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An alternative way to achieve Kepler's laws of equal areas and ellipses for the Earth

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Abstract
Kepler's laws of planetary motion are acknowledged as highly significant to the construction of universal gravitation. This study demonstrates different ways to derive the law of equal areas for the Earth by general geometrical and trigonometric methods, which are much simpler than the original derivation depicted by Kepler. The established law of equal area for the Earth was applied to analyse the angular velocity or the reciprocal of the distance—for the Earth's orbit around the Sun—and can be defined as a periodic function by analysing the available data, which help explain the law of ellipses for the Earth.

1. Introduction

It has been 400 years since Johannes Kepler (1571–1630) published his major masterpiece—*New Astronomy* [1]. This book proposed a concise description of the orbit of Mars. Mars orbits the Sun in an ellipse with the Sun located at one focus rather than in accordance with the long-held belief that Mars moved in a perfect circle. Also, the line connecting the Sun and Mars sweeps an equal area in an equal period of time. These discoveries are known as Kepler's first and second laws of planetary motion or the laws of ellipses and equal areas, respectively. These findings, of course, are dependent on the legacy of precise observations by Tycho Brahe (1546–1601). More importantly, it was Kepler's creativity that led him to apply geometric methods to convert astronomical data based on an Earth-centred model to a Sun-centred model.

Most of the relevant studies about Kepler's laws were derived algebraically from conservation principles of energy and angular momentum, which were not yet known to

Kepler [2–7]. The first part of this paper is intended to demonstrate a geometrical method, which is much simpler than the original depicted by Kepler, to obtain the law of equal areas using mainly the law of sines that every high school student learns. In the second part of this paper, the established law of equal areas for the Earth is applied to calculate the angular velocity of the Earth around the Sun. It can be shown to be a periodic function by analysing the data using current mathematical methods, and this characteristic may assist in understanding the law of ellipses for the Earth.

2. The law of equal areas for the Earth

To construct the law of equal areas for the Earth, we must first understand the position of the Earth relative to the Sun. However, how do we determine the Earth's location in the Universe? From thousands of years ago until the era of Kepler, all of the understanding about the motions of the planets and the Sun was based on observations taking the Earth as the origin from which the angles or positions of the planets and the Sun were measured. Specifically, man could only record the positions of the planets and the Sun as longitudes and latitudes and was unable to measure their distances from the Earth or calculate corresponding distance ratios. For this reason, Kepler, as a believer in the heliocentric model, needed to convert observational data based on the Earth as the origin to data based on the Sun as the origin, which required that he come up with a new approach. In Kepler's era, it was known that planets moved in the same plane. To locate an object on a plane, one must have at least two reference points as a basis. However, if the Sun is taken to be one fixed point, what can be used as a second reference point? If a fixed star is chosen, its distance cannot be determined, and if a nearby planet is used, its location in the sky is not stationary.

Nevertheless, Kepler noted that the time interval from the one opposition—where the Sun, Earth, and Mars are aligned—to the next opposition was about 780 days. Based on these data, he then determined the period of Mars' orbit around the Sun to be approximately 687 days. Because the position of Mars in the sky repeats every 687 days, this position can be used as a reference point. As a result, there are two fixed points to use as a benchmark. Finally, by applying the data of angular positions observed from the Earth, one can calculate the location of the Earth relative to the Sun.

The position is denoted by its ecliptic longitude where the longitude of the spring equinox is 0°, that of the summer solstice is 90° and so on. Kepler then chose the date of the opposition of Mars as a basic reference point. For convenience, one may select 5 am on 25 March 1950, when Mars was in opposition, i.e. the Sun (S), Earth (E) and Mars (M) were in a straight line as shown in figure 1. M will return to its original position every Martian year, and this can be treated as a second fixed point. E_i and E_j represent the positions of the Earth 1 Martian year before and after the opposition of Mars, respectively. From the observational data, these configurations occurred on 5 May 1948 and 8 February 1952, respectively. As a result, SE_iME_j forms a quadrilateral. If r_i and r_j are the lengths of the segments SE_i and SE_j, then r_i and r_j represent distances from the Sun to the Earth at two different times (figure 2).

The quadrilateral SE_iME_j can be treated as being made of $\angle SE_iM$ and $\angle SE_jM$, in which $\angle SE_iM$, $\angle SE_jM$, $\angle E_iMS$ and $\angle E_jMS$ can be observed as the longitudes of the Sun and Mars as seen from the Earth. The longitudes of the Sun and Mars were 44.7° and 144.9°, respectively, as seen from the Earth, E_i [8]. This implies the following equation:

$$\angle SE_iM = \mu_i = 144.9^\circ - 44.7^\circ = 100.2^\circ.$$

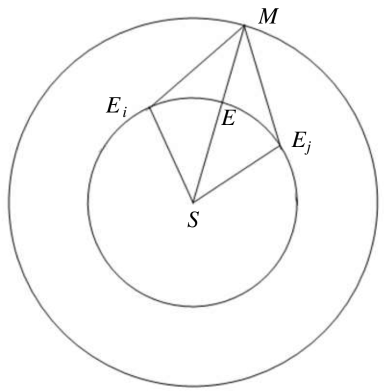

Figure 1. The positions of the Sun (S), Earth (E) and Mars (M). E_i and E_j represent the positions of the Earth 1 Martian year before and after the opposition of Mars, respectively.

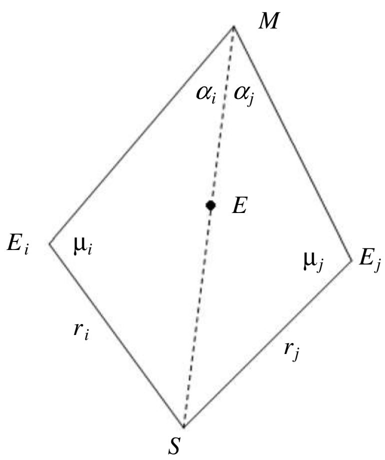

Figure 2. The quadrilateral formed by SE_iME_j. The angles μ_i, μ_j, α_i and α_j are observed.

Similarly, the longitudes of the Sun and Mars were 218.2° and 318.3°, respectively, as seen from the Earth, E_j [8]. This implies the following equation: **Q1**

$$\angle SE_jM = \mu_j = 318.3^\circ - 218.2^\circ = 100.1^\circ.$$

However, the longitude of Mars at the opposition of 25 March 1950 was 182.0°, and the longitude of Mars as seen from the Earth, E_i, on 5 May 1948 was 144.9°. Hence,

$$\angle E_iMS = \alpha_i = 182.0^\circ - 144.9^\circ = 37.1^\circ.$$

Similarly, the longitude of Mars as seen from the Earth, E_j, on 8 February 1954 was 218.2°.

$$\angle E_jMS = \alpha_j = 218.2^\circ - 182.0^\circ = 36.2^\circ.$$

The three interior angles in $\angle SE_iM$ and $\angle SE_jM$ are then all known, and SM is a common side. Therefore, by the law of sines,

$$\frac{r_i}{\sin\alpha_i} = \frac{SM}{\sin\mu_i}, \qquad \frac{r_j}{\sin\alpha_j} = \frac{SM}{\sin\mu_j}.$$

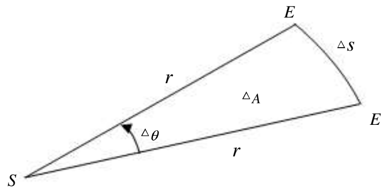

Figure 3. The law of equal areas for the Earth for a short period of time. The distance from the Earth to the Sun can be treated as a constant.

The relationship between r_i and r_j is as follows:

$$\frac{r_j}{r_i} = \frac{\sin\mu_i}{\sin\mu_j}\frac{\sin\alpha_j}{\sin\alpha_i}.$$

Originally, it was very hard to measure the actual distance between the Sun and the Earth. Now their ratio can be obtained from the corresponding angles $\angle SE_iM$ and $\angle SE_jM$ spanned by the lines connecting the Sun and Mars to the Earth 1 Martian year before and after the date of the opposition of Mars, respectively, and from the angles $\angle E_iMS$ and $\angle E_jMS$ spanned by the line of opposition of Mars and by the line connecting the Earth to Mars 1 Martian year before and after the date of the opposition of Mars, respectively. If ω_i and ω_j are the angular velocities of the Earth at E_i and E_j relative to the Sun, they can be calculated by the angles swept by the Earth in one day after the dates at E_i and E_j, respectively. Since the longitudes of the Earth as seen from the Sun on 5 May 1948 and 6 May 1948 are 224.897° and 225.865°, respectively, the angular speed $\omega_i = 225.866–224.897 = 0.969$. Similarly, $\omega_j = 139.561–138.549 = 1.012$.

The law of equal areas for the Earth means the line connecting the Earth to the Sun sweeps an equal area in the same period of time, as shown in figure 3. Namely,

$$\Delta A = \tfrac{1}{2}r^2\Delta\theta$$

and

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}r^2\frac{\Delta\theta}{\Delta t}.$$

So the areal velocity is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}r^2\omega,$$

where ω is the angular speed of the Earth around the Sun.

Hence, in order to prove the law of equal areas it is only necessary to show that the product of the square of the distance from the Earth to the Sun and the corresponding angular speed of the Earth is a constant, i.e.

$$r_i^2\omega_i = r_j^2\omega_j. \tag{1}$$

The position of the Earth E_i, which occurred 1 Martian year before the opposition of Mars as a reference point, is selected. Then six more positions of the Earth, E_j, are chosen, with three of them occurring 1, 2 and 3 Martian years before the time of the reference point, E_i. From observations of these E_j's, one can find the corresponding values of the ratios r_j^2/r_i^2 and ω_i/ω_j, as shown in table 1.

From the calculating results in the last two columns in the table, one may see that the difference between r_j^2/r_i^2 and ω_i/ω_j is very small, less than 1%, and can be treated as equal. In other words, $r_j^2\omega_j = r_i^2\omega_i$.

Table 1. r_j^2/r_i^2 and ω_i/ω_j obtained from the observatory data.

Time	μ_i	μ_j	α_i	α_j	ω_i	ω_j	r_j^2/r_i^2	ω_i/ω_j
13 September 1942		7.5°		4.6°		0.974	1.005	0.995
31 July 1944		33.8°		20.2°		0.956	1.026	1.014
18 June 1946		62.4°		33.2°		0.954	1.016	1.016
5 May 1948	100.2°		37.1°		0.969			
8 February 1952		100.1°		36.2°		1.012	0.959	0.958
26 December 1953		60.6°		31.5°		1.019	0.958	0.951
13 November 1955		30.3°		17.7°		1.007	0.967	0.962

Table 2. The values of ω_k, r_k^2/r_i^2 and ω_i/ω_k obtained on 30 different dates.

Dates	ω_i	ω_k	r_k^2/r_i^2	ω_i/ω_k	Dates	ω_i	ω_k	r_k^2/r_i^2	ω_i/ω_k
5 May 1948	0.969				5 May 1948	0.969			
26 October 1940		0.998	0.959	0.971	30 September 1957		0.983	0.982	0.986
9 December 1938		1.016	0.951	0.954	18 August 1959		0.961	1.008	1.008
21 January 1937		1.017	0.953	0.952	5 July 1961		0.953	1.016	1.016
6 March 1935		1.001	0.968	0.968	23 May 1963		0.962	1.005	1.008
18 April 1933		0.977	0.991	0.992	9 April 1965		0.982	0.988	0.987
1 June 1931		0.958	1.008	1.012	25 February 1967		1.005	0.958	0.964
14 July 1929		0.954	1.019	1.016	12 January 1969		1.019	0.950	0.951
27 August 1927		0.966	1.011	1.003	29 November 1970		1.014	0.953	0.956
9 October 1925		0.988	0.993	0.980	16 October 1972		0.992	0.974	0.976
22 November 1923		1.010	0.953	0.959	3 September 1974		0.969	0.999	1.000
4 January 1922		1.019	0.951	0.951	21 July 1976		0.955	1.012	1.015
17 February 1920		1.009	0.957	0.960	8 June 1978		0.957	1.010	1.013
1 April 1918		0.986	0.980	0.983	25 April 1980		0.973	0.991	0.996
14 May 1916		0.964	1.005	1.005	13 March 1982		0.997	0.974	0.972
27 June 1914		0.954	1.017	1.016	29 January 1984		1.016	0.951	0.954

All of the data shown in table 1 are based upon the positions of the Earth and its corresponding angular speed at the opposition of Mars on 25 March 1950, and 1 to 4 Martian years before and after, respectively. Similarly, these procedures can be continued for up to 30 more positions of the Earth, E_k, and its corresponding angular speed, ω_k, by including the data from 5 to 20 Martian years before and after the aforesaid opposition (table 2). The relationship $r_j^2\omega_j = r_i^2\omega_i$ still holds.

Because the periods of the Martian year, 687 days, and that of the Earth year, 365 days, are mutually prime, the positions of the Earth recorded every Martian year before and after 23 March 1950 will cover almost every position along the orbit of the Earth around the Sun. The positions selected by the method shown above are so dense as to approach generality. Due to this ergodicity, the law of equal areas may be established: $r^2\omega =$ constant.

Certainly, other oppositions of Mars can be examined, for example, 10 February 1916 and 12 February 1995. The difference between these two oppositions is only 1.8°. The procedures may be repeated as in table 2, and the relation of equal areas may be verified: $r^2\omega =$ constant. In retrospect, this method for establishing the law of equal areas is, on one hand, to apply the information on the oppositions of Mars as well as the period of Mars and, on the other hand, to use the mutual prime property between the 687 and 365 day periods of Mars and Earth, respectively, to guarantee ergodic distributions of the selected positions of the Earth.

It is very hard to directly measure the distances from the Earth or other planets to the Sun, or the ratio of the distances at two different positions. Nevertheless, it is relatively easy to observe the angles swept by the Earth. The establishment of the law of equal areas helps us, through measuring the angular speed of the planets, to derive the ratio of distances from the planets to the Sun at different times. This is the implicit meaning of the law of equal areas, which can be used effectively to determine distances from the planets to the Sun.

3. The law of ellipses for the Earth

After constructing the law of equal areas for the Earth, the next task is to establish the law of elliptical orbits for the Earth. Since the motion of the Earth around the Sun is regularly periodic, the distance function $r(\theta)$ from the Earth to the Sun, or the reciprocal of $r(\theta)$, can also be expressed as a periodic function of θ. Namely, it can be expressed as an infinite series of sines and cosines with different multiple angles as follows [9]:

$$\frac{1}{r} = a_0 + \sum_n a_n \cos n\theta + \sum_n b_n \sin n\theta \qquad (n = 1, 2, 3, \ldots).$$

In the ideal case, this function can be approximated by a single period of the trigonometric functions, i.e.

$$\frac{1}{r} = a_0 + a_1 \cos\theta + b_1 \sin\theta. \tag{2}$$

By applying the law of equal areas as shown in (1), or

$$\frac{1}{r} = c\sqrt{\omega},$$

where c is a proportional constant, the periodic function of the reciprocal of the distance may be expressed as follows:

$$\sqrt{\omega} = c_o + c_1 \cos\theta + c_2 \sin\theta.$$

By doing this, the observable angular speeds ω may be used to replace the unobservable distances r. In order to find the three unknown coefficients c_0, c_1 and c_2, as shown in the above equation, one has to choose three sets of data to set up simultaneous linear equations with three unknowns. One may randomly select three sets of data on 23 April 1998, 31 July 1998 and 2 October 1998 to form

$$\sqrt{\omega_1} = c_o + c_1 \cos\theta_1 + c_2 \sin\theta_1$$

$$\sqrt{\omega_2} = c_o + c_1 \cos\theta_2 + c_2 \sin\theta_2$$

$$\sqrt{\omega_3} = c_o + c_1 \cos\theta_3 + c_2 \sin\theta_3,$$

where θ_1, θ_2 and θ_3 are the inclined angles between the line connecting the Earth to the Sun at three different dates and the x-axis, which is set along the line connecting the Earth to the Sun on 27 January 1998. Hence $\theta_1 = 213.2°–127.4° = 85.8°$, as shown in table 3.

From the values of θ_1, θ_2, θ_3, ω_1, ω_2 and ω_3 in table 3, the solutions to the above simultaneous linear equations in three unknowns can be found. Their solutions are as follows:

$$c_0 = 0.993 \qquad c_1 = 0.015 \qquad c_2 = -0.007.$$

Table 3. The reference data for finding the periodic function of the reciprocal of the distance.

Time	Longitudes	θ_i	ω_i
27 January 1998	127.4°		
23 April 1998	213.2°	85.8°	0.975
3 July 1998	308.1°	180.7°	0.956
2 October 1998	9.1°	−118.3°	0.984

Table 4. Verifying the effectiveness for the periodic function of the distance at seven different positions.

Time	Longitudes	θ	ω	$\sqrt{\omega} - d$
12 February 1998	143.6°	16.2°	1.011	0.000
17 March 1998	176.7°	49.3°	0.995	0.000
6 May 1998	225.8°	98.4°	0.968	0.000
22 June 1998	270.9°	143.5°	0.954	0.000
18 August 1998	325.4°	198.0°	0.962	0.000
17 November 1998	55.0°	−72.4°	1.009	0.000
29 December 1998	97.6°	−29.8°	1.019	0.000

Hence, the periodic function for the square root of angular speed at three distinct positions of the Earth, or the reciprocal of the distances from three different positions of the Earth to the Sun, is as follows:

$$\sqrt{\omega} = 0.993 + 0.015\cos\theta - 0.007\sin\theta. \tag{3}$$

Furthermore, seven different positions of the Earth are randomly selected for seven different dates, and set $d = 0.993 + 0.015\cos\theta - 0.007\sin\theta$. By comparing the difference of $\sqrt{\omega}$ and d from the corresponding angular speed ω, whether the selected position of the Earth satisfies the periodic function of the distance can be verified, as shown in table 4. This table shows that $\sqrt{\omega}$ is identical to d, and this supports the validity of the periodic function in (3).

In fact, the periodic function shown in (2) and (3) is exactly the same as that for the path of an elliptical orbit. If one takes the line connecting the two foci to be the x-axis, then the path of elliptical orbit can be expressed as follows [10]:

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \qquad \text{or} \qquad \frac{1}{r} = B + A\cos\theta,$$

where e is eccentricity, a is the semi-major axis, $B = 1/a(1{-}e^2)$ and $A = e/a(1{-}e^2)$. If the x-axis is not the line connecting the two foci, then the equation for the ellipse is as follows:

$$\frac{1}{r} = B + A\cos(\theta - \theta_0) = B + d_1\cos\theta + d_2\sin\theta, \tag{4}$$

where θ_0 is the angle between the line connecting the perihelion to the origin and the x-axis, ${d_1}^2 + {d_2}^2 = A^2$, and $\sqrt{d_1^2 + d_2^2}/B = A/B = e$. Hence, the periodic function of the reciprocal of the distance shown in (2) or (3) has the form for the equation of ellipse, and the ratio of $\sqrt{c_1^2 + c_2^2}$ and c_0 is as follows:

$$\frac{\sqrt{c_1^2 + c_2^2}}{c_0} = \frac{\sqrt{(-0.007)^2 + 0.015^2}}{0.993} = 0.017,$$

which is exactly the same as the accepted value 0.017 for the eccentricity of the Earth's orbit. These can obviously show the equivalence of the periodicity of the reciprocal of the distance of the Earth to the equation of an ellipse, and verify firmly that the orbit of the Earth is exactly elliptical.

4. Conclusions

Applying the specific properties of the oppositions of Mars and the Martian year, the position of Mars relative to the Sun can be fixed in the celestial sphere. By means of the geometric relationship formed by these two fixed points and the motions of the Earth, one may overcome the obstacles due to the unobservable changes in the distances between the Earth and the Sun, and express them in terms of relatively easily measurable angles. Through these, the laws of equal areas and elliptical orbits for the Earth can be concisely established, which should confer a very deep and sincere admiration for Johannes Kepler and the insights he contributed some 400 years ago.

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