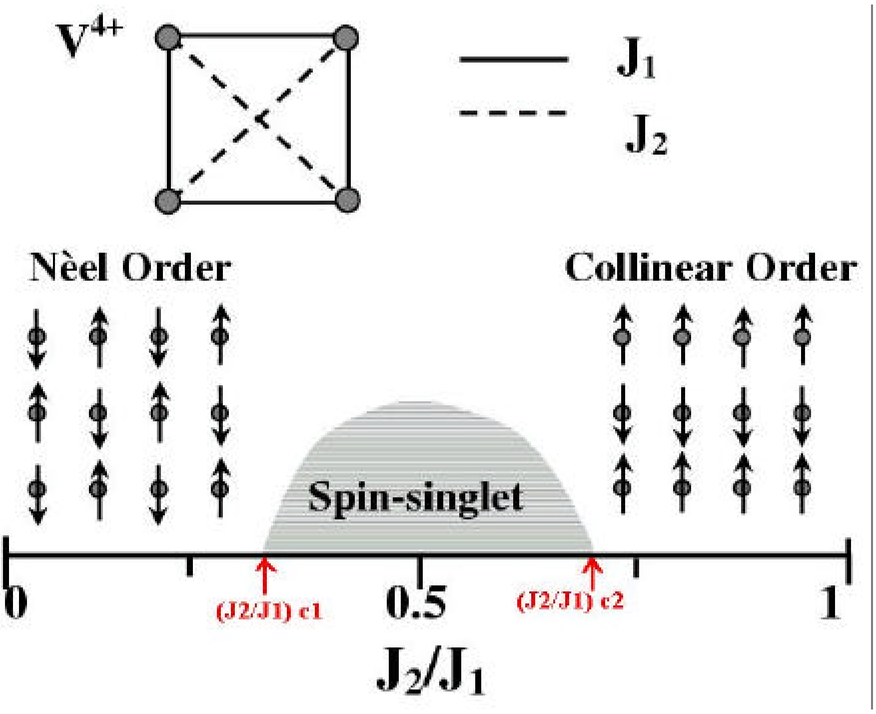


Both phases have gapless magnon excitations

Beyond semiclassical

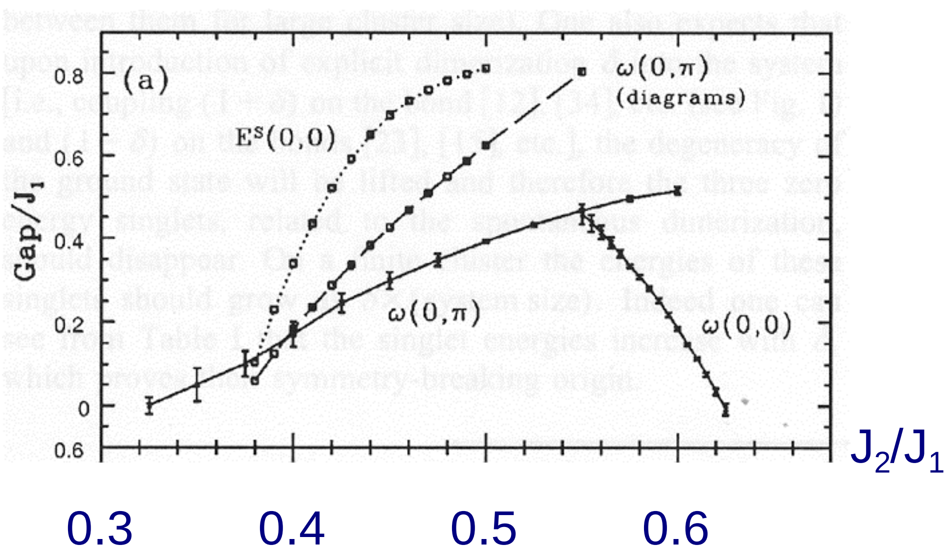
Spin-disordered phase near $J_2 = J_1/2$ ($0.38 < J_2/J_1 < 0.6$)

Spin liquid RVB (Caprioti, PRL 2001 using projected BCS)



Melzi et al., PRL 85,1318 (2000)

Gapful excitations within the disordered phase



Kotov et al. PRB 1999

Magnetization of the frustrated Heisenberg model on a square lattice

Weak B: No spin gap for Neel or collinear phase

$M \propto B$ linear

Intermediate B:

Possibility of magnetization plateau

- Gapful magnetic excitation,

  M doesn't change with B

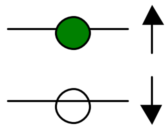
- Vanishing magnetic susceptibility $\chi=\partial M/\partial B$
- Discontinuity of $\partial E/\partial M$

Strong B: all three phases are saturated by the B field and become FM

Chern-Simons mean field approach

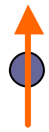
1. Spins as interacting bosons on the lattice

$$\begin{cases} S_i^- = b_i \\ S_i^+ = b_i^+ \\ S_i^z = b_i^+ b_i - 1/2 \end{cases}$$



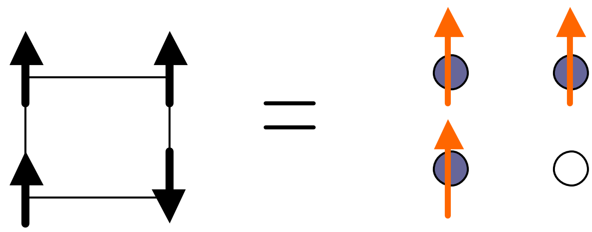
Plus hard-core constraint ($b_i^+ b_i = 0,1$)

2. Chern-Simons transformation in 2D

a boson  = a fermion 

attached with a flux quantum Φ_0

hard-core constraint automatically satisfied



3. Static mean-field approximation for the CS field

- $\langle f_i^+ f_i \rangle = \langle n_i \rangle = \Phi / \Phi_0$ ($\Phi_0 = 2\pi$ if $\hbar = 1$)
- $M = \langle S^z \rangle = \langle n \rangle - 1/2$

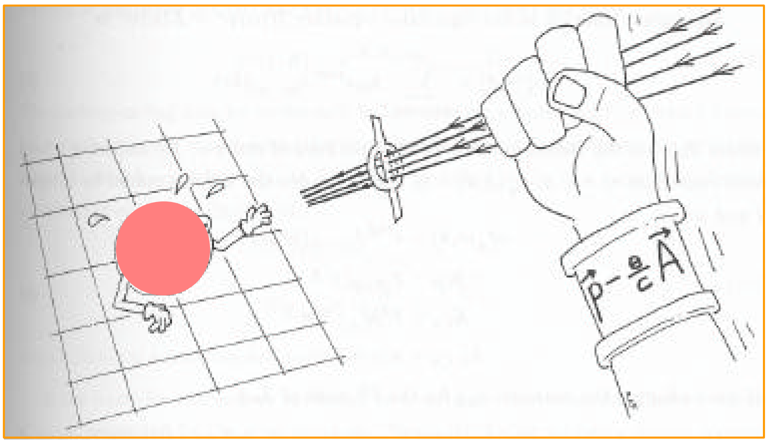
→ lattice fermions in an uniform CS gauge field spin

$$H = H_{xy} + H_z$$

$$H_{xy} = \frac{1}{2}\sum_{<i,j>} J_{ij}\left(S_i^+ S_j^- + S_j^+ S_i^-\right)$$

$$H_z = \sum_{<i,j>} J_{ij} S_i^z S_j^z$$

boson



From
D. Arovas,
Ph.D. thesis

fermion

$$H = H_{xy} + H_z$$

$$H_{xy} = \frac{1}{2}\sum_{<i,j>} J_{ij}^{\Phi}\left(f_i^+ f_j^- + f_j^+ f_i^-\right)$$

$$H_z = \sum_{<i,j>} J_{ij}\left(n_i - \frac{1}{2}\right)\left(n_j - \frac{1}{2}\right)$$

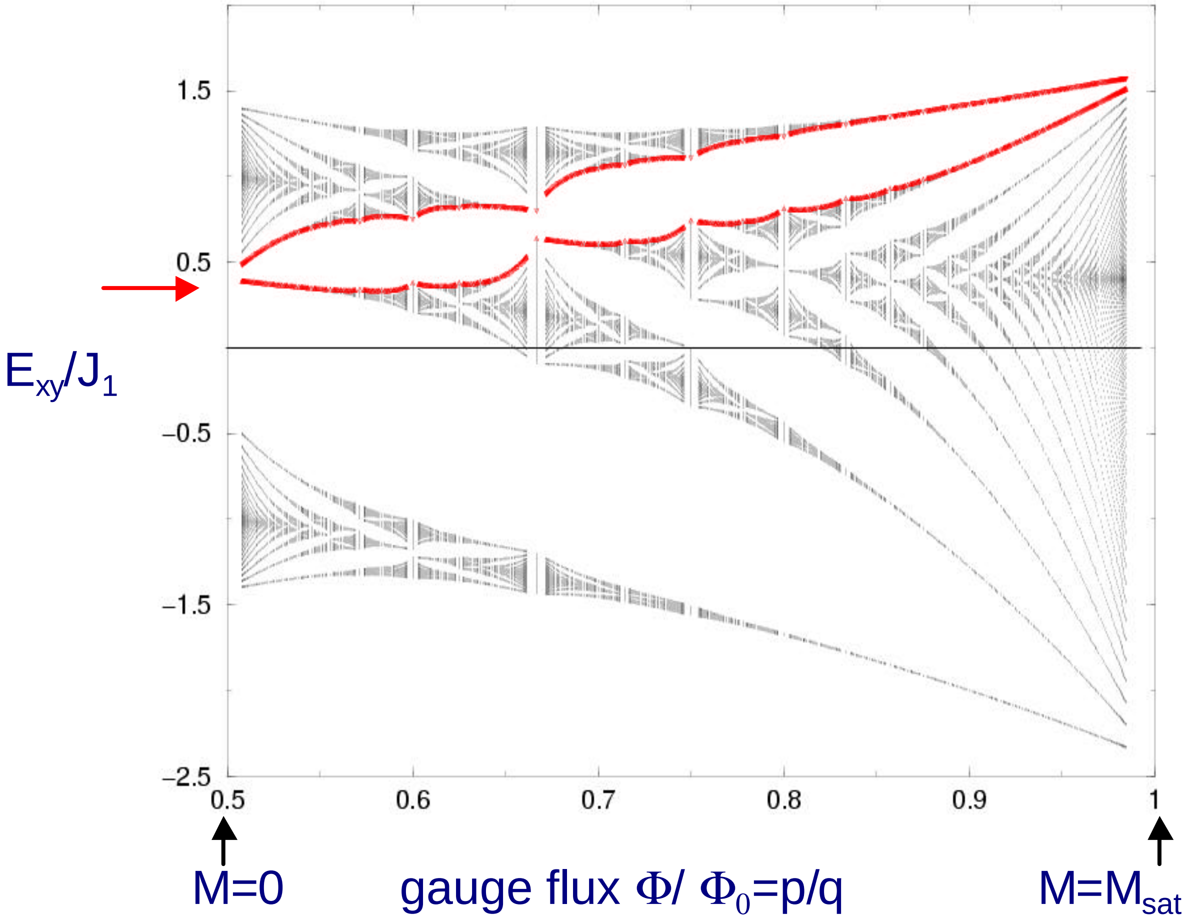
$$\begin{cases} n-1/2 = M \text{ (per site)} \\ \Phi/\Phi_0 = n = M+1/2 \end{cases}$$

→ $E(M)=E_{xy}(M)+E_z(M)$

minimize $F(M)= E(M) – BM$ to get the M-B curve

- Kinetic energy E_{xy}:

Hofstadter spectrum on a square lattice $J_2/J_1 = 0.2$



Each subband admits N/q states

→ N(p/q) fermions fill p subbands

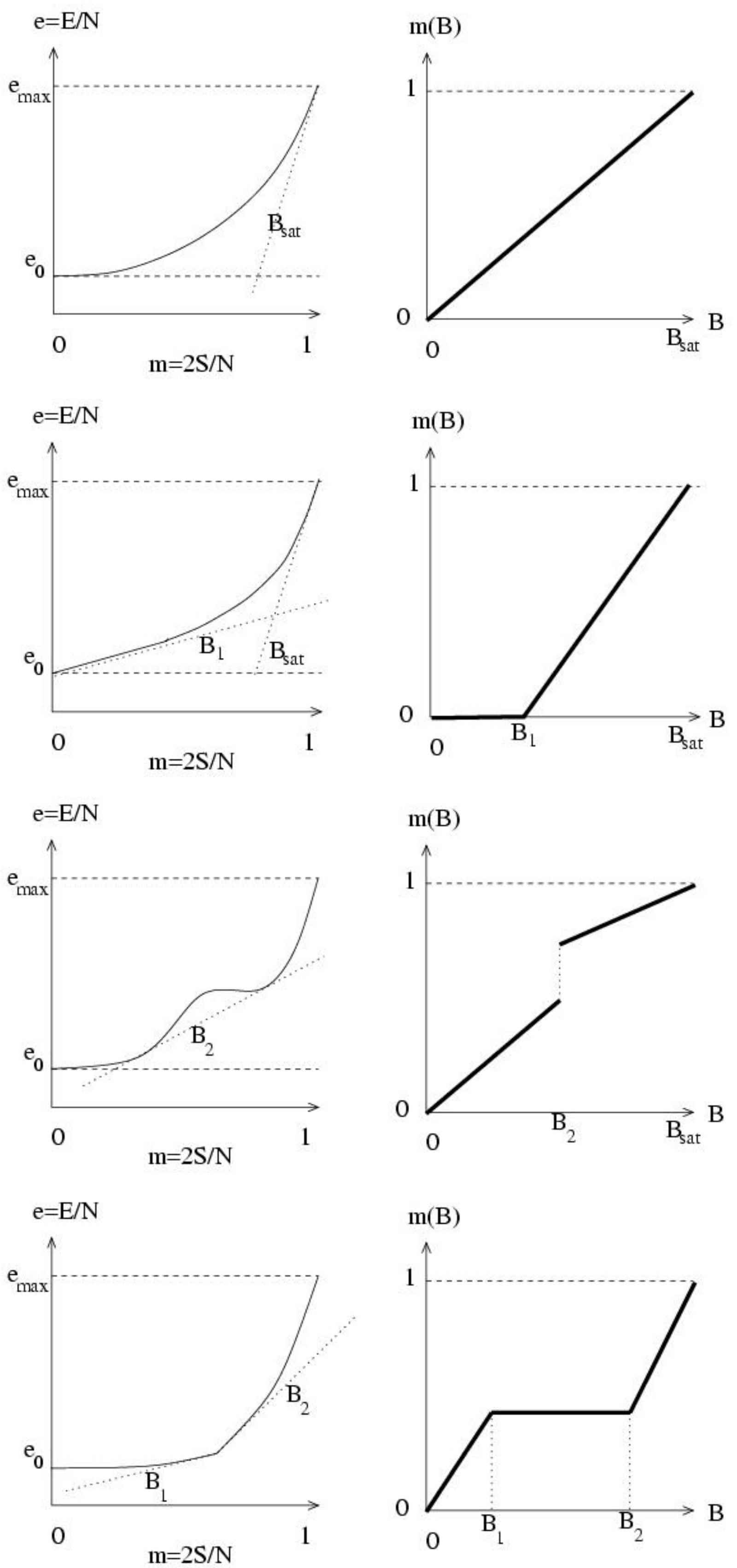
- Ising energy E_z (Hartree approximation)

$$E_z = 2N_sJ_1(n-1/2)^2+2N_sJ_2(n-1/2)^2$$

Total energy per site

$$e(M) = \frac{1}{N_s}\sum_{n,k} e_{n,k} + 2(J_1 + J_2)M^2$$

e(M) and magnetization curve (Lhuillier and Misguich)



Standard paramagnet

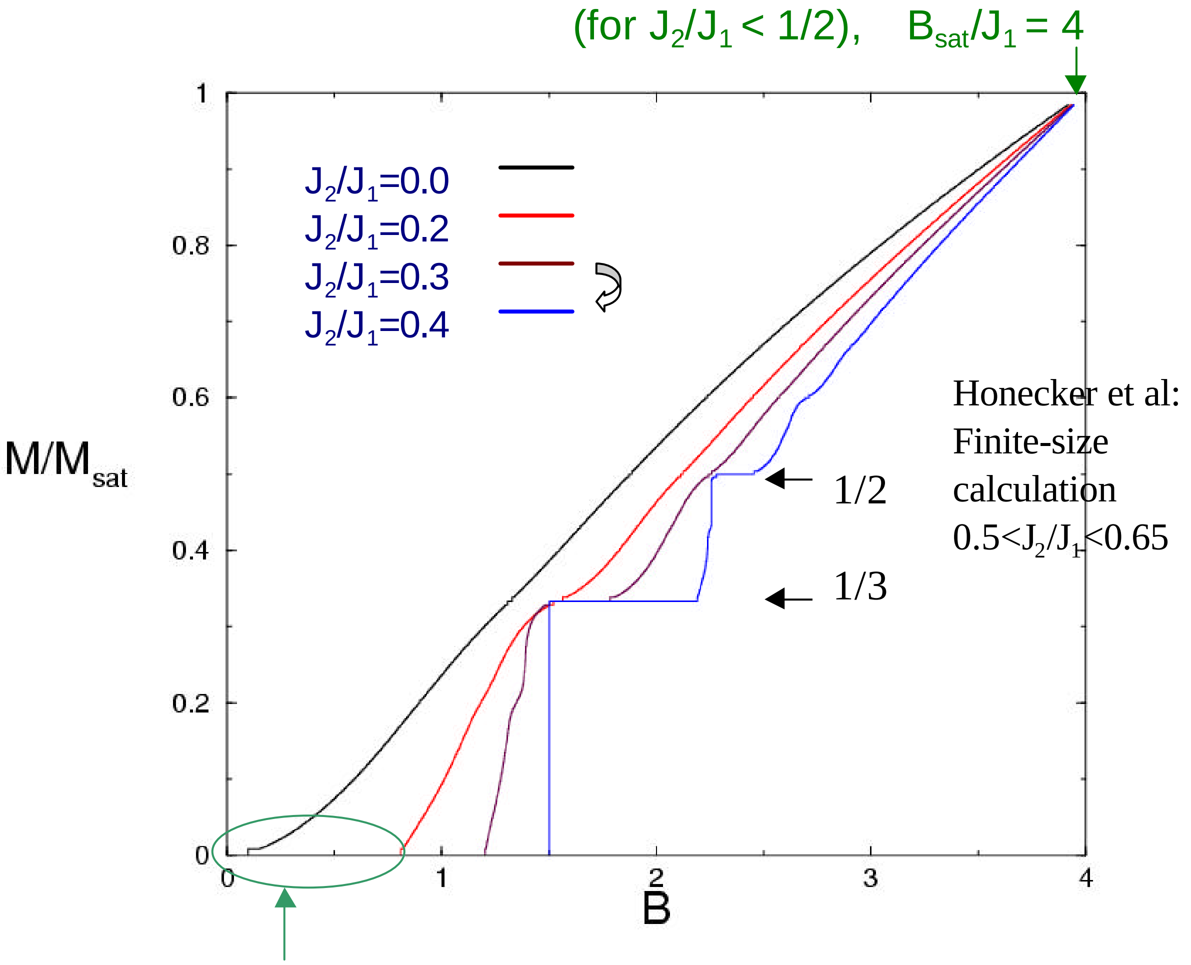
Spin gap

1^{st} order transition

plateau

B=∂E/∂M

Magnetization curves $J_2/J_1 = 1/2$:



fake spin gap (An artifact of the mean field approx.)

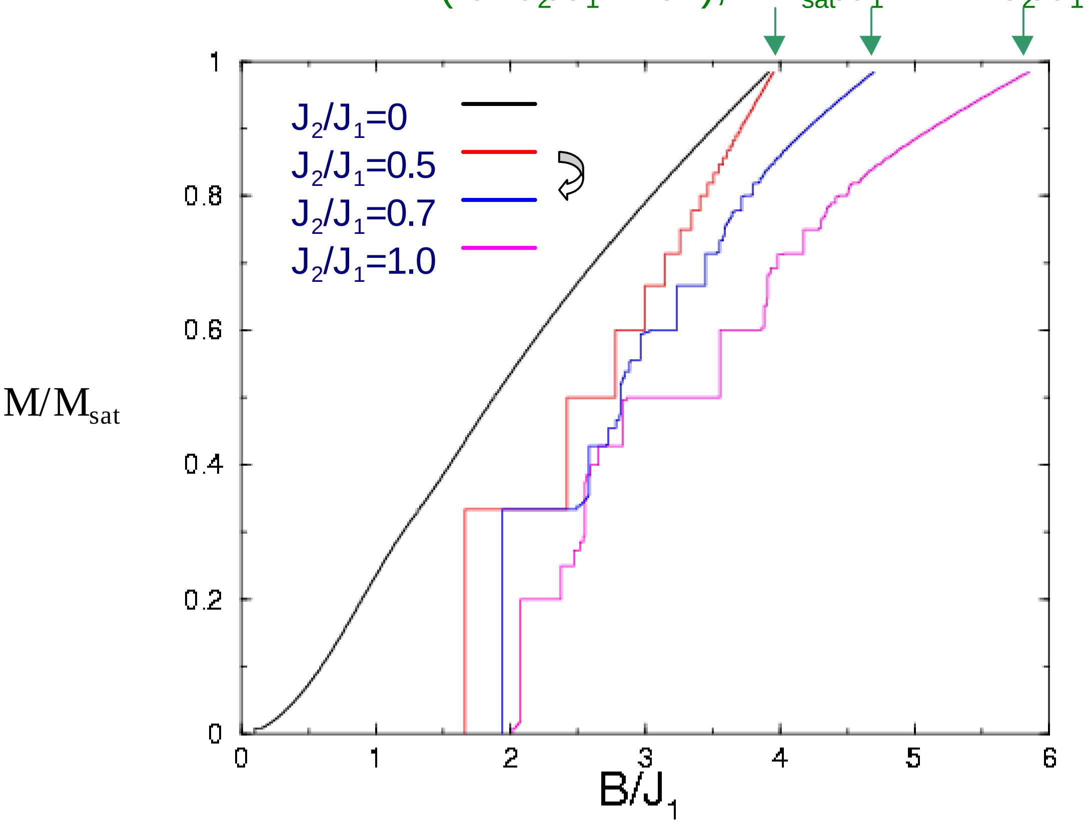
- plateau at 1/3 from $J_2/J_1 > 0.26795$ ($\sigma_T = -2$)
- plateau at 1/2 from $J_2/J_1 > 0.38268$ ($\sigma_T = -3$)

Topological nature of the quantized Hall conductance protects the plateaus from quantum fluctuations

Y.R. Yang, Warman, and S.M. Girvin, Phys. Rev. Lett. **70**, 2641 (1993).

$J_2/J_1 = 1/2$:

(for $J_2/J_1 = 1/2$), $B_{sat}/J_1 = 2+4J_2/J_1$



$J_2/J_1 = 0.5$: a series of plateaus at $M/M_{sat} = n/(n+2)$

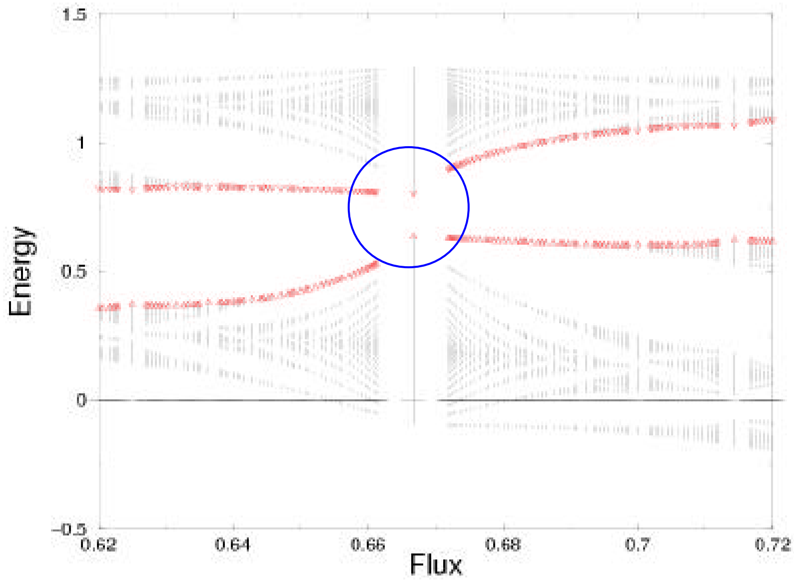
(no longer so in nonuniform mean field calculation)

$J_2/J_1 = 0.7$ and $J_2/J_1 = 1.0$:

- Irregular plateau structure, more studies using non-uniform mean field approx. are needed
- Main plateaus with simple fractions of M/M_{sat} might survive

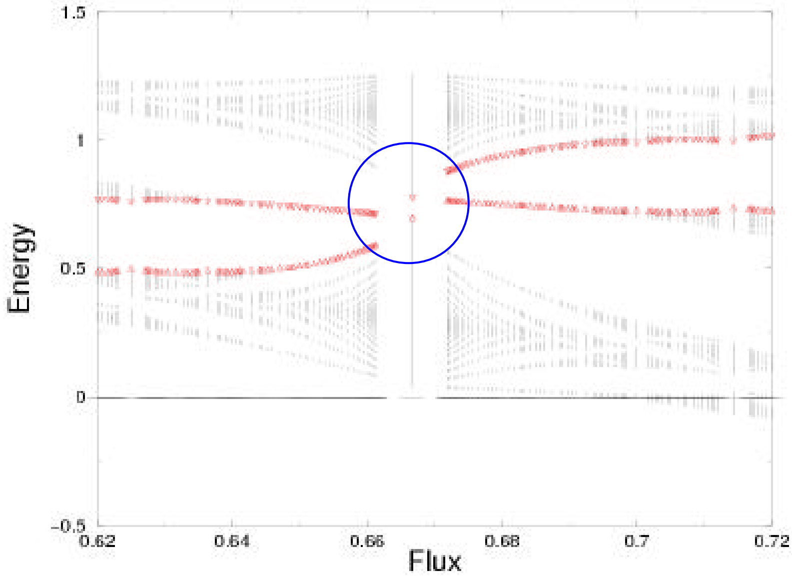
Magnetization plateau <-> Touch of energy bands

$J_2/J_1 = 0.2$



bands touch somewhere near $J_2/J_1 = 0.268$

$J_2/J_1 = 0.3$

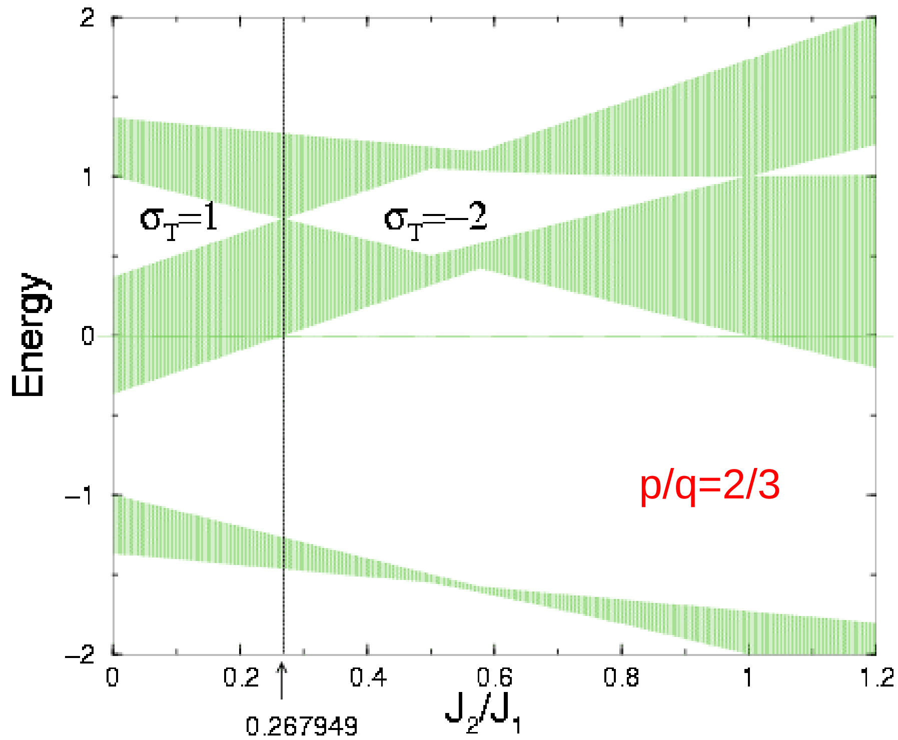


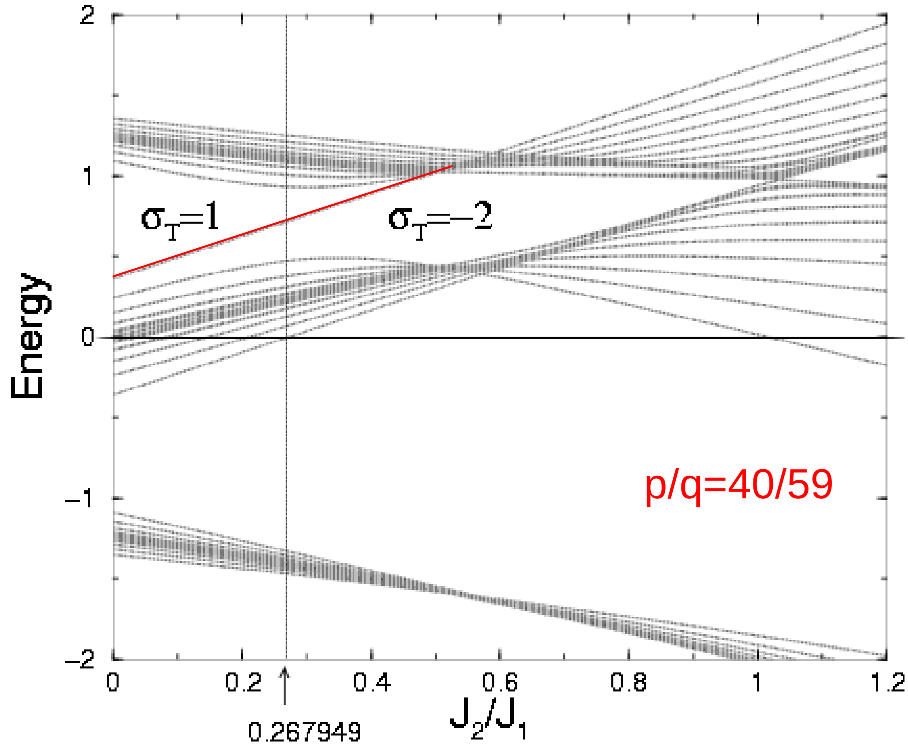
→ discontinuity of the fermi energy w.r.t. flux change

→ a jump for $B = \partial E/\partial M$

Jump of integer-valued Hall conductance induced by band-crossing " <-> Transport" of a subband

M.Y. Lee, M.C. Chang, and T.M. Hong, Phys. Rev. B**57**, 11895 (1998)





Summary (M.C. Chang and M.F. Yang, Phys. Rev. B, 2002)

Chern-Simons mean field result

- Saturation field B_{sat} coincides with exact result
- emergence of the $M/M_{sat} = 1/2$ plateau consistent with Neel/disorder phase boundary(J_2/J_1=0.3826)
- 1/3 plateau may indicate a phase transition for $J_2/J_1 < 0.38$. (validity of the mean field approx.?)
- irregular plateaus at $J_2/J_1 > 0.5$

  Still there in the nonuniform (collinear and Neel ordered) mean field calculations
- awaiting experimental confirmation