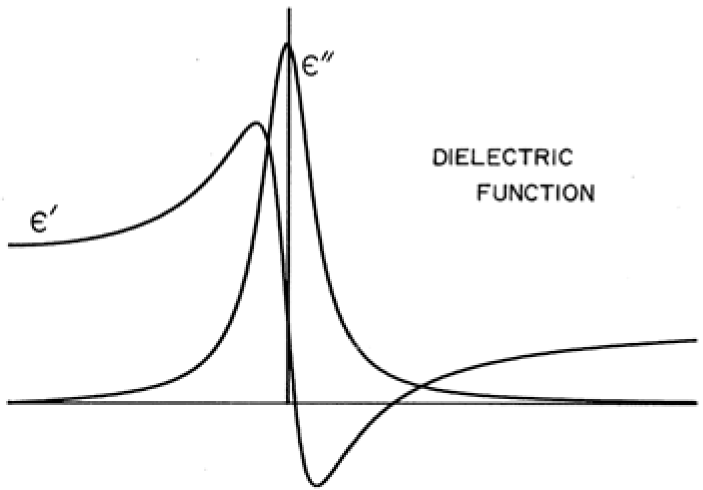
Chap 20 Phenomenological theory

- Dielectric function and EM wave propagation
- Drude model of the dielectric function
- Lorentz model of the dielectric function
- Kramers-Kronig relations
- Theory of linear response
    - Kubo-Greenwood formula



Dept of Phys

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Dielectric function

($\boldsymbol{r}$, t)-space | ($\boldsymbol{k}$, ω)-space

$$\nabla\cdot\vec{E}(\vec{r},t)=4\pi\rho(\vec{r},t)$$
$$\nabla\cdot\vec{D}(\vec{r},t)=4\pi\rho_{ext}(\vec{r},t)$$

$\Leftrightarrow$

$$i\vec{k}\cdot\vec{E}(\vec{k},\omega)=4\pi\rho(\vec{k},\omega)$$
$$i\vec{k}\cdot\vec{D}(\vec{k},\omega)=4\pi\rho_{ext}(\vec{k},\omega)$$

$$(\rho=\rho_{ext}+\rho_{ind})$$

Take the Fourier "shuttle" between 2 spaces:

$$\vec{E}(\vec{r},t)=\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{d\omega}{2\pi}\vec{E}(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{r}-\omega t)},\quad \text{same for } \vec{D}$$

$$\rho(\vec{r},t)=\int\frac{d^3\vec{k}}{(2\pi)^3}\frac{d\omega}{2\pi}\rho(\vec{k},\omega)e^{i(\vec{k}\cdot\vec{r}-\omega t)},\quad \text{same for } \rho_{ext}$$

$$\vec{D}(\vec{k},\omega)=\varepsilon(\vec{k},\omega)\vec{E}(\vec{k},\omega)$$ (by definition)

or $$\rho_{ext}(\vec{k},\omega)=\varepsilon(\vec{k},\omega)\rho(\vec{k},\omega)$$ (easier to calculate)

or $$\phi_{ext}(\vec{k},\omega)=\varepsilon(\vec{k},\omega)\phi(\vec{k},\omega)$$ $\because \vec{E}(\vec{k},\omega)=-i\vec{k}\phi(\vec{k},\omega)$... etc

Q: What is the relation between D(r,t) and E(r,t)?

EM wave propagation in solids

Maxwell equations

$$\nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\quad;\quad \nabla\cdot\vec{D}=4\pi\rho_{ext}$$

$$\nabla\times\vec{B}=+\frac{1}{c}\frac{\partial\vec{D}}{\partial t}+\frac{4\pi}{c}\vec{J}_{ext}\quad;\quad \nabla\cdot\vec{B}=0$$

$\Leftrightarrow$

$$i\vec{k}\times\vec{E}=+\frac{i\omega}{c}\vec{B}\quad;\quad \varepsilon_{ion}\vec{k}\cdot\vec{E}=4\pi\rho_{ext}$$

$$i\vec{k}\times\vec{B}=-\frac{i\omega}{c}\varepsilon_{ion}\vec{E}+\frac{4\pi}{c}\vec{J}_{ext};\quad i\vec{k}\cdot\vec{B}=0$$

$$\rightarrow \vec{k}\times\left(\vec{k}\times\vec{E}\right)=-\frac{\omega^2}{c^2}\varepsilon_{ion}\vec{E}-\frac{4\pi i\omega}{c^2}\sigma\vec{E}\qquad\left(\vec{J}=\sigma\vec{E}\right)$$

$$\vec{k}\times\left(\vec{k}\times\vec{E}\right)=\vec{k}\left(\vec{k}\cdot\vec{E}\right)-k^2\vec{E}$$

- Transverse wave

$$k^2=\frac{\omega^2}{c^2}\left(\varepsilon_{ion}+\frac{4\pi i\sigma}{\omega}\right)$$

$$\because \upsilon_p=\frac{\omega}{k}=\frac{c}{n},\ \text{refractive index } n=\sqrt{\varepsilon}$$

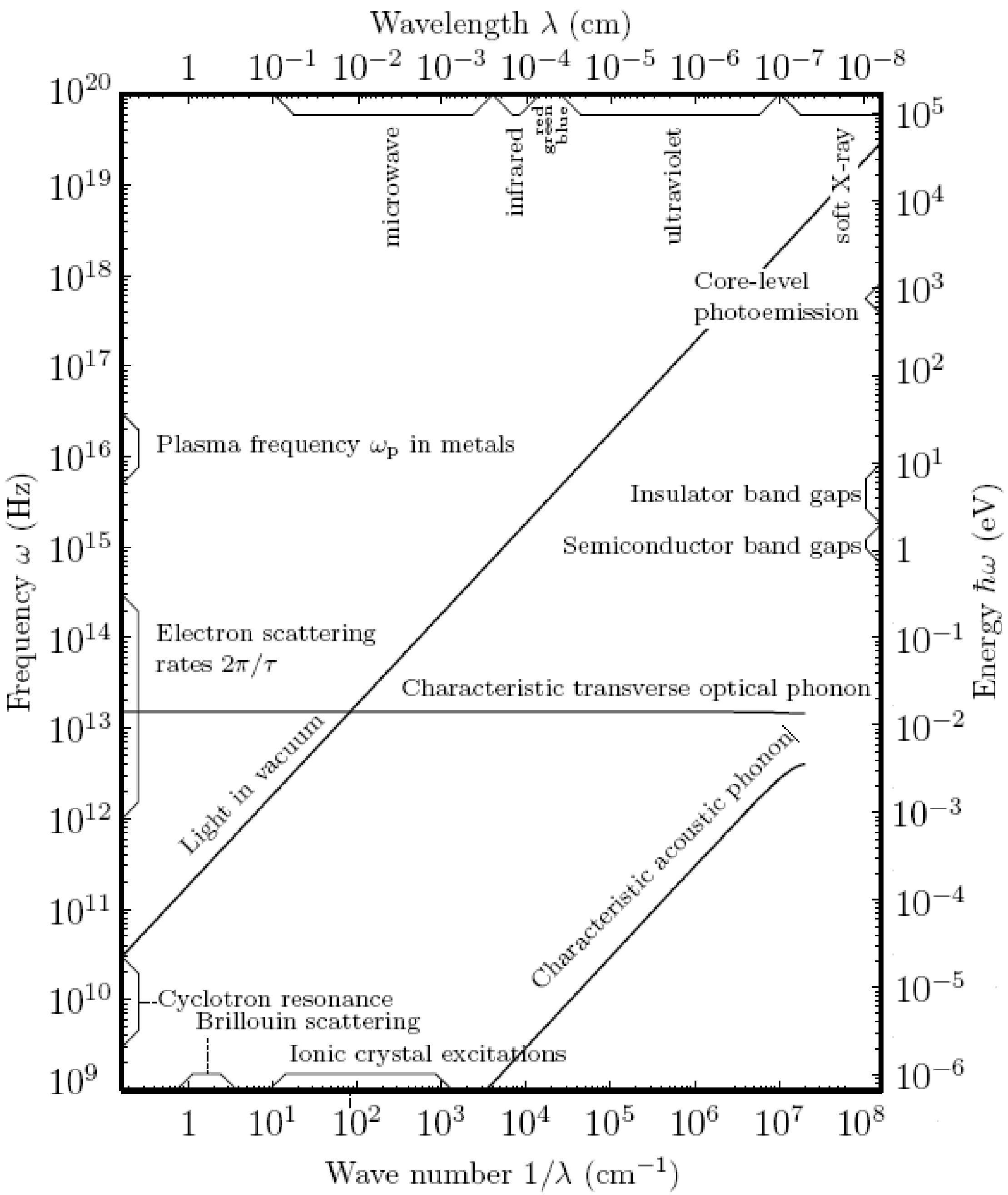
$$\therefore \varepsilon(\vec{k},\omega)=\varepsilon_{ion}+\frac{4\pi i\sigma}{\omega}$$

(in the following, let $\varepsilon_{\text{ion}}\sim 1$)

- Longitudinal wave

$$\varepsilon(\vec{k},\omega)=0$$

On the contrary, if $\varepsilon(\boldsymbol{k},\omega)\neq 0$, then the EM wave can only be transverse.



Drude model of AC conductivity

$$m_e \frac{d\langle \vec{v} \rangle}{dt} = -e\vec{E}(t) - m_e \frac{\langle \vec{v} \rangle}{\tau}$$

Assume

$$\vec{E}(t) = \vec{E}_0 e^{-i\omega t}$$

then

$$\langle \vec{v} \rangle = \vec{v}_0 e^{-i\omega t}$$

$$\rightarrow \langle \vec{v} \rangle = -\frac{e\tau / m_e}{1 - i\omega\tau} \vec{E}(t)$$

$$\rightarrow \vec{j} = -ne\langle \vec{v} \rangle = \sigma(\omega)\vec{E}$$

AC conductivity

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \quad \sigma_0 \equiv \frac{ne^2\tau}{m_e}$$

$$\varepsilon(\omega) = 1 + \frac{4\pi i \sigma}{\omega}$$

Lorentz model of the dielectric function

- Response of charged (independent) oscillators

For the ℓ-th oscillator (an atom or molecule with Z bound charges),

$$m_\ell\left(\frac{d^2}{dt^2}+\eta_\ell\frac{d}{dt}+\omega_\ell^2\right)\vec{r}_\ell(t)=-Z_\ell e\vec{E}_\omega e^{-i\omega t}$$

$$\vec{r}_\ell(t)=\vec{r}_{\ell\omega}e^{-i\omega t}\qquad \text{(for steady state)}$$

$$\rightarrow \vec{r}_{\ell\omega}=\frac{-Z_\ell e}{m_\ell\left(\omega_\ell^2-\omega^2-i\eta_\ell\omega\right)}\vec{E}_\omega$$

$$\rightarrow \vec{j}_\omega=\frac{1}{V}\sum_\ell(-Z_\ell e)\vec{v}_{\ell\omega}=\sigma(\omega)\vec{E}_\omega$$

- conductivity

$$\sigma(\omega)=\frac{1}{V}\sum_\ell\frac{-i\omega Z_\ell^2e^2}{m_\ell\left(\omega_\ell^2-\omega^2-i\eta_\ell\omega\right)}$$

$$\varepsilon(\omega)=1+\frac{4\pi i\sigma}{\omega}$$

$$=1+\frac{1}{V}\sum_\ell\frac{4\pi Z_\ell^2e^2/m_\ell}{\omega_\ell^2-\omega^2-i\eta_\ell\omega}$$

Or, electric polarization

$$\vec{P}_\omega=\frac{1}{V}\sum_\ell(-Z_\ell e)\vec{r}_{\ell\omega}=\chi_e(\omega)\vec{E}_\omega$$

electric susceptibility

$$\chi_e(\omega)=\frac{1}{V}\sum_\ell\frac{Z_\ell^2e^2}{m_\ell\left(\omega_\ell^2-\omega^2-i\eta_\ell\omega\right)}$$

$$\varepsilon(\omega)=1+4\pi\chi_e$$

will get the same result

N.P. Armitage 0908.1126

Conductivity of free electron gas

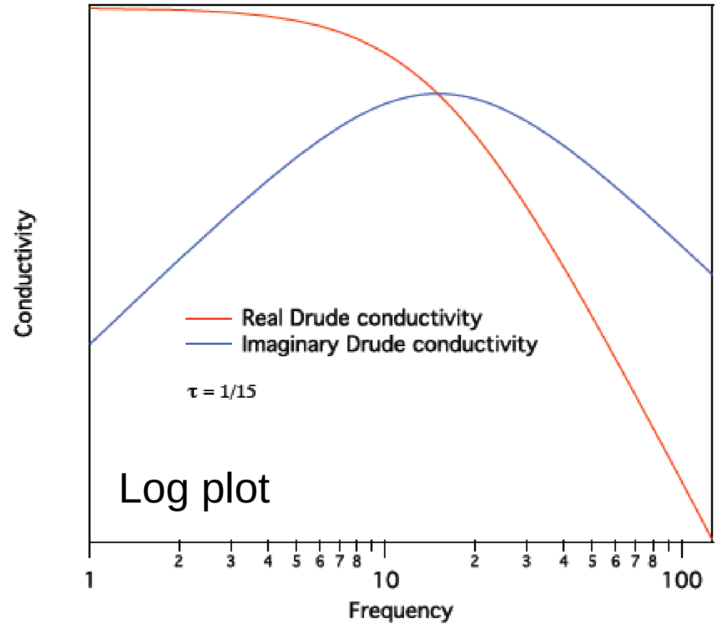
let ω_ℓ=0

$$\sigma(\omega)=\frac{1}{V}\sum_\ell \frac{i\omega e^2}{m\left(\omega^2+i\eta\omega\right)} \qquad \left(\eta=1/\tau\right)$$

$$=\frac{\sigma_0}{1-i\omega\tau}, \quad \sigma_0=\frac{ne^2\tau}{m}$$

Same as the Drude result

Q: what is the corresponding $\sigma(t)$?



- For a clean conductor at T=0

Plemelj formula (Landau QM p.156):

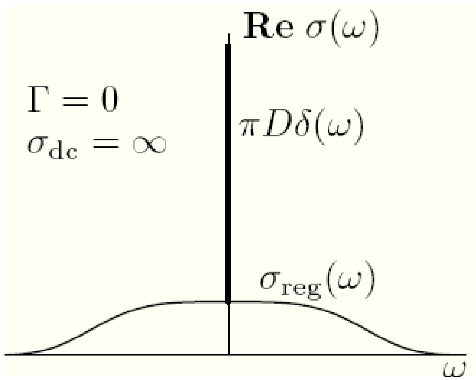
$$\lim_{\eta\to 0}\frac{1}{\omega\pm i\eta}=P\frac{1}{\omega}\mp i\pi\delta(\omega)$$

$$\therefore \sigma(\omega)=\frac{ne^2}{m}\left[\pi\delta(\omega)+\frac{i}{\omega}\right]$$

- Dielectric function

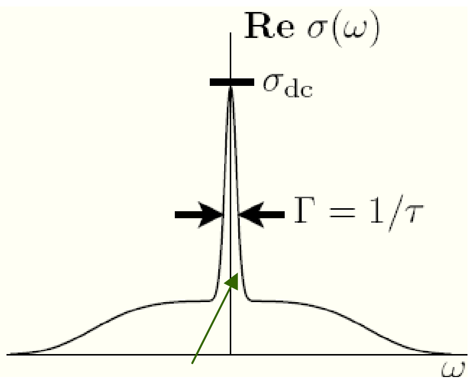
$$\varepsilon(\omega)=1+\frac{4\pi i\sigma}{\omega}$$

For a metal at T=0 (w/o disorder)



D: Drude weight

T≠0 or T=0 w/ disorder



Drude peak

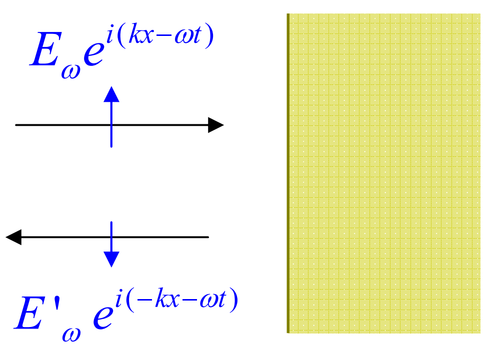
From Peter Jung's thesis

Reflectivity r and reflectance R

Response of a crystal to an EM field is characterized by $\varepsilon\,(k,\omega)$,
($k \sim 0$ compared to G/2)

- Experimentalists prefer to measure reflectivity r

- normal incidence



$$r(\omega) \equiv \frac{E'_\omega}{E_\omega}$$

$$r(\omega) = \frac{n-1}{n+1} \equiv \sqrt{R(\omega)}e^{i\theta(\omega)}$$

$$n(\omega) = \sqrt{\varepsilon(\omega)}$$

- It is easier to measure R than to measure θ

$\therefore$ measure R(ω) for every $\omega \rightarrow \theta(\omega)$ (with the help of KK relations)

$\rightarrow n(\omega)$

$\rightarrow \varepsilon(\omega)$

Lorentz model again

(for identical oscillators)

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\tau\omega\left(1 - \omega_0^2 / \omega^2\right)}$$

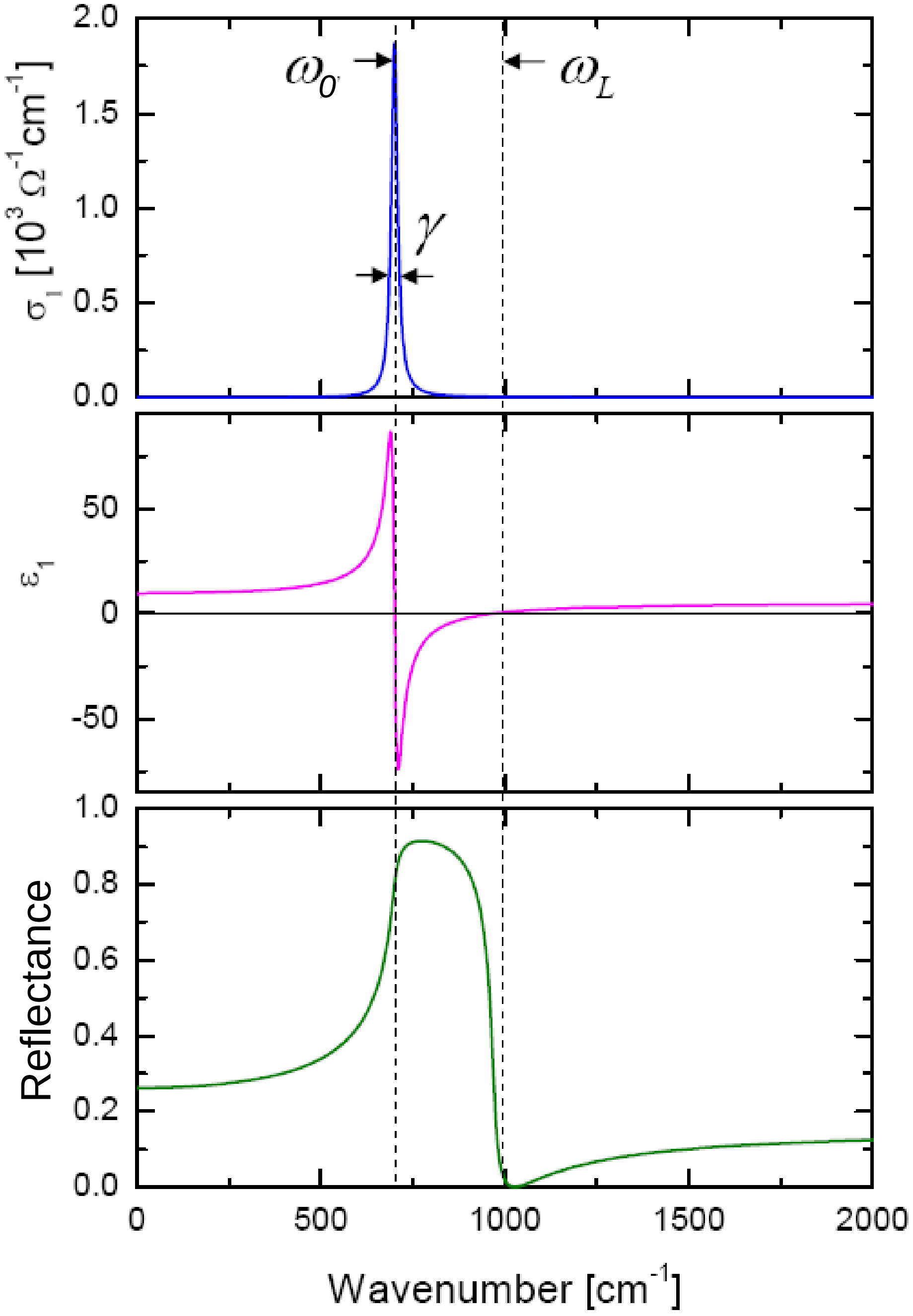
resonance at $\omega = \omega_0$

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega / \tau}$$

$\varepsilon'(\omega) = 0$ at $\omega = \omega_L$

$$\omega_L \cong \sqrt{\omega_p^2 + \omega_0^2}$$

$$\omega_p^2 \equiv \frac{4\pi n e^2}{m}$$



From Dr. A.B. Kuzmenko's slides

Kramers-Kronig relations (1926)

KK relation connects real part of the response function with the imaginary part

- Examples of response function:

$$j_\omega = \sigma(\omega) E_\omega$$
$$P_\omega = \chi_e(\omega) E_\omega$$
$$D_\omega = \varepsilon(\omega) E_\omega$$

- Properties of response function:

(α can be χ_e, or σ, or $\varepsilon - \varepsilon_{ion}$ ...)

- $\alpha(\omega)$ has no pole above (including) x-axis.

Due to causality, see Jackson Sec7.10

$\alpha = \alpha' + i\alpha''$

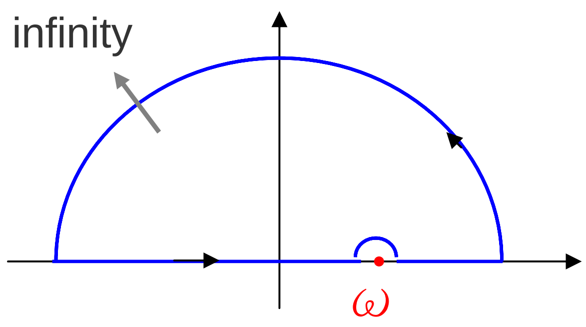
- $\alpha'(\omega)$ is even in ω, $\alpha''(\omega)$ is odd in ω.

since ε (t) is real

Therefore, we have

$$\alpha(\omega) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{\alpha(s)}{s-\omega} ds$$

$$\Rightarrow \begin{cases} \alpha'(\omega) = \dfrac{1}{\pi} P \displaystyle\int_{-\infty}^{\infty} \frac{\alpha''(s)}{s-\omega} ds \\ \alpha''(\omega) = -\dfrac{1}{\pi} P \displaystyle\int_{-\infty}^{\infty} \frac{\alpha'(s)}{s-\omega} ds \end{cases}$$



C.F. Bohren, What did Kramers and Kronig do and how did they do it? stacks.iop.org/EJP/31/573

• Kramers-Kronig relations

$$\alpha'(\omega) = \frac{2}{\pi}\int_0^\infty \frac{s\alpha''(s)}{s^2-\omega^2}ds$$
$$\alpha''(\omega) = -\frac{2\omega}{\pi}\int_0^\infty \frac{\alpha'(s)}{s^2-\omega^2}ds$$

• does not depend on any dynamic detail of the interaction

• the necessary and sufficient condition for its validity is causality

• A few sum rules:

$$\frac{2}{\pi}\int_0^\infty \frac{\alpha''(s)}{s}ds = \alpha'(0)$$

From $\alpha''(\omega >> 1)$

$$\frac{2}{\pi}\int_0^\infty \alpha'(s)ds = \lim_{\omega\to\infty}\omega\alpha''(\omega)$$

For conductivity, it's the Drude weight

and more…, e.g.,

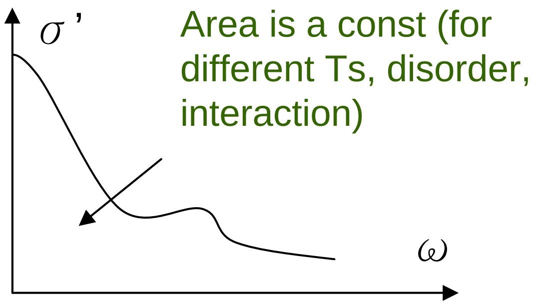
$$\frac{2}{\pi}\int_0^\infty s\varepsilon''(s)ds = \frac{4\pi ne^2}{m_0} \quad \left(=\omega_p^2\right)$$

(check, e.g., Lorentz model. Need exact form of ε for a general proof.)

Conductivity sum rule

$$\text{or} \quad \int_0^\infty \sigma'(s)ds = \frac{1}{8}\omega_p^2$$

Ferrell and Glover (1958)
(check Drude conductivity)



KK relation for reflection

$$E'_\omega = r(\omega) E_\omega$$

$$\ln r(\omega) = \ln R^{1/2}(\omega) + i\theta(\omega)$$

KK related

Why take log?

"That *ln* r satisfies the conditions for the validity of this relation follows from a causality argument." (Jahoda 1957)

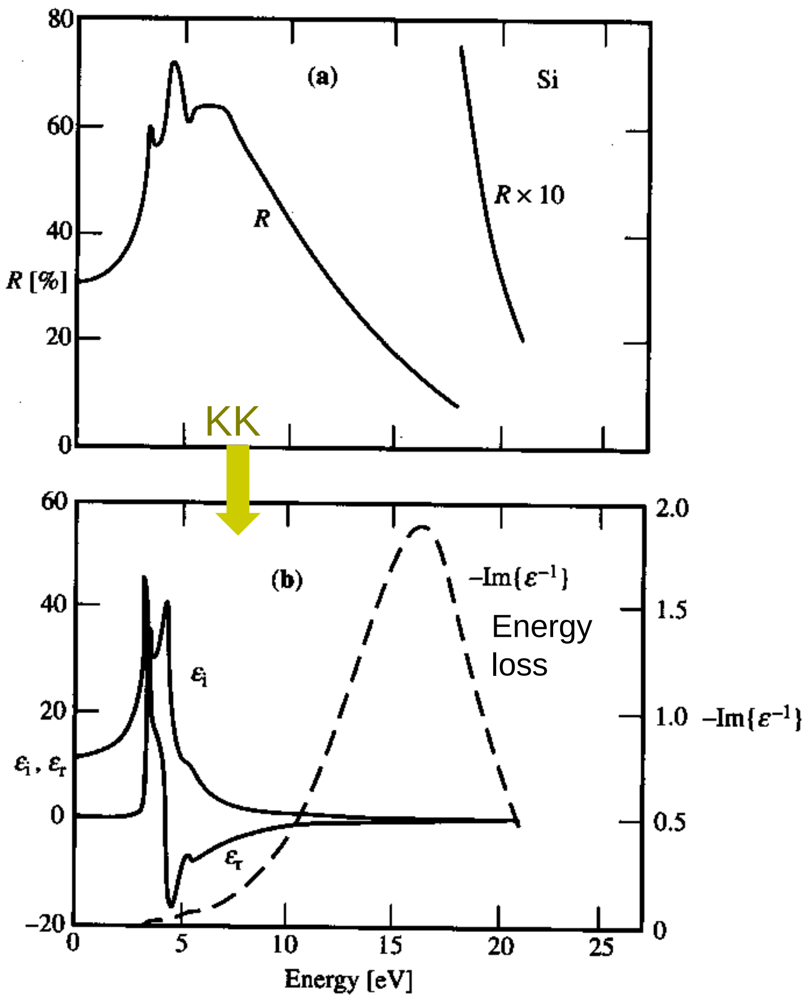
See Wooten, App G; Yu and Cardona p.252 for related discussion

Get the phase numerically

$$\theta(\omega) = -\frac{\omega}{\pi}\int_0^\infty \frac{\ln R(s)}{s^2-\omega^2} ds$$

$$= -\frac{1}{2\pi}\int_0^\infty \ln\left|\frac{s+\omega}{s-\omega}\right| \frac{d\ln R}{ds} ds$$

- constant R(s) doesn't contribute
- s>>ω, s<<ω don't contribute



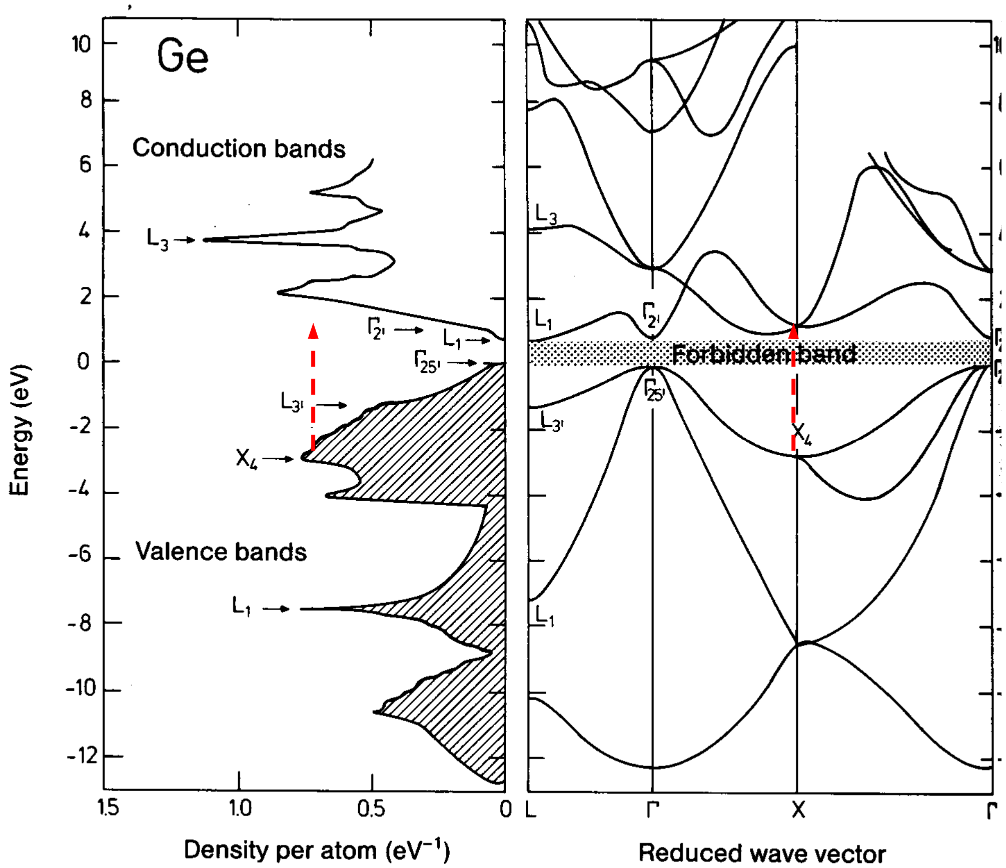
This trick is first used by Jahoda, Phys. Rev. 1957.

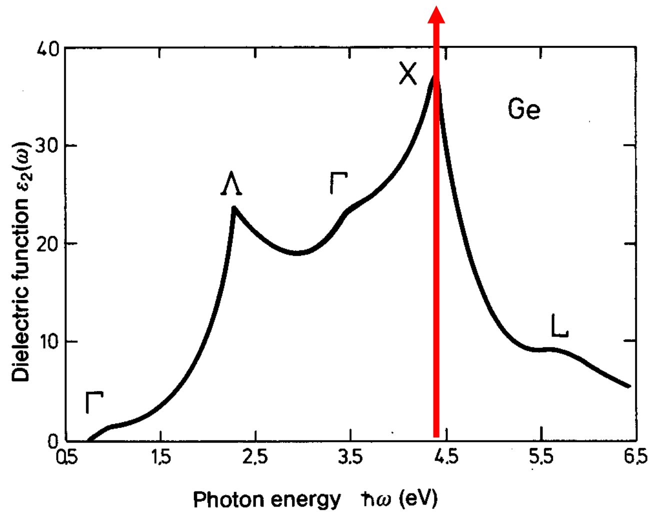
An application of the sum rule

Drude form: $\varepsilon''(\omega) = \dfrac{\omega_p^2}{\omega}\pi\delta(\omega)$ $\quad\Rightarrow\quad$ $\varepsilon'(\omega) = 1 - \dfrac{\omega_p^2}{\omega^2 - \omega_g^2}$

band gap absorption (due to nonzero ε'') can be very roughly approximated by $\Rightarrow \dfrac{\omega_p^2}{2\omega}\pi\delta(\omega-\omega_g)$ $\quad\therefore\quad$ $\varepsilon'(0) = 1 + \dfrac{\omega_p^2}{\omega_g^2}$ **(smaller band gap, larger dielectric constant)**

From sum rule ↑





	Si	Ge	GaAs	InP	GaP
ε'	12.0	16.0	10.9	9.6	9.1
Eg (theo)	4.8	4.3	5.2	5.2	5.75
Ex (exp't)	4.44	4.49	5.11	5.05	5.21

(ref: Cardona and Yu)

Figs from Ibach and Luth

Boltzmann approach to AC conductivity

$$\frac{\partial f}{\partial t}+\dot{\vec{r}}\cdot\frac{\partial f}{\partial \vec{r}}-e\vec{E}\cdot\frac{\partial f}{\hbar\partial \vec{k}}=-\frac{f-f^0}{\tau_\varepsilon}$$

$$\frac{\partial f_0}{\partial t}=\frac{\partial f_0}{\partial \vec{r}}=0$$

$$\rightarrow \frac{\partial \delta f}{\partial t}+\dot{\vec{r}}\cdot\frac{\partial \delta f}{\partial \vec{r}}-e\vec{E}\cdot\frac{\partial f_0}{\hbar\partial \vec{k}}=-\frac{\delta f}{\tau_\varepsilon}$$

assume $\vec{E}(\vec{r},t)=\vec{E}_0 e^{i(\vec{q}\cdot\vec{r}-\omega t)}$

$$\delta f=\Phi(\vec{k})e^{i(\vec{q}\cdot\vec{r}-\omega t)}$$

$$\rightarrow -i\omega\Phi+i\vec{q}\cdot\vec{v}\,\Phi-e\vec{E}_0\cdot\vec{v}\,\frac{\partial f_0}{\partial \varepsilon}=-\frac{\Phi}{\tau_\varepsilon}$$

$$\rightarrow \Phi=\frac{e\tau\vec{E}_0\cdot\vec{v}}{1-i\tau\left(\omega-\vec{q}\cdot\vec{v}\right)}\frac{\partial f_0}{\partial \varepsilon}$$

$$\vec{j}=-e\int[dk]\vec{v}\,\delta f$$

$$\rightarrow \boldsymbol{\sigma}(\vec{q},\omega)=e^2\int[dk]\frac{\tau\vec{v}\vec{v}}{1-i\tau\left(\omega-\vec{q}\cdot\vec{v}\right)}\left(-\frac{\partial f_0}{\partial \varepsilon}\right)$$

Note:

An improvement over the classical expression

$$\boldsymbol{\sigma}(0,0)=e^2\int[dk]\tau\vec{v}\vec{v}\left(-\frac{\partial f_0}{\partial \varepsilon}\right)\equiv\boldsymbol{\sigma}_{dc}$$

(see chap 17)

$$\boldsymbol{\sigma}(0,\omega)=\frac{\boldsymbol{\sigma}_{dc}}{1-i\omega\tau}$$

Time-dependent perturbation theory, a review

$H=H_0+H'(t)$

- Without H', $\left|\ell(t)\right\rangle = e^{-i\varepsilon_\ell^0 t/\hbar}\left|\ell\right\rangle$, where $H_0\left|\ell\right\rangle = \varepsilon_\ell^0\left|\ell\right\rangle$

- With H', $\left|\ell(t)\right\rangle \to \left|\tilde{\ell}(t)\right\rangle = \sum_m c_m(t)e^{-i\varepsilon_m t/\hbar}\left|m\right\rangle, \quad c_m(0)=\delta_{\ell m}$

$$i\hbar\left|\frac{d\tilde{\ell}(t)}{dt}\right\rangle = \left(H_0 + H'(t)\right)\left|\tilde{\ell}(t)\right\rangle$$

$$\to i\hbar\dot{c}_n = \sum_m \left\langle n\right|H'(t)\left|m\right\rangle e^{i\omega_{nm}t}c_m(t), \quad \omega_{nm}\equiv(\varepsilon_n^0-\varepsilon_m^0)/\hbar$$

To 1[st] order, $\cong \sum_m \left\langle n\right|H'(t)\left|m\right\rangle e^{i\omega_{nm}t}c_m^0(t), \quad c_m^0(t)=\delta_{\ell m}$

$$c_n^1(t)=\delta_{\ell n}+\frac{1}{i\hbar}\int^t dt'\left\langle n\right|H'(t')\left|\ell\right\rangle e^{i\omega_{n\ell}t'}$$

$$\to\left|\tilde{\ell}(t)\right\rangle = \sum_n c_n^1(t)e^{-i\omega_n t}\left|n\right\rangle$$

$$= e^{-i\omega_\ell t}\left|\ell\right\rangle + \frac{1}{i\hbar}\sum_n\int^t dt' e^{-i\omega_n(t-t')}\left\langle n\right|H'(t')\left|\ell\right\rangle e^{-i\omega_\ell t'}\left|n\right\rangle$$

or $\left|\tilde{\ell}(t)\right\rangle = e^{-i\omega_\ell t}\left|\ell\right\rangle + \frac{1}{i\hbar}\sum_n\int^t dt'\left\langle n\right|H'_I(t')\left|\ell\right\rangle e^{-i\omega_n t}\left|n\right\rangle$

Interaction representation

$H'_I(t)\equiv e^{iH_0 t/\hbar}H'(t)e^{-iH_0 t/\hbar}$

Theory of linear response

Assume $H'(t)=\int d^3r'\hat{B}(\vec{r}')\cdot h(\vec{r}',t)$

then

$$\left\langle \tilde{\ell}(t)\right|\hat{A}\left|\tilde{\ell}(t)\right\rangle = \langle \ell|\hat{A}|\ell\rangle + \frac{1}{i\hbar}\int^{t} dt'\langle \ell|\left[\hat{A}_I(t),\hat{H}'_I(t')\right]|\ell\rangle$$

$$= \langle \ell|\hat{A}|\ell\rangle + \frac{1}{i\hbar}\int^{t} dt'd^3r'\langle \ell|\left[\hat{A}_I(t),\hat{B}_I(\vec{r}',t')\right]|\ell\rangle h(\vec{r}',t')$$

In general, need to take thermal average

$$\langle \ell|\hat{O}|\ell\rangle \to \left\langle \hat{O}\right\rangle \equiv \sum_{\ell} f_\ell \langle \ell|\hat{O}|\ell\rangle, \quad f_\ell=\frac{e^{-\varepsilon_\ell^0/kT}}{Z}$$

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• Kubo formula (1957)

$$\delta\left\langle A(\vec{r},t)\right\rangle = \frac{1}{i\hbar}\int^{t} dt'd^3r'\left\langle\left[A_I(\vec{r},t),B_I(\vec{r}',t')\right]\right\rangle h(\vec{r}',t')$$

Response function

• Harmonic perturbation (1-particle):

$$H'(t)=U(\vec{r})e^{+\eta t}e^{-i\omega t}$$

$$=U(\vec{r})e^{-i\Omega t}, \quad \Omega\equiv\omega+i\eta!$$

(adiabatically turning on the perturbation)

then

$$\delta\left\langle A(t)\right\rangle = \frac{1}{i\hbar}\int_{-\infty}^{t} dt'\left\langle\left[A_I(t),U_I(t')\right]\right\rangle e^{-i\Omega t'}$$

$$=\sum_{\ell m}\frac{f_\ell - f_m}{\hbar(\omega_{\ell m}+\Omega)}\langle \ell|A|m\rangle\langle m|U|\ell\rangle e^{-i\Omega t}$$

$$\omega_{\ell m}\equiv\omega_\ell-\omega_m$$

Density response: susceptibility

$$H'(t) = -e\phi_{ext}(q,\omega)e^{i\vec{q}\cdot\vec{r}-i\Omega t} \leftrightarrow U(\vec{r}) = -e\phi_{ext}(q,\omega)e^{i\vec{q}\cdot\vec{r}}$$

- Charge density

$$\langle\hat{\rho}(\vec{r},t)\rangle = -e\sum_\ell f_\ell \left|\langle\vec{r}|\tilde{\ell}(t)\rangle\right|^2$$

$$=\sum_\ell f_\ell \langle\tilde{\ell}(t)|-e\delta(\hat{\vec{r}}-\vec{r})|\tilde{\ell}(t)\rangle$$

$$\to \hat{\rho}(\vec{r}) = -e\delta(\hat{\vec{r}}-\vec{r})$$

- Change of charge density (Prob. 5)

$$\delta\langle\hat{\rho}(\vec{r},t)\rangle = \sum_{\ell m}\frac{f_\ell - f_m}{\hbar(\omega_{\ell m}+\Omega)}\langle\ell|\hat{\rho}(\vec{r})|m\rangle\langle m|\hat{U}(\vec{r})|\ell\rangle e^{-i\Omega t}$$

- Take plane waves as the unperturbed states

$$\langle\vec{r}|\ell\rangle \to \langle\vec{r}|\vec{k}\sigma\rangle = \frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{r}}\chi_\sigma$$

$$\to \delta\rho(\vec{q},\omega) = \chi(\vec{q},\omega)\phi_{ext}(q,\omega)$$

$$\boxed{\chi(\vec{q},\omega) = \frac{e^2}{V}\sum_{k\sigma}\frac{f_k - f_{k+q}}{\hbar(\omega_k - \omega_{k+q} + \omega + i\eta)}}$$

$$\rho = \rho_{ext} + \delta\rho$$

$$q^2\phi_{ext} = 4\pi\rho_{ext}$$

$$\therefore \rho = \left(1 + 4\pi\frac{\chi}{q^2}\right)\rho_{ext}$$

(α = $4\pi\frac{\chi}{q^2}$)

- Random Phase Approx. (RPA)

$$\rho = \left[1+\alpha+\alpha^2+\cdots\right]\rho_{ext}$$

$$= \frac{1}{1-4\pi\frac{\chi}{q^2}}\rho_{ext} = \frac{\rho_{ext}}{\varepsilon}$$

$\to$ dielectric function

$$\varepsilon(\vec{q},\omega) = 1 - \frac{4\pi\chi}{q^2}$$

This χ is not the electric susceptibility mentioned earlier

Long wavelength limit (H.W.)

$$\varepsilon(\vec{q},\omega) = 1 - \frac{4\pi e^2}{q^2 V}\sum_{k\sigma}\frac{f_k - f_{k+q}}{\hbar\left(\omega_k - \omega_{k+q} + \Omega\right)}$$

$$= 1 - \frac{4\pi e^2}{\hbar q^2 V}\sum_{k\sigma} f_k \frac{2(\omega_k - \omega_{k+q})}{\left(\omega_k - \omega_{k+q}\right)^2 - \Omega^2}$$

- For $\omega \gg v_F q$

$$\varepsilon(\vec{q},\omega) \cong 1 - \frac{4\pi e^2}{V}\sum_{k\sigma}\frac{f_k}{m\Omega^2}\left[1 + 3\frac{\hbar^2(\vec{q}\cdot\vec{k})^2}{m^2\Omega^2}\right]$$

$$= 1 - \frac{4\pi e^2}{m\Omega^2}\frac{1}{V}\sum_{k\sigma} f_k\left(1 + \frac{\hbar^2 q^2}{m^2\Omega^2}k^2\right)$$

$$\frac{1}{V}\sum_{k\sigma} f_k k^2 = \frac{3}{5} n k_F^2 \qquad \text{(at T=0)}$$

$$\therefore\ \varepsilon(\vec{q}\to 0,\omega) = 1 - \frac{\omega_p^2}{\omega^2}\left[1 + \frac{3}{5}\left(\frac{\hbar k_F}{m\omega}\right)^2 q^2\right] \qquad \omega_p^2 \equiv \frac{4\pi n e^2}{m}$$

Static limit $(\omega \ll v_F q)$

$$\chi(\vec{q},0)=\frac{e^2}{V}\sum_{k\sigma}\frac{f_k-f_{k+q}}{\hbar\left(\omega_k-\omega_{k+q}\right)}$$

- For long wavelength

$$\chi(\vec{q})\approx-\frac{2e^2}{V}\sum_{\vec{k}}\left(-\frac{\partial f_k}{\partial\varepsilon}\right)$$

$$=-e^2 g(\varepsilon_F),\quad \text{or}\ \ e^2\frac{\partial n}{\partial\varepsilon}\quad (=\chi_{TF})$$

- For general wave length, we have

$$\chi(\vec{q})=-e^2 g(\varepsilon_F)\left[\frac{1}{2}+\frac{1-x^2}{4x}\ln\left(\frac{1+x}{1-x}\right)\right]$$

$$=\chi_{TF}F(x),\ \ x=q/2k_F$$

- It can be shown (not easy) that for $k_F r \gg 1$

$$\phi_{ind}(\vec{r})=\int\frac{d^3\vec{q}}{(2\pi)^3}\left[\frac{\phi_{ext}(\vec{q})}{\varepsilon(\vec{q})}-\phi_{ext}(\vec{q})\right]e^{i\vec{q}\cdot\vec{r}}$$

$$\approx\frac{c}{r^3}\cos 2k_F r$$

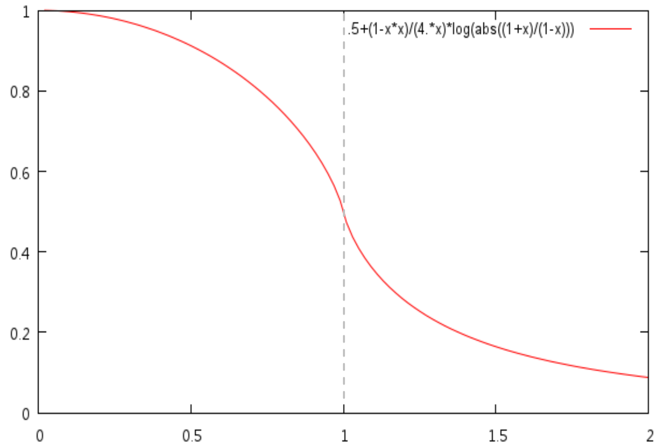
Friedel oscillation

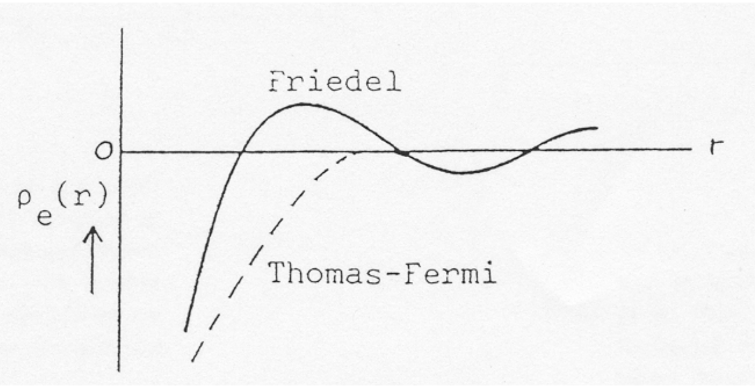
Note:

ε ($\boldsymbol{q},\omega$) is nonanalytic at (0,0)!

$$\lim_{q\to0}\lim_{\omega\to0}\varepsilon(\vec{q},\omega)\neq\lim_{\omega\to0}\lim_{q\to0}\varepsilon(\vec{q},\omega)$$

Lindhard function F(x)





Haug and Schmitt-Rink, Prog. Q. Elect., 1984

Current response: conductivity

Vector potential of an uniform electric field $\vec{E}(t) = -\frac{1}{c}\frac{\partial \vec{A}(t)}{\partial t}$

$$\vec{E}(t) = \vec{E}_\omega e^{-i\Omega t}, \text{ then } \vec{A}(t) = \vec{A}_\omega e^{-i\Omega t}; \vec{E}_\omega = \frac{i\Omega}{c}\vec{A}_\omega$$

$$H = \frac{1}{2m_0}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + V_{latt} = H_0 + \frac{e}{m_0 c}\vec{A}\cdot\vec{p} + O(A^2), \quad \left(\text{choose } \nabla\cdot\vec{A} = 0\right)$$

$$H' = \frac{e}{m_0 c}\vec{p}\cdot\vec{A}_\omega e^{-i\Omega t} \quad \leftrightarrow \quad U = \frac{e}{m_0 c}\vec{p}\cdot\vec{A}_\omega$$

$$\vec{j} = \frac{-e/V}{m_0}\left(\vec{p} + \frac{e}{c}\vec{A}\right)$$

Diamagnetic current **Paramagnetic current**

$$\rightarrow \delta\left\langle j_\alpha(t)\right\rangle = -\frac{e^2}{m_0 c V}\left\langle A_\alpha(t)\right\rangle + \frac{1}{V}\sum_{\ell m}\frac{f_\ell - f_m}{\hbar(\omega_{\ell m} + \Omega)}\left\langle \ell\right|\frac{-e}{m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)_\alpha\left|m\right\rangle\left\langle m\right|\hat{U}\left|\ell\right\rangle e^{-i\Omega t}$$

$$\delta\left\langle j_\alpha(t)\right\rangle = \delta\left\langle j_\alpha(\omega)\right\rangle e^{-i\Omega t}$$

$$\rightarrow \delta\left\langle j_\alpha(\omega)\right\rangle = -\frac{e^2}{i\Omega m_0 V}\sum_\ell f_\ell E_\alpha(\omega) - \frac{e^2}{i\Omega m_0^2 V}\sum_{\ell m}\frac{f_\ell - f_m}{\hbar(\omega_{\ell m} + \Omega)}\left\langle \ell\right| p_\alpha\left|m\right\rangle\left\langle m\right| p_\beta\left|\ell\right\rangle E_\beta(\omega)$$

$$= \sigma_{\alpha\beta}(\omega)E_\beta(\omega)$$

Note: $\sigma_{\alpha\beta}(\vec{q},\omega) : p_\alpha, p_\beta \rightarrow p_\alpha e^{-i\vec{q}\cdot\vec{r}}, p_\beta e^{i\vec{q}\cdot\vec{r}}$

A brief summary:

- Classical (Drude, Lorentz)

$$\sigma(\omega)=\frac{\sigma_0}{1-i\omega\tau}$$

- Semiclasical (Boltzmann)

$$\sigma_{\alpha\beta}(\vec{q},\omega)=e^2\int[dk]\left(-\frac{\partial f_0}{\partial\varepsilon}\right)\frac{\tau v_\alpha v_\beta}{1-i\tau\left(\omega-\vec{q}\cdot\vec{v}\right)}$$

- Quantum (Kubo-Greenwood)

$$\sigma_{\alpha\beta}(\vec{q},\omega)=-\frac{e^2}{i\Omega}\left[\frac{n}{m_0}\delta_{\alpha\beta}+\frac{1}{V}\sum_{\ell m}\frac{f_\ell-f_m}{\hbar(\omega_{\ell m}+\Omega)}v^\alpha_{\ell m}v^\beta_{m\ell}\right]$$

ℓ = (n, k, σ)
m = (n', $k+q$, σ')

$$v^\beta_{m\ell}=\frac{1}{m_0}\left\langle m\left|p_\beta e^{i\vec{q}\cdot\vec{r}}\right|\ell\right\rangle$$

- interband transitions

- Quantum manybody (Kubo-Greenwood, in 2nd quantized form)

- Relaxation time
- electron localization
- electron interaction, phonons

…

Descendant of the Kubo-Greenwood formula (I)

$$\sigma_{\alpha\beta}(0,\omega) = -\frac{e^2}{i\Omega m_0}\left[n\delta_{\alpha\beta} + \frac{1}{m_0 V}\sum_{\ell m}\frac{f_\ell - f_m}{\hbar(\omega_{\ell m}+\Omega)} p^{\alpha}_{\ell m} p^{\beta}_{m\ell} \right]$$

- $$\frac{1}{\omega_{\ell m}\pm\Omega} = \frac{1}{\omega_{\ell m}}\left(1 \mp \frac{\Omega}{\omega_{\ell m}\pm\Omega}\right)$$

H.W.: Verify this. Hint:

$p_\alpha = (im/\hbar)[H, r_\alpha]$

$[r_\alpha, p_\beta] = i\hbar\delta_{\alpha\beta}$

- f-sum rule (note: many sum rules are called f-sum rules)

$$n\delta_{\alpha\beta} + \frac{1}{m_0 V}\sum_{\ell m}\frac{f_\ell - f_m}{\hbar\omega_{\ell m}} p^{\alpha}_{\ell m} p^{\beta}_{m\ell} = 0$$

A result of gauge invariance.
See Tremblay A.M.- N-corps, p.102
Akkermans and Montambaux, p.299
P.Allen's article, p.174

$$\rightarrow \sigma_{\alpha\beta}(\omega) = \frac{e^2}{iV}\sum_{\ell m}\frac{f_\ell - f_m}{\hbar\omega_{\ell m}}\frac{v^{\alpha}_{\ell m} v^{\beta}_{m\ell}}{\omega_{\ell m}+\Omega}$$

Grosso and Parravicini, p.430
$q \neq 0$ allowed

- real and imaginary parts

$$\begin{cases} \sigma'_{\alpha\alpha}(\omega) = -\frac{\pi e^2}{V}\sum_{\ell m}\frac{f_\ell - f_m}{\hbar\omega_{\ell m}}\left|v^{\alpha}_{\ell m}\right|^2 \delta\left(\omega + \omega_{\ell m}\right) \\ \sigma''_{\alpha\alpha}(\omega) = -\frac{e^2}{V}\sum_{\ell m}\frac{f_\ell - f_m}{\hbar\omega_{\ell m}}\frac{\left|v^{\alpha}_{\ell m}\right|^2}{\omega_{\ell m}+\omega} \end{cases}$$

P.Allen's article, p.173
$\alpha \neq \beta$ is allowed

Check

$$\int_{-\infty}^{\infty}\sigma_{\alpha\alpha}'(s)ds = \frac{1}{4}\omega_p^2$$

Consider a perfect crystal. The current matrix elements are k-diagonal.
For a semiconductor or insulator, this is the usual optical inter-band conductivity. For a metal, we have additional diagonal elements within a band.

- Intraband conductivity

$$\left\langle \vec{k}' \middle| e^{i\vec{q}\cdot\vec{r}} p_\beta \middle| \vec{k} \right\rangle \cong m_0 v_\beta \delta_{k'.k+q}$$

$$\sigma_{\alpha\beta}(\vec{q},\omega) \cong \frac{2e^2}{iV} \sum_k \frac{f_k - f_{k+q}}{\hbar\left(\omega_k - \omega_{k+q}\right)} \frac{v_\alpha v_\beta}{\omega_k - \omega_{k+q} + \omega + i\eta}$$

$$\text{for } q \ll k, \quad \frac{\partial f}{\partial \varepsilon} \cong \frac{f_{k+q} - f_k}{\varepsilon_{k+q} - \varepsilon_k}$$

$$\therefore \ \sigma_{\alpha\beta}(\vec{q},\omega) \cong ie^2 \int [dk] \left(-\frac{\partial f}{\partial \varepsilon}\right) \frac{v_\alpha v_\beta}{\omega_k - \omega_{k+q} + \omega + i\eta},$$

$$\omega_k - \omega_{k+q} \cong -\vec{q}\cdot\vec{v}; \quad \eta = \tau^{-1}$$

$$\rightarrow \sigma_{\alpha\beta}(\vec{q},\omega) \cong e^2 \int [dk] \left(-\frac{\partial f}{\partial \varepsilon}\right) \frac{v_\alpha v_\beta \tau}{1 - i\tau\left(\omega - \vec{q}\cdot\vec{v}\right)}$$

redue to
Boltzmann result

$$\sigma_{\alpha\beta}(0,\omega) \cong \frac{ie^2}{\omega + i\tau^{-1}} \int [dk] \left(-\frac{\partial f}{\partial \varepsilon}\right) v_\alpha v_\beta = \frac{ie^2}{\omega + i\tau^{-1}} \frac{n}{m_\alpha^{op}} \delta_{\alpha\beta}$$

Note:

$$\sigma_{\alpha\alpha}^{DC} = \lim_{\omega\to 0} \lim_{q\to 0} \sigma_{\alpha\alpha}(\vec{q},\omega)$$

See P.Allen's article, p.167, 174

Grosso and Parravicini, p.414

Recall ch 17 (also see Marder p.694)

$$v_\alpha v_\beta \left(-\frac{\partial f_0}{\partial \varepsilon} \right) = v_\alpha \frac{\partial \varepsilon}{\hbar \partial k_\beta} \left(-\frac{\partial f_0}{\partial \varepsilon} \right) = v_\alpha \left(-\frac{\partial f_0}{\hbar \partial k_\beta} \right)$$

$$\therefore \int [dk] v_\alpha v_\beta \left(-\frac{\partial f_0}{\partial \varepsilon} \right)$$

$$= \int [dk] \left[-\frac{\partial (f_0 v_\alpha)}{\hbar \partial k_\beta} + f_0 \frac{\partial v_\alpha}{\hbar \partial k_\beta} \right]$$

$$= \int [dk] f_0 \cdot m^*_{\alpha\beta}(\vec{k})^{-1}$$

$$\equiv \frac{n}{m^{op}_{\alpha\beta}} \delta_{\alpha\beta}$$ aka: optical effective mass

- $m^{op} = m^*$ if the carriers are near band bottom
- $1/m^{op} = 0$ for a filled band
- This seems to be an effect related to the exclusion principle.

Less carriers are able to transit in a fuller band.
Again see F. Wooten p.78.

supplementary

Descendant of the Kubo-Greenwood formula (II)

$$\sigma_{\alpha\beta}(\omega) = -\frac{e^2}{i\Omega m_0}\left[n\delta_{\alpha\beta} + \frac{1}{m_0 V}\sum_{\ell m}\frac{f_\ell - f_m}{\hbar(\omega_{\ell m}+\Omega)} p^\alpha_{\ell m} p^\beta_{m\ell} \right]$$

$$\bullet \frac{1}{\Omega}\frac{1}{\omega_{\ell m}\pm\Omega} = \frac{1}{\Omega}\frac{1}{\omega_{\ell m}} \mp \frac{1}{\omega_{\ell m}}\frac{1}{\omega_{\ell m}\pm\Omega}$$

$$\rightarrow \sigma_{\alpha\beta}(\omega) = \frac{ine^2}{\Omega}\left(\frac{1}{m^{op}}\right)_{\alpha\beta}\delta_{\alpha\beta} + \frac{e^2}{im_0^2}\frac{1}{V}\sum_{\ell m}\frac{f_\ell - f_m}{\hbar\omega_{\ell m}(\omega_{\ell m}+\Omega)} p^\alpha_{\ell m} p^\beta_{m\ell}$$

$$\bullet \frac{1}{\omega_{\ell m}(\omega_{\ell m}+\Omega)} = \frac{1}{\omega_{\ell m}^2 - \Omega^2} + \frac{\Omega}{\omega_{\ell m}}\frac{1}{\omega_{\ell m}^2-\Omega^2}$$

$$\therefore \text{ 2nd term in } \sigma_{\alpha\beta}(\omega) = \frac{e^2}{im_0^2}\frac{1}{\hbar V}\left[\sum_{\ell m}(f_\ell - f_m)\frac{p^\alpha_{\ell m} p^\beta_{m\ell}}{\omega_{\ell m}^2-\Omega^2} + \sum_{\ell m}(f_\ell - f_m)\frac{\Omega}{\omega_{\ell m}}\frac{p^\alpha_{\ell m} p^\beta_{m\ell}}{\omega_{\ell m}^2-\Omega^2} \right]$$

$$\begin{cases} \sigma^2_{\alpha\beta}(\omega) = \dfrac{e^2}{m_0^2}\dfrac{1}{\hbar V}\sum_{\ell m}(f_\ell - f_m)\dfrac{\mathrm{Im}\left(p^\alpha_{\ell m} p^\beta_{m\ell}\right)}{\omega_{\ell m}^2-\Omega^2} \\ \sigma^3_{\alpha\beta}(\omega) = \dfrac{e^2}{im_0^2}\dfrac{1}{\hbar V}\sum_{\ell m}(f_\ell - f_m)\dfrac{\mathrm{Re}\left(p^\alpha_{\ell m} p^\beta_{m\ell}\right)}{\omega_{\ell m}^2-\Omega^2}\dfrac{\Omega}{\omega_{\ell m}} \end{cases}$$

AC (optical) Hall conductivity
e.g., see Rashba PRB 2004

Wang and Callaway PRB 1974

supplementary

DC Hall conductivity for a filled Landau subband (Thouless et al, 1982)

$$\sigma_{\alpha\neq\beta}^{DC} = 0 + \frac{e^2}{im_0^2}\frac{1}{\hbar V}\sum_{\ell m} f_\ell \frac{p_{\ell m}^\alpha p_{m\ell}^\beta - p_{\ell m}^\beta p_{m\ell}^\alpha}{\omega_{\ell m}^2} + 0$$

(one-band) $$= \frac{2e^2}{i\hbar V}\sum_k f_k \left(\left\langle \frac{\partial u_{nk}}{\partial k_\alpha} \middle| \frac{\partial u_{nk}}{\partial k_\beta} \right\rangle - \left\langle \frac{\partial u_{nk}}{\partial k_\beta} \middle| \frac{\partial u_{nk}}{\partial k_\alpha} \right\rangle \right)$$

$$= \frac{e^2}{\hbar} \int_{filled} [dk]\, \Omega_{n\gamma}(\vec{k}) \quad (\alpha,\beta,\gamma \text{ are cyclic})$$

- This integral has a close connection with topology (1st Chern number)

$\rightarrow$ quantization of the quantum Hall conductance

- Recall an alternative, semiclassical approach

$$\frac{d\vec{r}}{dt} = \frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \vec{k}} - \dot{\vec{k}} \times \vec{\Omega}(\vec{k})$$

$$\vec{j} = -e\int [dk]\, \vec{v}_k f_0$$

$$= -\frac{e}{\hbar}\int [dk] \frac{\partial \varepsilon}{\partial \vec{k}} f_0 + \frac{e^2}{\hbar}\int [dk]\, \vec{\Omega}(\vec{k}) f_0 \times \vec{E}$$

$$\rightarrow \sigma_{xy} = \frac{e^2}{\hbar} \int_{filled} [dk]\, \Omega_z(\vec{k})$$

supplementary

Onsager relation for the response function

Kubo formula
$$\delta\langle A(\vec r,t)\rangle=\int_{-\infty}^{\infty}dt'd^3r'\,\chi^R_{AB}(\vec r-\vec r',t-t')h(\vec r',t')$$

- Retarded response function

$$\chi^R_{AB}(\vec k,t-t')\equiv\frac{1}{i\hbar}\theta(t-t')\left\langle\left[A(-\vec k,t),B(\vec k,t')\right]\right\rangle$$

- Time reversal

$$\langle \ell|A|m\rangle\to\langle m|A(-t)|\ell\rangle$$
$$=\eta_A\langle m|A(t)|\ell\rangle,\quad \eta_A=\pm$$
$$\left\langle A(-\vec k,t)B(\vec k,0)\right\rangle\to\eta_A\left\langle B(-\vec k,0)A(\vec k,t)\right\rangle=\eta_A\eta_B\left\langle B(-\vec k,t)A(\vec k,0)\right\rangle$$
$$\therefore\ \chi^R_{AB}(\vec k,t)\to\eta_A\eta_B\frac{1}{i\hbar}\theta(t)\left\langle\left[B(-\vec k,t),A(\vec k,0)\right]\right\rangle=\eta_A\eta_B\chi^R_{BA}(\vec k,t)$$
$$\chi^R_{AB}(\vec k,\omega)\to\eta_A\eta_B\chi^R_{BA}(\vec k,\omega)$$ Onsager relation

- By the way, another response function

$$\chi_{AB}(\vec k,t-t')\equiv\frac{1}{i\hbar}\left\langle\left[A(-\vec k,t),B(\vec k,t')\right]\right\rangle$$

aka: generalized susceptibility

$$\chi_{AB}(\vec k,\omega)=\frac{1}{i\hbar}\int_{-\infty}^{\infty}dt\left\langle\left[A(-\vec k,t),B(\vec k,0)\right]\right\rangle e^{i\omega t}$$
$$=\begin{cases}2i\chi''^R_{AB}(\vec k,\omega)\ \text{if}\ \eta_A=-\\ 2\chi'^R_{AB}(\vec k,\omega)\ \text{if}\ \eta_A=+\end{cases}\qquad\leftarrow\chi^R_{AB}(-\omega)=\chi^{R\,*}_{AB}(\omega)$$

See L.P. Levy, p.154, 164

Fluctuation-dissipation theorem (Callen and Welton 1951, Takahashi, Kubo 1952 …)

Thus random impacts of surrounding molecules generally cause two kinds of effect: firstly, they act as a random driving force on the Brownian particle or the mirror to maintain its incessant irregular motion, and, secondly, they give rise to the frictional force for a forced motion. The first is the *systematic* part of the effect and the second is the *random* part. This in turn means that the frictional force and the random force must be related, because both come from the same origin. This internal relationship between the systematic and the random parts of *microscopic forces* is, in fact, a very general matter, which is manifested in the so-called *fluctuation–dissipation theorem.*

As we shall see in the following, this theorem states a general relationship between the response of a given system to an external disturbance and the internal fluctuation of the system in the absence of the disturbance. Such a response is characterized by a response function or equivalently by an admittance, or an impedance. The internal fluctuation is characterized by a correlation function of relevant physical quantities of the system fluctuating in thermal equilibrium, or equivalently by their fluctuation spectra. The fluctuation–dissipation theorem can thus be used in two ways: it can predict the characteristics of the fluctuation or the noise intrinsic to the system from the known characteristics of the admittance or the impedance, or it can be used as the basic formula to derive the admittance from the analysis of thermal fluctuations of the system. The Nyquist theorem is a classical example of the first category (Nyquist 1928), whereas, perhaps, Onsager's proof of the symmetry of kinetic coefficients is the oldest example of the second (Onsager 1931).

Kubo Rep. Prog. Phys. 1966

Fluctuation-dissipation theorem

- Dynamic structure factor

aka: correlation function

$$S_{AB}(\vec{k},\omega) \equiv \int_{-\infty}^{\infty} dt \left\langle A(-\vec{k},t)B(\vec{k},0)\right\rangle e^{i\omega t} \qquad \beta \equiv \frac{1}{k_B T}$$

$$\left\langle B(\vec{k},0)A(-\vec{k},t)\right\rangle = \cdots = \left\langle A(-\vec{k},t-i\beta\hbar)B(\vec{k},0)\right\rangle$$

← use cyclic symm of trace (Kubo-Martin-Schwinger identity)

$$\therefore \chi_{AB}(\vec{k},\omega) = \frac{1}{i\hbar}\left(1-e^{-\hbar\omega/k_B T}\right) S_{AB}(\vec{k},\omega)$$

$$\begin{cases} \chi''^{R}_{AB}(\omega) = -\dfrac{1}{2\hbar}\left(1-e^{-\hbar\omega/k_B T}\right) S_{AB}(\omega) & \text{if } \eta_A = - \\ \chi'^{R}_{AB}(\omega) = -\dfrac{i}{2\hbar}\left(1-e^{-\hbar\omega/k_B T}\right) S_{AB}(\omega) & \text{if } \eta_A = + \end{cases}$$

(aka: Callen-Welton formula)

"+" sign if H' ∼ −B.h

- Classical form ($\eta_A = -$)

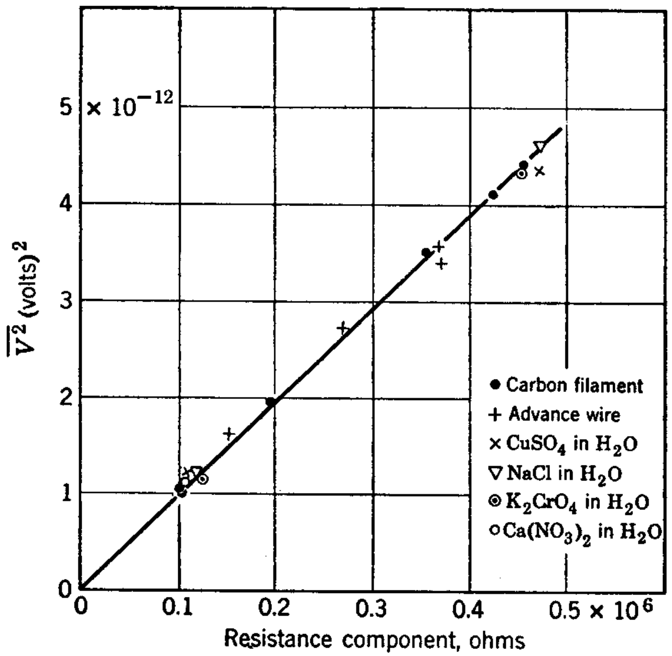
$$\chi''^{R}_{AB}(\omega) = -\frac{\omega}{2k_B T} S_{AB}(\omega)$$

Close connection between the spontaneous fluctuations in the system (dynamic structure factor), and the response of the system to external perturbations (susceptibility).

See L.P. Levy, p.155

supplementary

Johnson noise (1928) Bell Lab



The slope is independent of material, shape, and conduction mechanism

Nyquist theorem for electronic noise (1928)

$$H' = \frac{1}{c}\int d^3r'\vec{j}\cdot\vec{A}_\omega e^{-i\omega t}, \quad \vec{E}_\omega = \frac{i\omega}{c}\vec{A}_\omega$$

$$\delta\langle j_\alpha(\vec{r},t)\rangle = \frac{1}{i\hbar c}\int^t dt'd^3r'\langle[j_\alpha(\vec{r},t), j_\beta(\vec{r}',t')]\rangle A_\beta e^{-i\omega t'}$$

$$= -\frac{1}{\hbar\omega}\int^t dt'd^3r'\langle[j_\alpha(\vec{r},t), j_\beta(\vec{r}',t')]\rangle e^{-i\omega t'}E_\beta$$

$$\to \sigma_{\alpha\beta}(\vec{k},t-t') = -\frac{1}{\hbar\omega}\theta(t-t')\langle[j_\alpha(-\vec{k},t), j_\beta(\vec{k},t')]\rangle$$

$$\sigma_{\alpha\beta}(\vec{k},\omega) = \frac{1}{i\omega}\chi^R_{j_\alpha j_\beta}(\vec{k},\omega)$$

$$\to S_{j_\alpha j_\beta}(\omega) = \frac{2\hbar}{1-e^{-\hbar\omega/k_BT}}\sigma'_{\alpha\beta}(\vec{k},\omega)$$

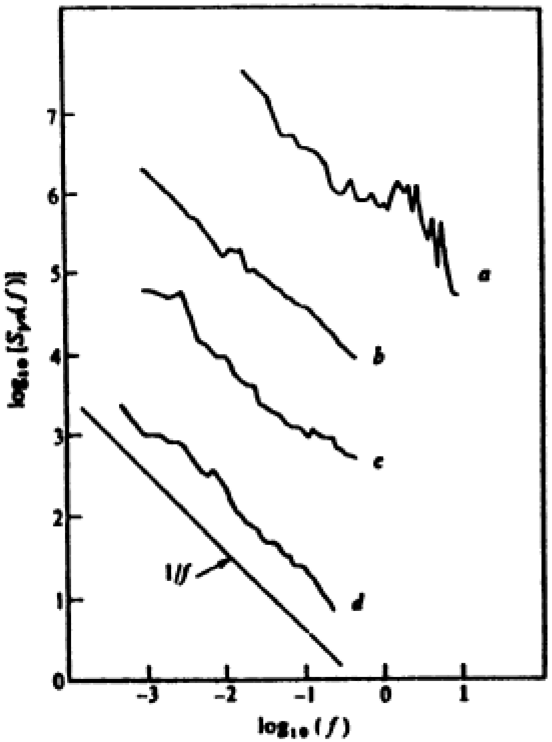
quantum (shot n.) ↓ ↓ thermal (Johnson n.)

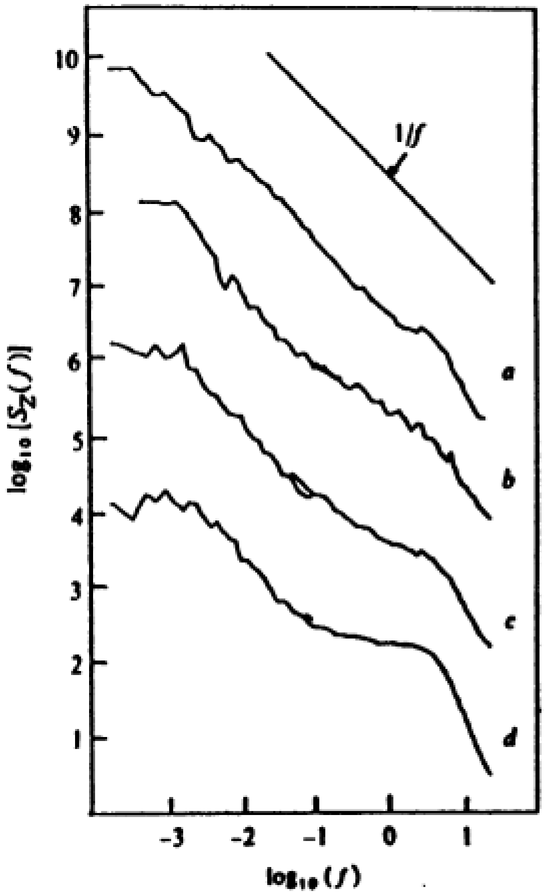
$$\text{or } \sigma'_{\alpha\beta}(\vec{k},\omega) = \frac{1-e^{-\hbar\omega/k_BT}}{2\hbar}\int_{-\infty}^{\infty} dt\langle j_\alpha(-\vec{k},t)j_\beta(\vec{k},0)\rangle e^{i\omega t}$$

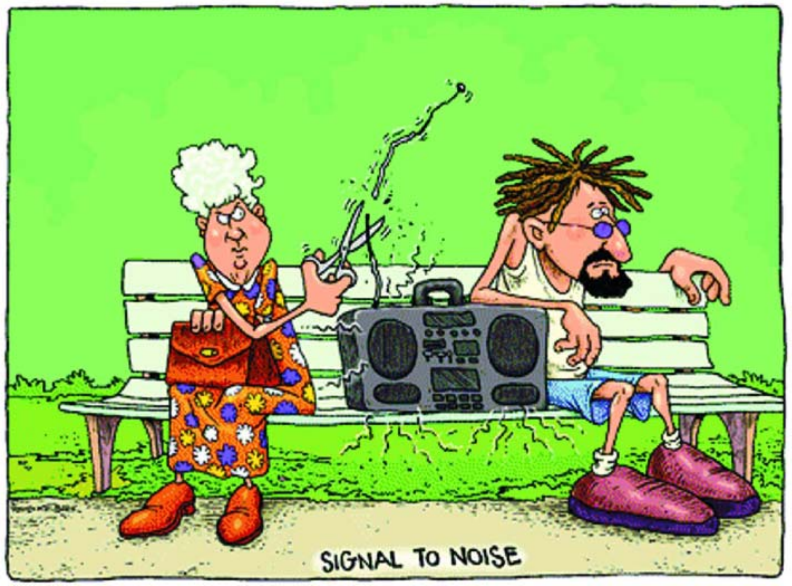
Non-equil. property **Equilibrium property**

(This offers one way to determine the Boltzmann constant.)

Zagoskin, p.116







Loudness (left) and pitch (right) fluctuation spectra vs. frequency (Hz) (log-log scale), for a. Scott Joplin piano rags; b. classical radio station; c. rock station; d. news-and-talk station.

Voss and Clark, Nature 1975

C. Beenakker & C. Schönenberger, Phys Today May 2003