Chap 17 Transport phenomena and Fermi liquid theory

- Boltzmann equation
- Onsager reciprocal relation
- Thermal electric phenomena
  - Seebeck, Peltier, Thomson…
- Classical Hall effect, anomalous Hall effect
- Theory of Fermi liquid
  - e-e interaction and Pauli exclusion principle
  - specific heat, effective mass
  - 1st sound and zero sound

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Boltzmann equation

- Distribution function: f(**r**,**k**,t) (“g” in Marder’s)

$$f(\vec{r},\vec{k},t)d^3r\cdot 2\frac{d^3k}{(2\pi)^3} =$$ Number of electrons within d^3r and d^3k around (**r**,**k**) at time t

For example, $\vec{j}(\vec{r},t) = -e\int 2\frac{d^3k}{(2\pi)^3}\dot{\vec{r}}f(\vec{r},\vec{k},t)$

- Evolution of the distribution function

$$t \to t+dt$$

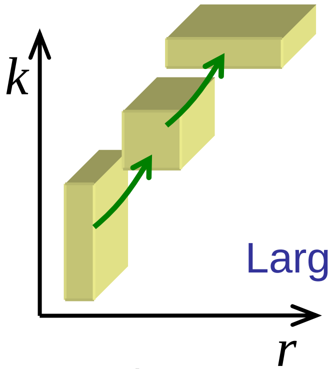
$$f(\vec{r},\vec{k},t) \to f(\vec{r}+d\vec{r},\vec{k}+d\vec{k},t+dt)$$

$$=f(\vec{r},\vec{k},t)+\frac{\partial f}{\partial \vec{r}}\cdot d\vec{r}+\frac{\partial f}{\partial \vec{k}}\cdot \vec{k}+\frac{\partial f}{\partial t}dt$$

<u>without collision</u>

the phase-space density does not change in the comoving frame

$$\frac{df}{dt} = \frac{\partial f}{\partial t}+\dot{\vec{r}}\cdot\frac{\partial f}{\partial \vec{r}}+\dot{\vec{k}}\cdot\frac{\partial f}{\partial \vec{k}} = 0$$



Larger △r ↔ smaller △k

Phase space is incompressible (Liouville’s Theorem)

supplementary

With collision
(due to disorder… etc): $$\frac{\partial f}{\partial t}+\dot{\vec{r}}\cdot\frac{\partial f}{\partial \vec{r}}+\dot{\vec{k}}\cdot\frac{\partial f}{\partial \vec{k}}=\left(\frac{\partial f}{\partial t}\right)_{coll}$$ Boltzmann eq.

~ Source/drain

- Transition rate for an electron at k → k’:

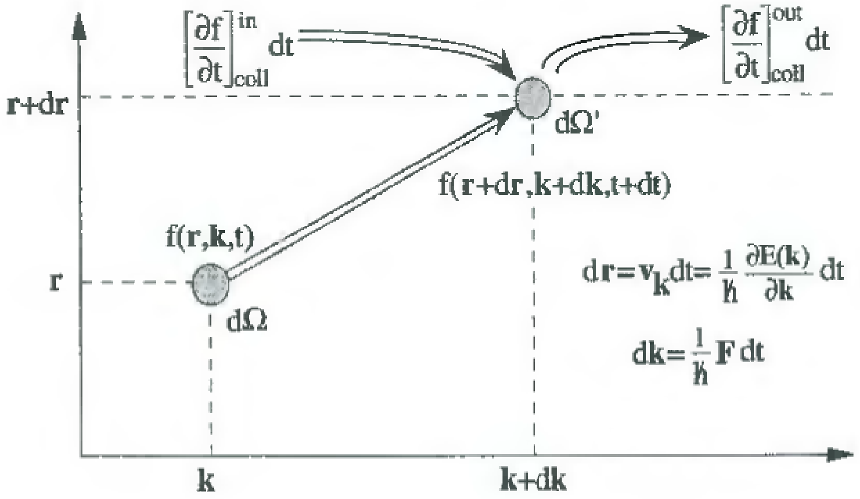
$W_{k,k'}$ (calculated by Fermi golden rule)

In a crowded space, one needs to consider occupancy and summation

$$W_{k,k'}\rightarrow\sum_{k'}f_k(1-f_{k'})W_{k,k'}$$

- On the other hand, the transition rate for an electron to be scattered into k

$$\sum_{k'}(1-f_k)f_{k'}W_{k',k}$$



Grosso, SSP, p.404

$$\therefore\left(\frac{\partial f}{\partial t}\right)_{coll}=\sum_{k'}\left[(1-f_k)f_{k'}W_{k',k}-f_k(1-f_{k'})W_{k,k'}\right]$$

$$=\sum_{k'}(f_{k'}-f_k)W_{k',k}\qquad \text{if } W_{k,k'}=W_{k',k}$$ (for elastic scattering)

see Marder, Sec 18.2 for more

supplementary

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = \sum_{k'} (f_{k'} - f_k) W_{k',k}$$

if $f_k = f_k^0 + \vec{C}\cdot\vec{k}$, then

(valid for uniform *E*, *B*, *T* fields)

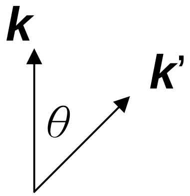
$$\left(\frac{\partial f}{\partial t}\right)_{coll} = -\vec{C}\cdot\sum_{k'}(\vec{k}-\vec{k}')W_{k',k}$$

$(k = k')$

$$= -\vec{C}\cdot\vec{k}\sum_{k'}(1-\hat{k}\cdot\hat{k}')W_{k',k}$$

$$= -\frac{f_k - f_k^0}{\tau_\varepsilon}$$

Only the component of ***k***' // ***k*** contribute to the integral (if W depends only on θ)



where $\frac{1}{\tau_\varepsilon} = \sum_{k'}(1-\hat{k}\cdot\hat{k}')W_{k',k}$

Transport relaxation time

Note: for inelastic scatterings, detailed balance requires

$$(1- f_k^0)\, f_{k'}^0 W_{k',k} = f_k^0 (1- f_{k'}^0) W_{k,k'}$$

$$f^0(\vec{k}) = \frac{1}{e^{\left[\varepsilon(\vec{k})-\mu\right]/k_B T}+1}$$

$$\rightarrow e^{\frac{\varepsilon(\vec{k})-\mu}{k_B T}} W_{k',k} = e^{\frac{\varepsilon(\vec{k}')-\mu}{k_B T}} W_{k,k'}$$

- Relaxation time approximation

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = -\frac{f(\vec{r},\vec{k},t)-f_0(\vec{r},\vec{k})}{\tau\left(\varepsilon(\vec{k})\right)}$$

Unperturbed equilibrium dist

Relaxation (allows energy dependence)

$$f_0(\vec{r},\vec{k})=\frac{1}{\exp\left(\frac{\varepsilon(\vec{k})-\mu(\vec{r})}{k_B T(\vec{r})}\right)+1}$$

Density and temperature gradients are allowed

- Boltzmann eq.

$$\frac{\partial f}{\partial t}+\dot{\vec{r}}\cdot\frac{\partial f}{\partial \vec{r}}+\dot{\vec{k}}\cdot\frac{\partial f}{\partial \vec{k}}=\frac{df}{dt}=-\frac{f-f^0}{\tau_\varepsilon}$$

$$\rightarrow \quad f_{rk}(t)=\frac{1}{\tau}\int_{-\infty}^{t} dt' f^0_{rk}(t') e^{-(t-t')/\tau}$$

$$\text{or} \quad = f^0_{rk}-\int_{-\infty}^{t} dt' e^{-(t-t')/\tau}\frac{df^0_{rk}(t')}{dt'}$$

$$\rightarrow \quad f_{rk}(t)= f^0_{rk}+\int_{-\infty}^{t} dt' e^{-(t-t')/\tau}\vec{v}_k\cdot\left[\frac{\partial \mu}{\partial \vec{r}}+\frac{(\varepsilon-\mu)}{T}\frac{\partial T}{\partial \vec{r}}-\hbar\dot{\vec{k}}\right]\frac{\partial f_0}{\partial \varepsilon}$$

$$= f_0+\delta f$$

Let's call this "Chamber's formulation"

$$\frac{df_0}{dt}=\dot{\vec{r}}\cdot\frac{\partial f_0}{\partial \vec{r}}+\dot{\vec{k}}\cdot\frac{\partial f_0}{\partial \vec{k}}$$

$$\frac{\partial f_0}{\partial \vec{r}}=\left.\frac{\partial f_0}{\partial \mu}\right|_T\frac{\partial \mu}{\partial \vec{r}}+\left.\frac{\partial f_0}{\partial T}\right|_\mu\frac{\partial T}{\partial \vec{r}}$$

$$=\left[-\frac{\partial \mu}{\partial \vec{r}}-\frac{(\varepsilon-\mu)}{T}\frac{\partial T}{\partial \vec{r}}\right]\frac{\partial f_0}{\partial \varepsilon}$$

$$\frac{\partial f_0}{\partial \vec{k}}=\hbar\dot{\vec{r}}\frac{\partial f_0}{\partial \varepsilon}$$

1st, consider a system with electric field and temperature gradient, but no magnetic field

$$\dot{\vec{r}} = \frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \vec{k}}$$

$$\hbar\dot{\vec{k}} = -e\vec{E}$$

- Electric current density $\vec{j}(\vec{r},t) = -e\int [dk]\dot{\vec{r}} f(\vec{r},\vec{k},t), \quad [dk] \equiv 2\frac{d^3k}{(2\pi)^3}$

$$= -e\int [dk]\cancel{\vec{v}_k} f_0 - e\int [dk]\vec{v}_k \delta f$$

For steady perturbations $\delta f = \left(\int_{-\infty}^{t} dt' e^{-(t-t')/\tau}\right)\vec{v}\cdot\left[e\vec{E} + \frac{\partial\mu}{\partial\vec{r}} + \frac{(\varepsilon-\mu)}{T}\frac{\partial T}{\partial\vec{r}}\right]\frac{\partial f_0}{\partial\varepsilon}$

$$= \tau\vec{v}\cdot\left[e\vec{G} + \frac{(\varepsilon-\mu)}{T}\frac{\partial T}{\partial\vec{r}}\right]\frac{\partial f_0}{\partial\varepsilon}, \qquad \vec{G} \equiv \vec{E} + \frac{1}{e}\frac{\partial\mu}{\partial\vec{r}}$$

Electrochemical "force"

$$\Rightarrow \vec{j}(\vec{r},t) = e\int [dk]\tau\vec{v}\,\vec{v}\cdot\left[e\vec{G} + \frac{(\varepsilon-\mu)}{T}\frac{\partial T}{\partial\vec{r}}\right]\left(-\frac{\partial f_0}{\partial\varepsilon}\right)$$

$$= \mathbf{L}_{11}\vec{G} + \mathbf{L}_{12}\left(-\frac{\nabla T}{T}\right), \qquad \mathbf{L}_{11} \text{ is } \boldsymbol{\sigma}$$

σ is conductivity tensor

- Thermal current density $\vec{j}^Q(\vec{r},t) = \int [dk](\varepsilon-\mu)\dot{\vec{r}}\delta f$

$$= -\int [dk]\tau(\varepsilon-\mu)\vec{v}\,\vec{v}\cdot\left[e\vec{G} + \frac{(\varepsilon-\mu)}{T}\frac{\partial T}{\partial\vec{r}}\right]\left(-\frac{\partial f_0}{\partial\varepsilon}\right)$$

$$= \mathbf{L}_{21}\vec{G} + \mathbf{L}_{22}\left(-\frac{\nabla T}{T}\right), \qquad \mathbf{L}_{22} \text{ is } \boldsymbol{\kappa}T$$

κ is thermal conductivity tensor

Coefficients of transport (matrices)

$$\mathbf{L}_{11} = e^2 \int [dk] \tau \vec{v}\,\vec{v} \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \qquad \mathbf{L}_{12} = -e \int [dk] \tau (\varepsilon_k - \mu) \vec{v}\,\vec{v} \left(-\frac{\partial f_0}{\partial \varepsilon} \right)$$

$$\mathbf{L}_{21} = -e \int [dk] \tau (\varepsilon_k - \mu) \vec{v}\,\vec{v} \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \qquad \mathbf{L}_{22} = \int [dk] \tau (\varepsilon_k - \mu)^2 \vec{v}\,\vec{v} \left(-\frac{\partial f_0}{\partial \varepsilon} \right)$$

They are of the form

$$\Lambda_{ij}^{(\nu)} = e^2 \int [dk] \tau (\varepsilon_k - \mu)^\nu v_i v_j \left(-\frac{\partial f_0}{\partial \varepsilon} \right)$$

$$\mathbf{L}_{11} = \mathbf{\Lambda}^{(0)}$$

$$\mathbf{L}_{12} = \mathbf{L}_{21} = \frac{1}{(-e)} \mathbf{\Lambda}^{(1)}$$

One example of Onsager relation

$$\mathbf{L}_{22} = \frac{1}{(-e)^2} \mathbf{\Lambda}^{(2)}$$

Define energy-resolved conductivity

$$\sigma_{ij}(\varepsilon) = e^2 \int [dk] \tau_\varepsilon v_i v_j \delta(\varepsilon_k - \varepsilon)$$

then

$$\Lambda_{ij}^{(\nu)} = \int d\varepsilon (\varepsilon - \mu)^\nu \left(-\frac{\partial f_0}{\partial \varepsilon} \right) \sigma_{ij}(\varepsilon)$$

T=0,

$$\boxed{\begin{aligned} \mathbf{\Lambda}^{(0)} &= \boldsymbol{\sigma}(\varepsilon_F) \\ \mathbf{\Lambda}^{(1)} &= \frac{\pi^2}{3} (k_B T)^2 \boldsymbol{\sigma}'(\varepsilon_F) \\ \mathbf{\Lambda}^{(2)} &= \frac{\pi^2}{3} (k_B T)^2 \boldsymbol{\sigma}(\varepsilon_F) \end{aligned}}$$

H.W.

More on the conductivity

consider ▽T=0.

Ohm's law: $\vec{j} = \boldsymbol{\sigma}\vec{G}$ $\qquad \vec{G} \equiv \vec{E} + \frac{1}{e}\frac{\partial \mu}{\partial \vec{r}}$

Fick's law for "diffusion current": $\vec{j} = -\mathbf{D}\nabla\rho$

- For electron gas at low T,

$$\mu \cong \varepsilon_F = \frac{\hbar^2}{2m^*}\left(3\pi^2 n\right)^{2/3} \qquad \left(\rho = -en\right)$$

$$\frac{\partial \varepsilon_F}{\partial \vec{r}} = \frac{2}{3}\frac{\varepsilon_F}{n}\frac{\partial n}{\partial \vec{r}} = \frac{2}{3}\frac{\varepsilon_F}{\rho}\frac{\partial \rho}{\partial \vec{r}}$$

$$\therefore \vec{j} = -\mathbf{D}\frac{3\rho}{2\varepsilon_F}\frac{\partial \varepsilon_F}{\partial \vec{r}}$$

$$\leftrightarrow \quad \vec{j} = \frac{\boldsymbol{\sigma}}{e}\frac{\partial \varepsilon_F}{\partial \vec{r}}$$

$$\rightarrow \quad \mathbf{D} = \frac{1}{(-e)}\frac{2\varepsilon_F}{3\rho}\boldsymbol{\sigma} = \frac{1}{e^2 g(\varepsilon_F)}\boldsymbol{\sigma}$$

Einstein relation
(for degenerate electron gas)

- At T=0, $\left(-\frac{\partial f_0}{\partial \varepsilon}\right) = \delta(\varepsilon - \varepsilon_F)$

$$\sigma_{ij} = e^2 \int [dk]\tau_\varepsilon v_i v_j \delta(\varepsilon - \varepsilon_F)$$

$$= e^2 \tau_{\varepsilon_F} g(\varepsilon_F)\left\langle v_i v_j \right\rangle_{FS}$$

$\left\langle v_i v_j \right\rangle_{FS}$ is an integral over the FS

$$D_{ij} = \tau_{\varepsilon_F}\left\langle v_i v_j \right\rangle_{FS}$$

For isotropic diffusion,

$$D_{xx} = \tau_{\varepsilon_F}\frac{1}{3}\left\langle v^2 \right\rangle_{FS} = \frac{1}{3}v_F^2 \tau_{\varepsilon_F}$$

- Alternative form of conductivity

$$\sigma_{ij} = e^2 \int [dk] \tau v_i v_j \left(-\frac{\partial f_0}{\partial \varepsilon} \right)$$

$$v_i v_j \left(-\frac{\partial f_0}{\partial \varepsilon} \right) = v_i \frac{\partial \varepsilon}{\hbar \partial k_j} \left(-\frac{\partial f_0}{\partial \varepsilon} \right) = v_i \left(-\frac{\partial f_0}{\hbar \partial k_j} \right)$$

$$\therefore \sigma_{ij} = e^2 \tau \int [dk] \left[-\frac{\partial (f_0 v_i)}{\hbar \partial k_j} + f_0 \frac{\partial v_i}{\hbar \partial k_j} \right]$$

If τ a const.

$$= e^2 \tau \int [dk] \, f_0 \cdot m_{ij}^{*-1}$$

$$\to \frac{ne^2 \tau}{m^{op}} \delta_{ij}$$

$\sim$ free electron gas

$$\sum_{k\sigma} v_i v_j \left(-\frac{\partial f_0}{\partial \varepsilon} \right) = \sum_{k\sigma} \frac{f_0}{m_{ij}^*(k)}$$

$$\equiv \frac{N}{m_{ij}^{op}}$$

Optical effective mass
= m* if the carriers are
in a parabolic band

- Fourier's law on thermal conduction

$$\vec{j}^Q = -\boldsymbol{\kappa} \nabla T = \mathbf{L}_{22} \left(-\frac{\nabla T}{T} \right)$$

Note: this would induce electric current.
To remedy this, see Marder, p.496.

$$\boldsymbol{\kappa} = \frac{\mathbf{L}_{22}}{T} = \frac{\pi^2}{3} \frac{k_B^2 T}{e^2} \boldsymbol{\sigma}$$

(Wiedemann-Franz law for metals)

supplementary

More on the Onsager reciprocal relation (1931)

互易關係

- Transport processes near equilibrium (linear transport regime)

$$j_i = \sigma_{ij}\left(-\frac{\partial \phi}{\partial x_j}\right) \qquad \text{(Ohm's Law)}$$

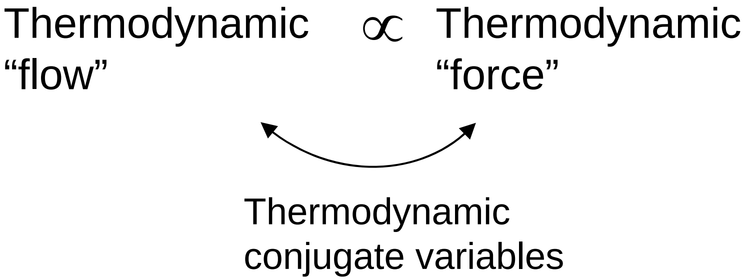
$$j_i = D_{ij}\left(-\frac{\partial \rho}{\partial x_j}\right) \qquad \text{(Fick's Law)}$$

$$j_i^Q = \kappa_{ij}\left(-\frac{\partial T}{\partial x_j}\right) \qquad \text{(Fourier's Law)}$$

…

- They are of the form

$$J_i = L_{ij} X_j$$



Thermodynamic "flow" $\propto$ Thermodynamic "force"

Thermodynamic conjugate variables

- "kinetic coefficient" **L** is symmetric:

For example, if E_x drives a current J_y, then E_y will drive a current J_x.

- The Onsager relation is a result of fluctuation-dissipation theorem, plus time reversal symmetry.

- A specific example:

the conductivity tensor of a crystal is symmetric, whatever the crystal symmetry is.

supplementary

Simultaneous irreversible processes

For example,

$$\begin{pmatrix}\vec{j}\\ \vec{j}^{Q}\end{pmatrix}=\begin{pmatrix}\mathbf{L}_{11} & \mathbf{L}_{12}\\ \mathbf{L}_{21} & \mathbf{L}_{22}\end{pmatrix}\begin{pmatrix}\vec{G}\\ -\nabla T/T\end{pmatrix}$$

also of the form:

$$J_i = L_{ij}\left(-\frac{\partial Y}{\partial x_j}\right)$$

Same symmetry relation applies to this larger α matrix

→ $\mathbf{L}^{T}_{12}=\mathbf{L}_{21}$

- if force 1 (e.g., a temperature gradient) drives a flow 2 (diffusion current), then force 2 (density gradient) will drive a flow 1 (heat current) !

Precursors of the symmetry relation
(D.G. Miller, J Stat Phys 1995)

- Stokes (1851), anisotropic heat conduction
- Kelvin (1854), thermoelectric effect
- …

Note:

For many transport processes near equilibrium, the entropy production is a product of flow and force

$$T\dot{S}_{irr} = J_i X_i$$

$$\dot{S}_{irr} = \frac{L_{ij}}{T} X_i X_j > 0$$

∴Entropy production ~ a thermodynamic potential

Nature likes to stay at the lowest potential

→ minimum entropy production

(only in the linear regime)

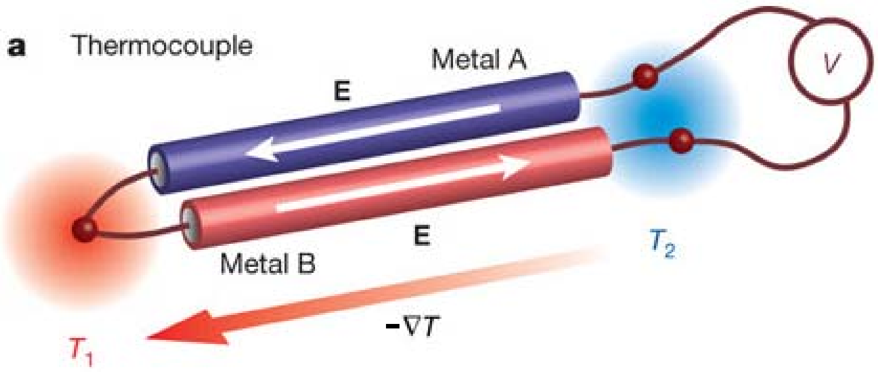
1968

http://www.ntnu.no/ub/spesialsamlingene/tekhist/tek5/eindex.htm

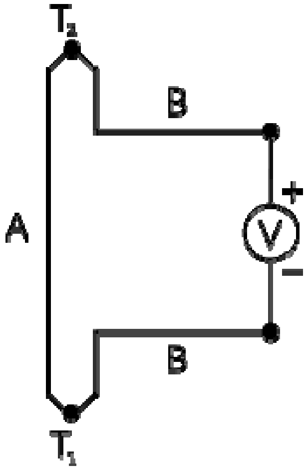
Thermoelectric coupling

(1) Seebeck effect (1821)

Seebeck found that a compass needle would be deflected by a closed loop formed by two metals joined in two places, with a temperature difference between the junctions.



or



- In the absence of electric current

Longitudinal *T* gradient →

electrochemical potential

(in metals or semiconductors)

$$\begin{pmatrix} 0 \\ \vec{j}^{Q} \end{pmatrix} = \begin{pmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{pmatrix} \begin{pmatrix} \vec{G} \\ -\nabla T / T \end{pmatrix}$$

$\rightarrow \vec{G} = \alpha \nabla T$ 溫差導致電位差

- Seebeck coefficient

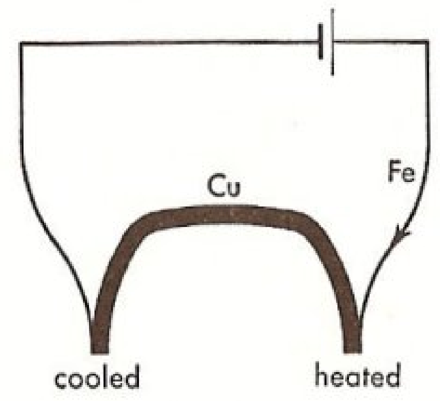
(aka thermoelectric power) 熱電功率

$$\alpha = (\mathbf{L}_{11})^{-1} \frac{\mathbf{L}_{12}}{T} = -\frac{\pi^2}{3} \frac{k_B^2 T}{e} \frac{\sigma'}{\sigma}$$

typical values observed: a few μV / K
(Bi: ~ 100 μV / K)

(2) Peltier effect (1834)

Peltier found that the junctions of dissimilar metals were heated or cooled, depending upon the direction of electrical current.



- In a bi-metallic circuit without *T* gradient, a current flow would induce a heat flow (in metals or semiconductors)

$$\begin{pmatrix} \vec{j} \\ \vec{j}^{\,Q} \end{pmatrix} = \begin{pmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{pmatrix} \begin{pmatrix} \vec{G} \\ 0 \end{pmatrix}$$

$$\rightarrow \vec{j}^{\,Q} = \Pi \, \vec{j}$$

$$\Pi = \mathbf{L}_{21} \left(\mathbf{L}_{11} \right)^{-1} = \alpha T$$

(If $\mathbf{L}_{11}$ and $\mathbf{L}_{21}$ commute)

- In practical applications of Seebeck/Peltier, the figure of merit (dimensionless) is (Prob 7)

$$ZT = \frac{\alpha^2 \sigma}{\kappa} T$$

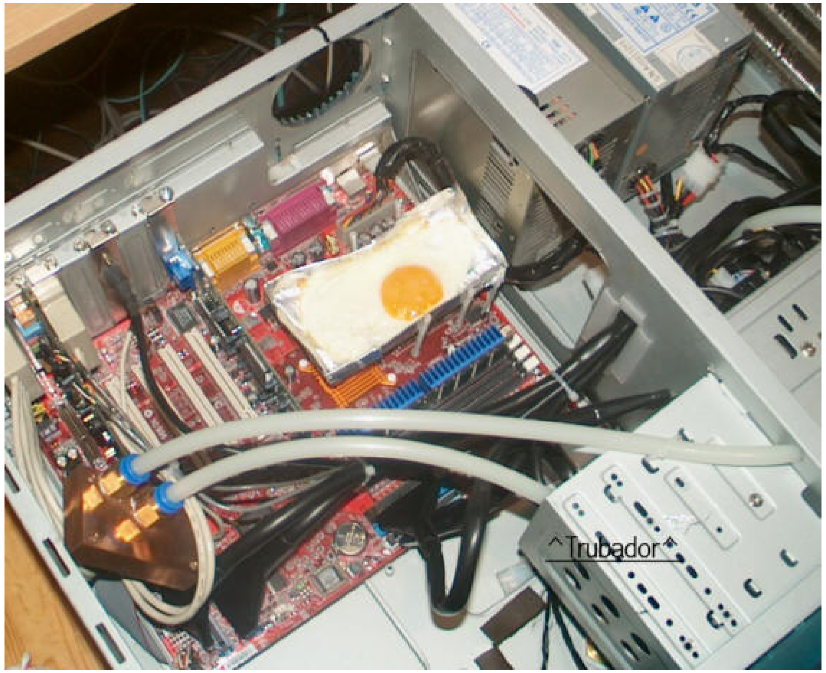
High electric conductivity *and* low thermal conductivity is good.
(>< Wiedemann-Franz law)

- Bi_2Te_3 ZT ~0.6 at room temperature

If ZT~4, then thermoelectric refrigerators will be competitive with traditional refrigerators.

Thermoelectric cooling

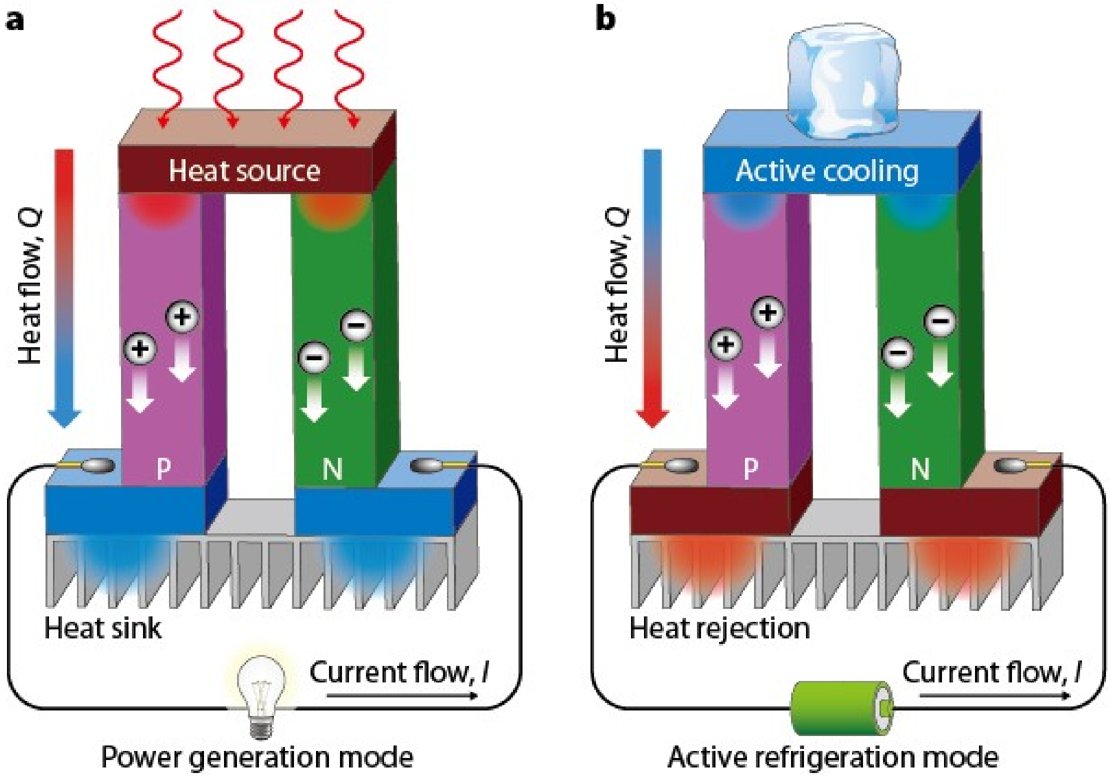
Peltier element:
Bismuth Telluride
(p/n type connected in series)



advantages

- Solid state heating and cooling – no liquids. CFC free.
- Compact instrument
- Fast response time for good temperature control

Seebeck vs Peltier



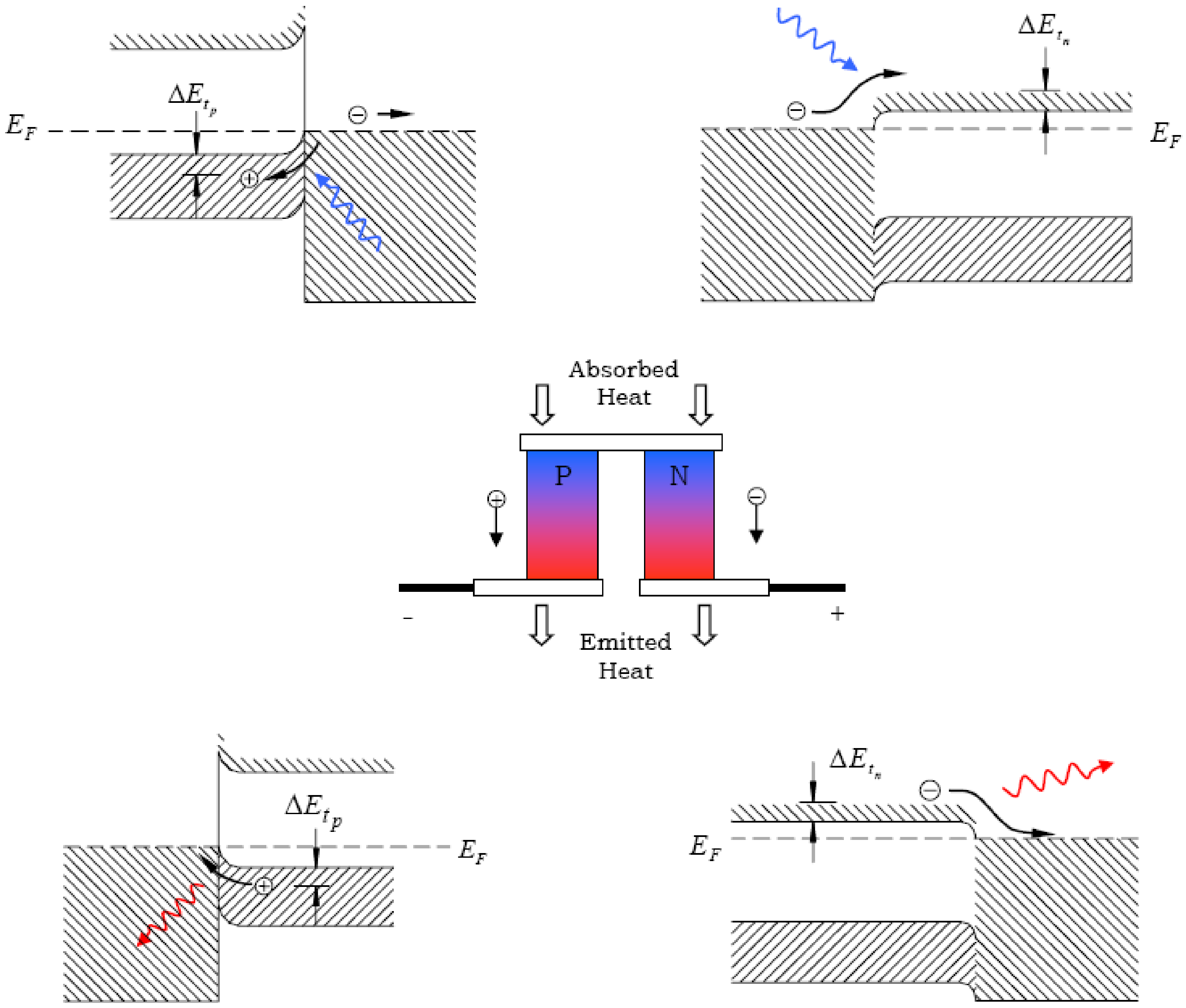
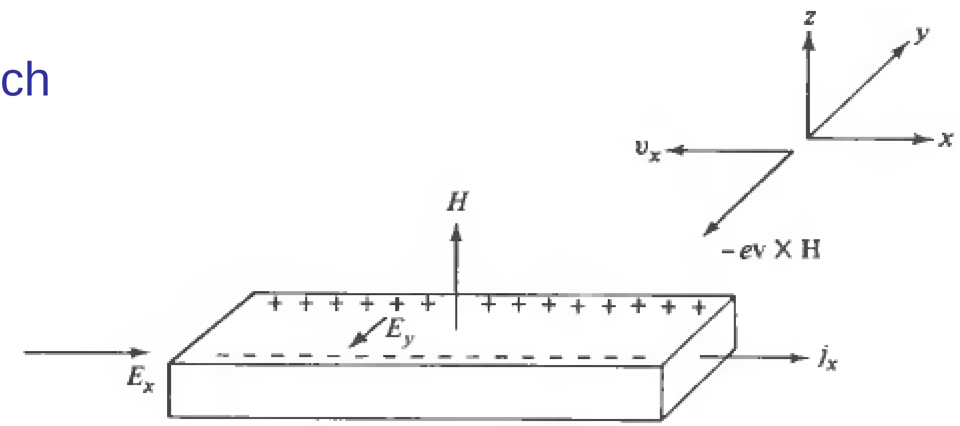


Figure 4. Thermal energy absorption and emission as electrons and holes cross the junctions between thermoelectric material and metal.

Hall effect (1879): classical approach



$$m^*\frac{d\vec{v}}{dt} = -e\vec{E} - e\frac{\vec{v}}{c}\times\vec{B} - m^*\frac{\vec{v}}{\tau}$$

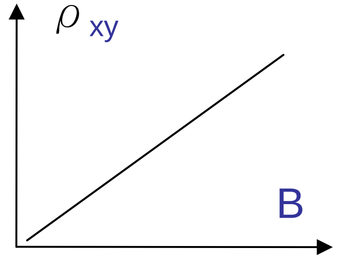
$\vec{B} = B\hat{z};\ d\vec{v}/dt = \vec{0}$ at steady state

➡ $$\begin{pmatrix} m^*/\tau & eB/c \\ -eB/c & m^*/\tau \end{pmatrix}\begin{pmatrix} v_x \\ v_y \end{pmatrix} = -e\begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$\vec{j} = -en\vec{v}$

➡ $$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \frac{m^*}{ne^2\tau} & \frac{B}{nec} \\ -\frac{B}{nec} & \frac{m^*}{ne^2\tau} \end{pmatrix}\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \rho_0\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}\begin{pmatrix} j_x \\ j_y \end{pmatrix}$$

$$\rho_0 = \frac{m^*}{ne^2\tau},\ \omega_c = \frac{eB}{m^*c}$$



$$\boldsymbol{\sigma} = \boldsymbol{\rho}^{-1} = \frac{\sigma_0}{1+(\omega_c\tau)^2}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}$$

$$\sigma_0 = \frac{ne^2\tau}{m^*}$$

$$\xrightarrow{\omega_c\tau<<1}\sigma_0\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}$$

$$\xrightarrow{\omega_c\tau>>1}\begin{pmatrix} 0 & -nec/B \\ nec/B & 0 \end{pmatrix}$$

Hall effect: semiclassical approach

$$\hbar \frac{d\vec{k}}{dt} = -e\vec{E} - \frac{e}{c}\vec{v}_k \times \vec{B}$$

Recall “Chamber’s formulation” (without density and T gradients)

$$\delta f = \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \vec{v}_k \cdot \left[-\hbar \dot{\vec{k}} \right] \frac{\partial f_0}{\partial \varepsilon}$$

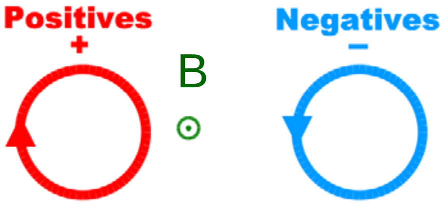
$$= -e \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \vec{v}_k \cdot \left(\vec{E} + \frac{1}{c} \cancel{\vec{v}_k \times} \vec{B} \right) \left(-\frac{\partial f_0}{\partial \varepsilon} \right)$$

We can now only count on v_k for magnetic effect

$$\hbar \dot{\vec{k}} = -e\vec{E} - \frac{e}{c}\vec{v} \times \vec{B}$$

$$\rightarrow \quad \frac{\vec{v}_\perp}{c} = -\frac{\hbar}{e}\frac{\vec{B} \times \dot{\vec{k}}}{B^2} + \frac{\vec{E} \times \vec{B}}{B^2}$$

ExB drift



Assume E ⊥ B

$$\frac{\vec{v}}{c} \cdot \vec{E} = -\frac{\hbar}{e}\vec{E} \cdot \frac{\vec{B} \times \dot{\vec{k}}}{B^2} = -\frac{\hbar}{e}\frac{\vec{E} \times \vec{B}}{B^2} \cdot \dot{\vec{k}}$$

$$e^{-(t-t')/\tau} \dot{\vec{k}}(t') = \frac{d}{dt'}\left(e^{-(t-t')/\tau} \vec{k}(t') \right) - \frac{1}{\tau} e^{-(t-t')/\tau} \vec{k}(t')$$

$$\rightarrow \delta f = c\frac{\vec{E} \times \vec{B}}{B^2} \cdot \hbar \left(\vec{k}(t) - \left\langle \vec{k}(t) \right\rangle \right) \left(-\frac{\partial f_0}{\partial \varepsilon} \right), \text{ where } \left\langle \vec{k}(t) \right\rangle \equiv \frac{1}{\tau} \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \vec{k}(t)$$

$$\vec{j} = -e\int[dk]\vec{v}_k\left(\cancel{f_0}+\delta f\right)$$

$$=ce\frac{\vec{E}\times\vec{B}}{B^2}\int[dk]\left(\vec{k}-\left\langle\vec{k}\right\rangle\right)\frac{\partial f_0}{\partial\vec{k}}$$

$$=ce\frac{\vec{E}\times\vec{B}}{B^2}\int[dk]\left\{\frac{\partial}{\partial\vec{k}}\left[\left(\vec{k}-\left\langle\vec{k}\right\rangle\right)f_0\right]-f_0\right\}$$

- For $\omega_c\tau$ >>1, the first term is zero

$$\therefore\ \vec{j} = -ec\frac{\vec{E}\times\vec{B}}{B^2}\int[dk]f_0 = -nec\frac{\vec{E}\times\vec{B}}{B^2}$$

$$\rightarrow \boldsymbol{\sigma} = \begin{pmatrix} 0 & -\frac{nec}{B} & 0 \\ \frac{nec}{B} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \sigma_0\begin{pmatrix} 0 & -\frac{1}{\omega_c\tau} & 0 \\ \frac{1}{\omega_c\tau} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Q1:How do we get the next order terms?

$$\boldsymbol{\sigma} = \sigma_0\begin{pmatrix} \frac{1}{(\omega_c\tau)^2} & -\frac{1}{\omega_c\tau} & 0 \\ \frac{1}{\omega_c\tau} & \frac{1}{(\omega_c\tau)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Q2: What if the orbit is not closed?

If the open orbit is along the x-direction (in real space), then

$$\boldsymbol{\sigma} = \sigma_0\begin{pmatrix} \gamma & -\frac{1}{\omega_c\tau} & 0 \\ \frac{1}{\omega_c\tau} & \frac{1}{(\omega_c\tau)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

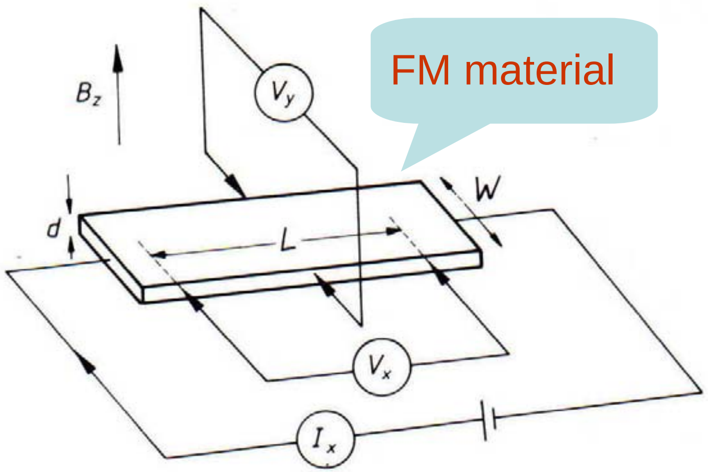
See Kitel, QTS, p.244

Q3: What about $\omega_c\tau$ <<1?

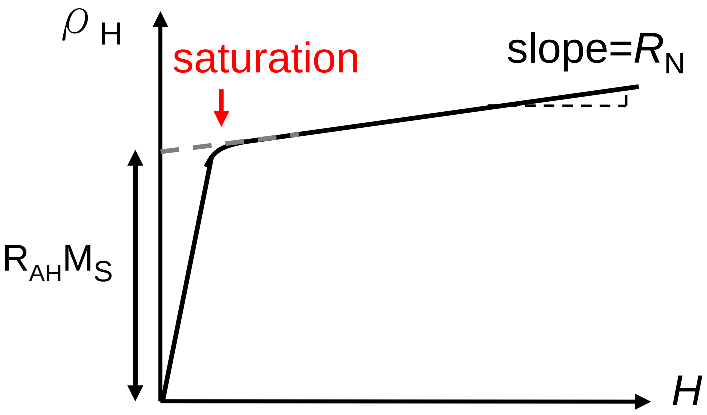
supplementary

Anomalous Hall effect (Edwin Hall, 1881):

Hall effect in ferromagnetic (FM) materials



FM material



The usual Lorentz force term

$$\rho_H = R_N H + \rho_{AH}(H),$$

Anomalous term

$$\rho_{AH}(H) \equiv R_{AH} M(H)$$

Ingredients required for a successful theory:

- magnetization (majority spin)
- spin-orbit coupling
(to couple the *majority-spin* direction to transverse orbital direction)

supplementary

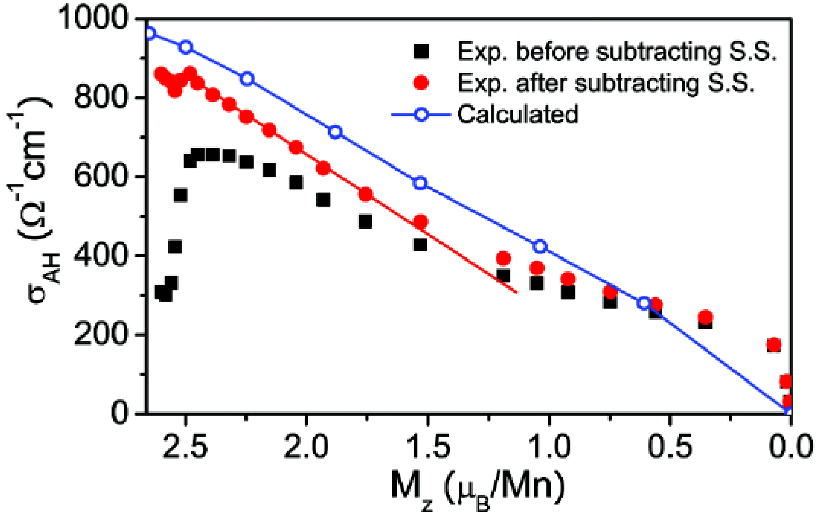
“Intrinsic” AHE due to the Berry curvature

$$\begin{cases} \hbar\dfrac{d\vec{k}}{dt} = -e\vec{E} \\ \dfrac{d\vec{r}}{dt} = \dfrac{1}{\hbar}\dfrac{\partial\varepsilon}{\partial\vec{k}} - \dot{\vec{k}}\times\vec{\Omega}(\vec{k}) \end{cases}$$

$$= \frac{1}{\hbar}\frac{\partial\varepsilon}{\partial\vec{k}} + \frac{e}{\hbar}\vec{E}\times\vec{\Omega}(\vec{k})$$

A transverse current

Mn$_5$Ge$_3$



Zeng et al PRL 2006

After averaging over long-wavelength spin fluctuations, the calculated anomalous Hall conductivity is roughly linear in M. The S.S. refers to skew scattering.

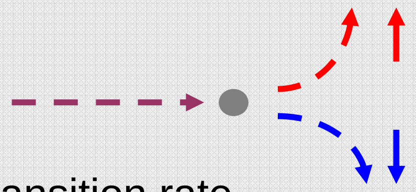
To leading order,

$$\vec{j} = -e\int[dk]\vec{v}_k f_0$$

$$= -\frac{e}{\hbar}\int[dk]\cancel{\frac{\partial\varepsilon}{\partial\vec{k}}} f_0 + \frac{e^2}{\hbar}\int[dk]\vec{\Omega}(\vec{k}) f_0 \times \vec{E}$$

$$\rightarrow \sigma_{AH} = \frac{e^2}{\hbar}\int_{filled}[dk]\,\Omega_z(\vec{k})$$

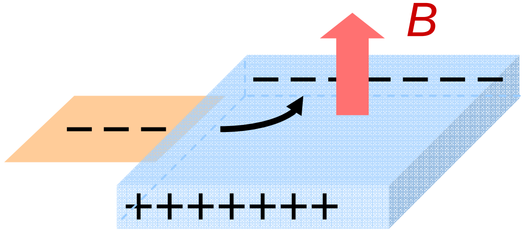
- Skew scattering from an impurity



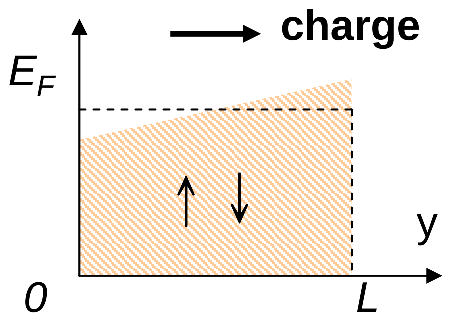
transition rate

$$W_{\vec{k}s\to\vec{k}'s'} \approx \lambda_{SO}\vec{\sigma}_{s's}\cdot\vec{k}'\times\vec{k}$$

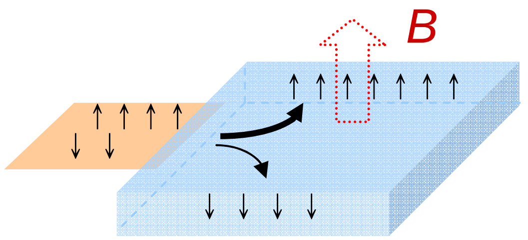
- classical Hall effect



✓ Lorentz force

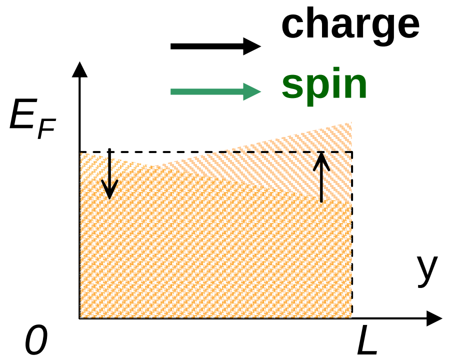


- anomalous Hall effect

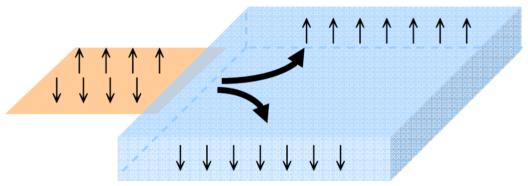


✓ Berry curvature (int)

✓ Skew scattering (ext)

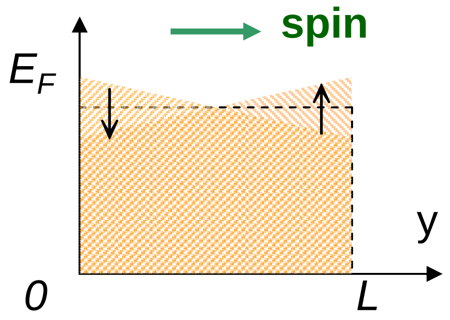


- spin Hall effect



✓ Berry curvature (int)

✓ Skew scattering (ext)



No magnetic field required !

Thermo-galvano-magnetic phenomena

$$\vec{j}^{\,e}=\boldsymbol{\sigma}(\vec{B})\vec{E}+\boldsymbol{\alpha}(\vec{B})\left(-\nabla T\right)$$

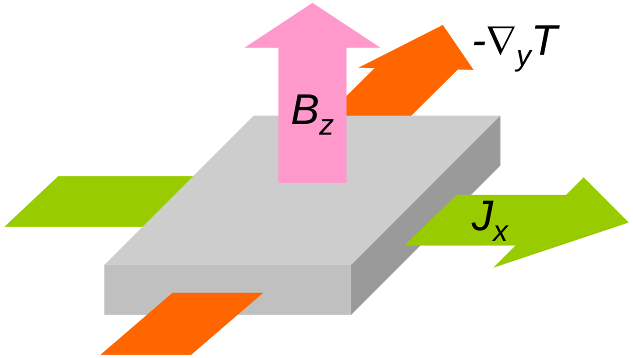
$$\vec{j}^{\,Q}=\boldsymbol{\beta}(\vec{B})\vec{E}+\boldsymbol{\kappa}(\vec{B})\left(-\nabla T\right)$$

Onsager relations

$$\boldsymbol{\sigma}^{T}(-\vec{B})=\boldsymbol{\sigma}(\vec{B})$$

$$\boldsymbol{\beta}^{T}(-\vec{B})=\boldsymbol{\alpha}(\vec{B})$$

$$\boldsymbol{\kappa}^{T}(-\vec{B})=\boldsymbol{\kappa}(\vec{B})$$



- Expand to first order in B,

Ohm	Hall	Nernst
1826	1879	1886

The effect of B on thermo-induced electric current

$$\Rightarrow\quad \vec{j}^{\,e}=\sigma\vec{E}+R\vec{E}\times\vec{B}+\alpha\left(-\nabla T\right)+N\left(-\nabla T\right)\times\vec{B}$$

Onsager relations

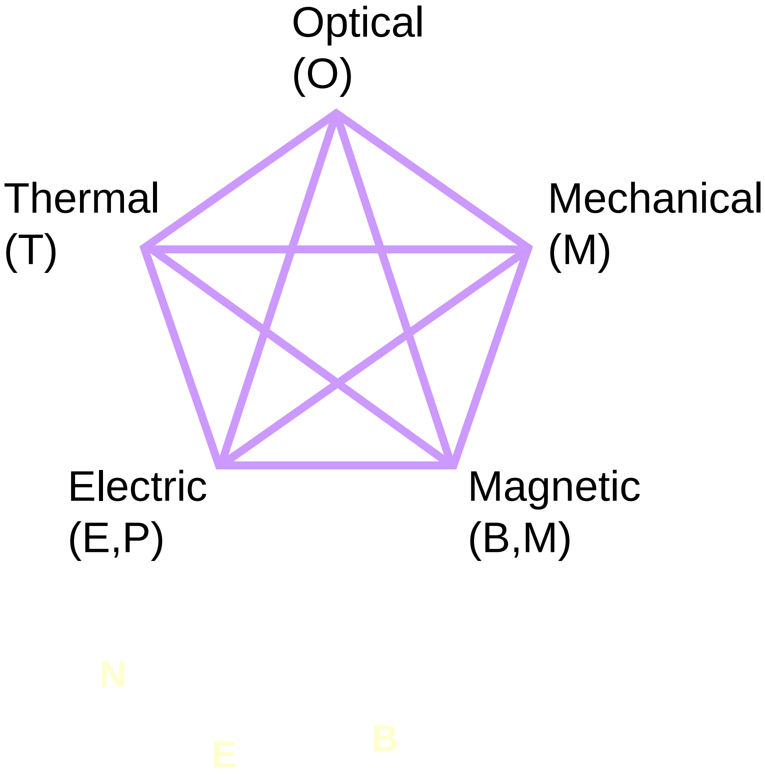
$$\vec{j}^{\,Q}=\beta\vec{E}+N\vec{E}\times\vec{B}+\kappa\left(-\nabla T\right)+L\left(-\nabla T\right)\times\vec{B}$$

The effect of B on electric-induced thermo current

Ettingshausen	Fourier	Leduc-Righi (Thermal Hall effect)
1886	1807	1887

Landau, ED of continuous media, p.101

Beyond thermo-galvano-magnetic phenomena



Optical
(O)

Thermal
(T)

Mechanical
(M)

Electric
(E,P)

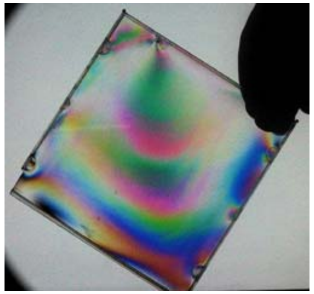
Magnetic
(B,M)

N

E

B

- E-T: Thomson effect, Peltier/Seebeck effect
- E-B: Hall effect, magneto-electric material
- E-B-T: Nernst/Ettingshausen effect, Leduc-Righi effect
- E-O, B-O: Kerr effect, Faraday effect, photovoltaic effect, photoelectric effect
- E-M, B-M: piezoelectric effect/electrostriction, piezomagnetic effect/magnetostriction
- M-O: photoelasticity
- ...



solid state refrigerator

solid state sensor

solid state motor, artificial muscle

...

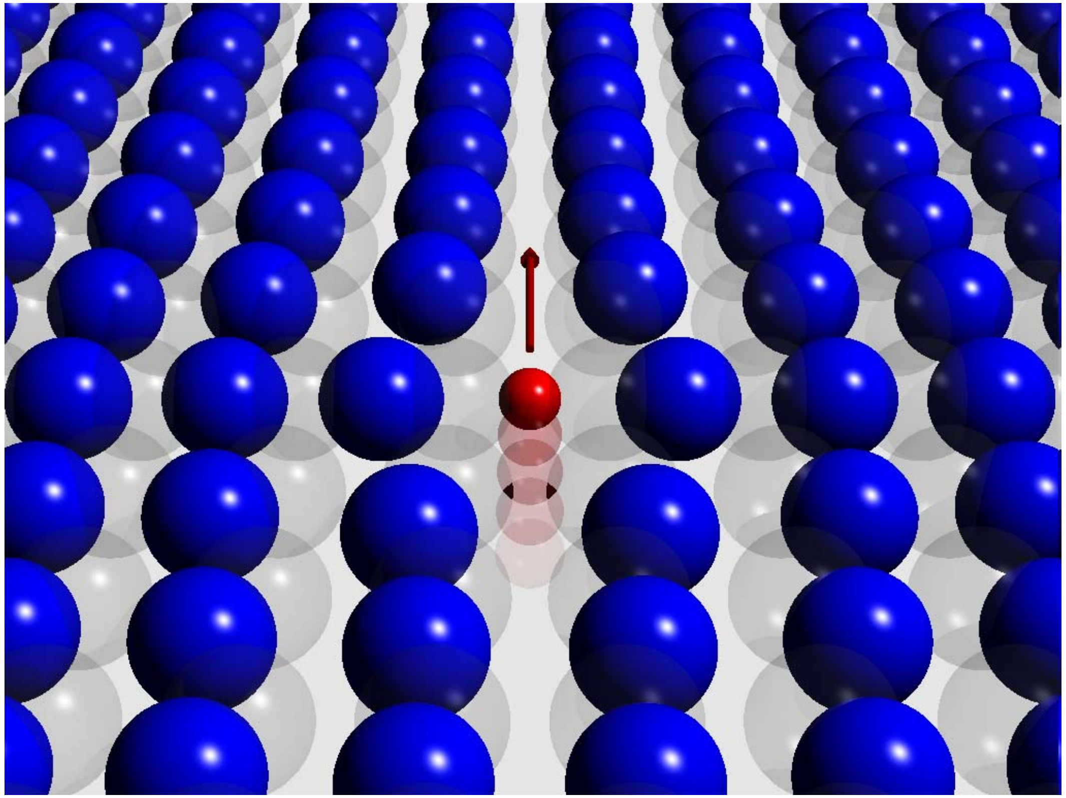
Landau and Lifshitz, Electrodynamics of continuous media

Scheibner, 4 review articles in *IRE Transations on component parts*, 1961, 1962

TABLE 1-3 Physical and Chemical Transduction Principles. (from "Expanding the vision of sensor materials" 1995)

Input (Primary) Signal	Output (Secondary) Signals					
	Mechanical	Thermal	Electrical	Magnetic	Radiant	Chemical
Mechanical	(Fluid) Mechanical effects; e.g., diaphragm, gravity balance. Acoustic effects; e.g., echo sounder.	Friction effects; e.g., friction calorimeter. Cooling effects; e.g., thermal flow meter.	Piezoelectricity. Piezoresistivity. Resistive. Capacitive. Induced effect.	Magneto-mechanical effects; e.g., piezomagnetic effect.	Photoelastic systems (stress-induced birefringence). Interferometer. Sagnac effect. Doppler effect.	
Thermal	Thermal expansion; e.g., bimetallic strip, liquid-in-glass and gas thermometers. Resonant frequency. Radiometer effect; e.g., light mill.		Seebeck effect. Thermo-resistance. Pyroelectricity. Thermal (Johnson) noise.		Thermo-optical effects; e.g., liquid crystals. Radiant emission.	Reaction activation; e.g., thermal dissociation.
Electrical	Electrokinetic and electro-mechanical effects; e.g., piezoelectricity, electrometer, and Ampere's Law.	Joule (resistive) heating. Peltier effect.	Charge collectors. Langmuir probe.	Biot-Savart's Law.	Electro-optical effects; e.g., Kerr effect, Pockels effect. Electro-luminescence.	Electrolysis. Electro-migration.
Magnetic	Magneto-mechanical effects; e.g., magnetostriction, and magnetometer.	Thermo-magnetic effects; e.g., Righi-Leduc effect. Galvano-magnetic effects; e.g., Ettingshausen effect.	Thermo-magnetic effects; e.g., Ettingshausen-Nernst effect. Galvano-magnetic effects; e.g., Hall effect, and magneto-resistance.	Magneto-optical effects; e.g., Faraday effect, and Cotton-Mouton effect.		

Landau’s theory of Fermi liquid

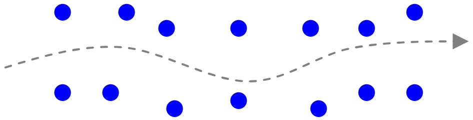


Why e-e interaction can usually be ignored in metals?

- $$K \sim \frac{\hbar^2}{m}\frac{1}{r^2}, \quad U \sim \frac{e^2}{r}$$

$$\frac{U}{K} \sim \frac{me^2}{\hbar^2} r = \frac{r}{a_B}$$

Typically, 2 < U/K < 5

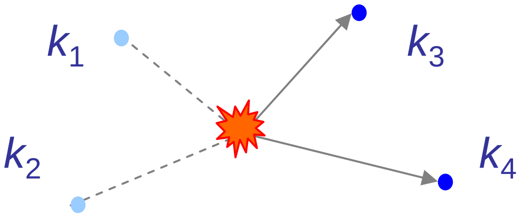


- Average e-e separation in a metal is about 2 A

Experiments find e mean free path about 10000 A (at 300K)

At 1 K, it can move 10 cm without being scattered! Why?

- A collision event:



- Calculate the e-e scattering rate using Fermi's golden rule:

$$\frac{1}{\tau} = \frac{2\pi}{\hbar}\sum_{i,f}\left|\langle f \,|V_{ee}|\, i\rangle\right|^2 \delta(E_i - E_f)$$

Scattering amplitude $\left|\langle f \,|V_{ee}|\, i\rangle\right|^2 = \left|\langle k_3, k_4 \,|V_{ee}|\, k_1, k_2\rangle\right|^2$

$$E_i = E_1 + E_2; \quad E_f = E_3 + E_4$$

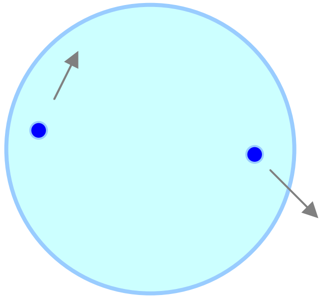
The summation is over all possible initial and final states that obey energy and momentum conservation

Pauli principle reduces available states for the following reasons:

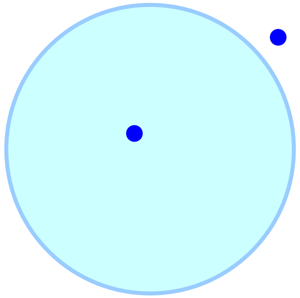
If the scattering amplitude $|V_{ee}|^2$ is roughly of the same order for all k's, then

$$\tau^{-1} \sim |V_{ee}|^2 \sum_{k_1,k_2} \sum_{k_3,k_4} 1$$

$E_1+E_2=E_3+E_4$;
$\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4$

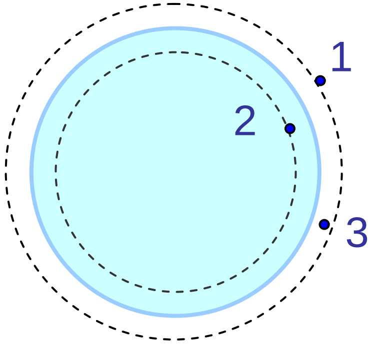


- 2 e's inside the FS cannot scatter with each other (energy conservation + Pauli principle), at least one of them must be outside of the FS.



Let electron 1 be outside the FS:

- One e is "shallow" outside, the other is "deep" inside also cannot scatter with each other, since the "deep" e has nowhere to go.



- If $|E_2| < E_1$, then $E_3+E_4 > 0$ (let $E_F=0$)

But since $E_1+E_2 = E_3+E_4$, 3 and 4 cannot be very far from the FS if 1 is close to the FS.

Let's fix E_1, and study possible initial and final states.

(let the state of electron 1 be fixed)

- number of initial states = (volume of E_2 shell)/Δ^3k

  number of final states = (volume of E_3 shell)/Δ^3k

  (E_4 is uniquely determined)

- $\tau^{-1} \sim V(E_2)/\Delta^3k \text{ x } V(E_3)/\Delta^3k$ ← number of states for scatterings

$$V(E_2) \cong 4\pi k_F^2 \,|\, k_2 - k_F \,|$$

$$V(E_3) \cong 4\pi k_F^2 \,|\, k_3 - k_F \,|$$

$\therefore \tau^{-1} \sim (4\pi/\Delta^3 k)^2 \, k_F^2|k_2-k_F| \times k_F^2|k_3-k_F|$

Total number of states for particle 2 and 3 = $[(4/3)\pi k_F^3/ \Delta^3 k]^2$

- The fraction of states that "can" participate in the scatterings

$= (9/k_F^2)\ |k_2-k_F| \times |k_3-k_F|$

$\sim (E_1/E_F)^2$ (1951, V. Wessikopf)

Finite temperature:

$\sim (kT/E_F)^2 \sim 10^{-4}$ at room temperature

→ e-e scattering rate $\propto T^2$

- need very low T (a few K) and very pure sample to eliminate thermal and impurity scatterings before the effect of e-e scattering can be observed.

In general

$$\tau^{-1} \sim \varepsilon^2 + \pi^2 \left(k_B T\right)^2$$

Landau's theory of the Fermi liquid (1956)

assumptions

- Strongly interacting fermion system

→ weakly interacting quasi-particle (QP) system

∼ a particle plus its surrounding, finite life-time

- 1-1 correspondence between fermions and QPs (fermion, spin-1/2, charge -e).

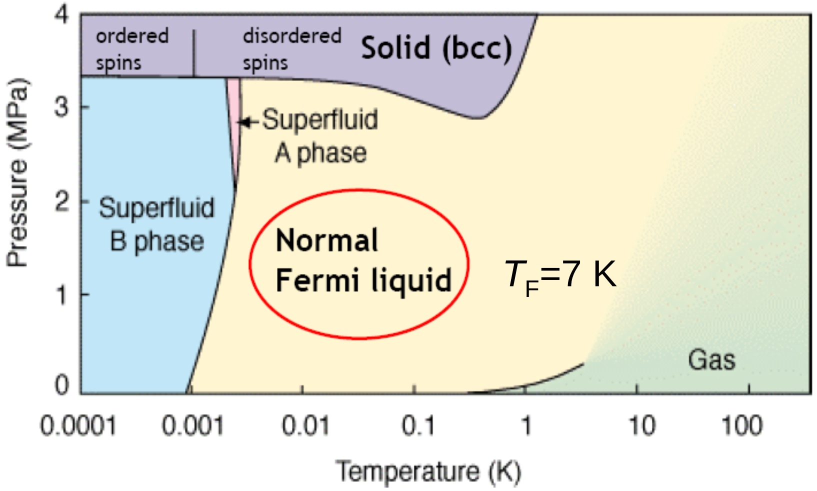
Q: Is this trivial?

- adiabatic continuity: As we turn off the interaction, the QPs smoothly change back to noninteracting fermions.

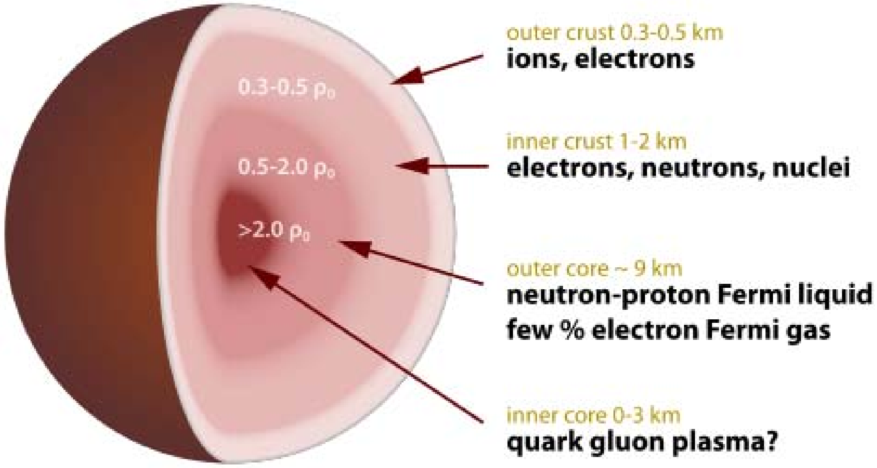

1962

- The following analysis applies to a *neutral*, *isotropic* FL, such as He-3.

He-3



Another application:



Similarity and difference with free electron gas

- QP distribution (at eq.) $f_{k\sigma} = \frac{1}{e^{(\varepsilon_{k\sigma}-\mu)/k_B T}+1}$

For a justification, see Marder

$$\text{at } T = 0, \quad f_{k\sigma} \rightarrow f^0_{k\sigma} = \theta(\varepsilon_F - \varepsilon_{k\sigma}) \quad (\mu = \varepsilon_F \quad \text{at } T = 0)$$

$$\rightarrow \theta(\varepsilon_F - \varepsilon^0_{k\sigma})$$ ← if no other ext perturbations

k_F is not changed by interaction!

- Due to external perturbations the distribution will deviate from the *manybody ground state* (no perturbation) *at T=0*
  - Thermal $\delta f_{k\sigma} = f_{k\sigma}(\varepsilon_{k\sigma}) - \theta(\varepsilon_F - \varepsilon^0_{k\sigma})$
  - Non-thermal (T=0)
    (density perturbation, magnetic field… etc) $\delta f_{k\sigma} = \theta(\mu - \varepsilon_{k\sigma}) - \theta(\varepsilon_F - \varepsilon^0_{k\sigma})$
  - In general $\delta f_{k\sigma} = f_{k\sigma}(\varepsilon_{k\sigma}) - \theta(\varepsilon_F - \varepsilon^0_{k\sigma})$
- QP energy $\varepsilon_{k\sigma} = \varepsilon^0_k + \sum_{k'\sigma'} u^{\sigma\sigma'}_{kk'} \delta f_{k'\sigma'}$

In absence of other QPs

$u^{\sigma\sigma'}_{kk'}$ **is an effective interaction between QPs near FS** $\left(u^{\sigma\sigma'}_{kk'} = u^{\sigma'\sigma}_{k'k}\right)$

- Total number $N = \sum_{k\sigma} f_{k\sigma} \xrightarrow{1-1} \frac{N}{V} = \frac{k_F^3}{3\pi^2}$

- Total energy

$$E[\delta f] = E[0] + \sum_{k\sigma} \varepsilon_{k\sigma} \delta f_{k\sigma}$$

$$= E[0] + \sum_{k\sigma} \varepsilon^0_{k\sigma} \delta f_{k\sigma} + \frac{1}{2} \sum_{\substack{k\sigma \\ k'\sigma'}} u^{\sigma\sigma'}_{kk'} \delta f_{k\sigma} \delta f_{k'\sigma'} + O(\delta f^3)$$

This form is not good for charged FL (with long-range interaction)

- If there is no magnetic field, nor magnetic order, then $\varepsilon_{k\sigma}$ is independent of σ, and $u^{\sigma\sigma'}_{kk'}$ depends only on the relative spin directions.

$$u^{\sigma\sigma'}_{k_1 k_2} = \langle k_1, k_2 | V_{ee} | k_1, k_2 \rangle$$ (forward scattering amplitude)

For example,

$$u^{\sigma\sigma'}_{kk'} = \begin{cases} \frac{1}{V} \frac{4\pi e^2}{\varepsilon(\vec{k} - \vec{k}', 0) \left| \vec{k} - \vec{k}' \right|^2} & \text{if } \sigma = \sigma' \\ 0 & \text{if } \sigma \neq \sigma' \end{cases}$$

Quinn, p.384
(recall the Fock interaction in ch 9)

- Fermi velocity $\vec{v}_F \equiv \left.\frac{\partial \varepsilon^0_{k\sigma}}{\hbar \partial \vec{k}}\right|_{k=k_F}$

- Effective mass $m^* \equiv \frac{\hbar k_F}{v_F}$

- DOS

$$D^*(\varepsilon_F) \equiv \frac{1}{V}\sum_{k\sigma}\delta(\varepsilon^0_{k\sigma}-\varepsilon_F)$$

Note: The use of $\varepsilon^0_{k\sigma}$ follows Coleman's note, Baym and Pethick etc, but not Marder's

See ch 7

$$= \frac{1}{4\pi^3}\oint \frac{dS_\varepsilon}{\left|\nabla_k \varepsilon^0_k\right|} = \frac{m^* k_F}{\pi^2 \hbar^2}$$

- Specific heat

$$dE = \sum_{k\sigma}\varepsilon_{k\sigma}\delta f_{k\sigma} \cong \sum_{k\sigma}\varepsilon^0_{k\sigma}\delta f_{k\sigma}, \quad \delta f_{k\sigma} \cong \frac{\partial f(\varepsilon^0_{k\sigma})}{\partial T}\delta T$$

(to lowest order)

$$\to C_V = \left.\frac{\partial E}{\partial T}\right|_V = \sum_{k\sigma}\varepsilon^0_{k\sigma}\frac{\partial f(\varepsilon^0_{k\sigma})}{\partial T}$$

$$= \sum_{k\sigma}\int d\varepsilon\,\delta(\varepsilon^0_{k\sigma}-\varepsilon)\varepsilon\frac{\partial f(\varepsilon)}{\partial T}$$

$$= V\int d\varepsilon\, D^*(\varepsilon)\varepsilon\frac{\partial f(\varepsilon^0_{k\sigma})}{\partial T}$$

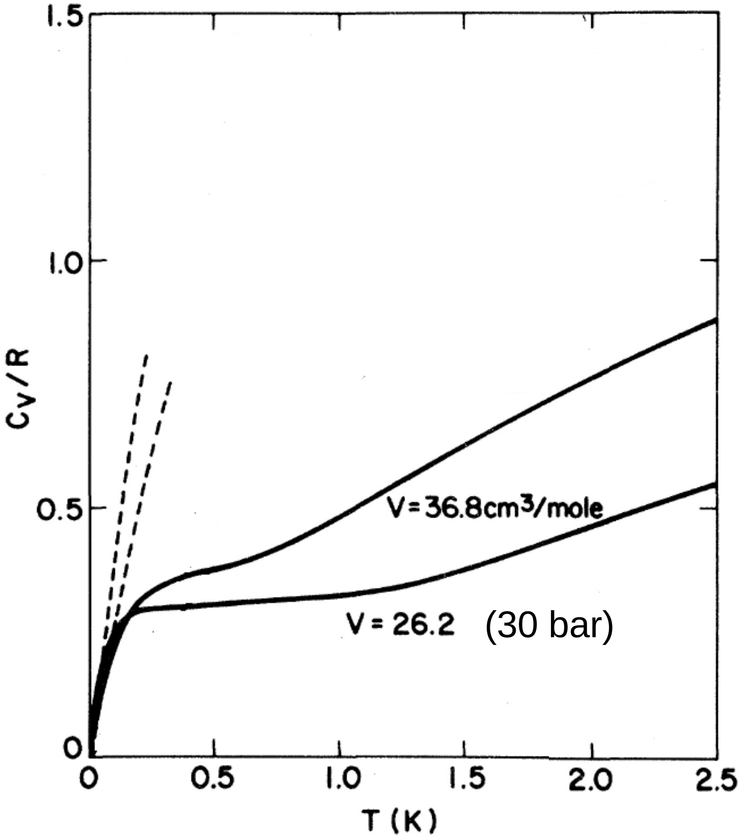
See ch 6

$$\cong VD^*(\varepsilon_F)\int d\varepsilon\,\varepsilon\frac{\partial f(\varepsilon^0_{k\sigma})}{\partial T} = \frac{\pi^2}{3}k_B^2 T V D^*(\varepsilon_F)$$

same as non-interacting result except for the effective mass.

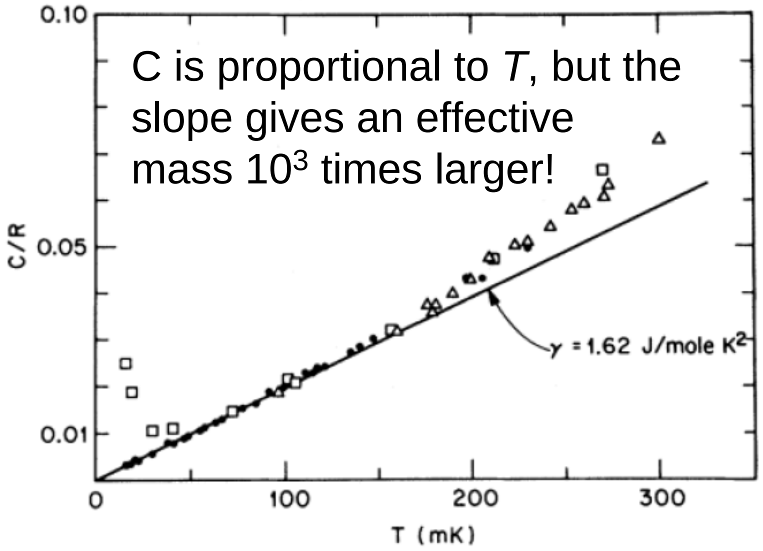
Heavy fermion material (CeAl$_3$)

He-3

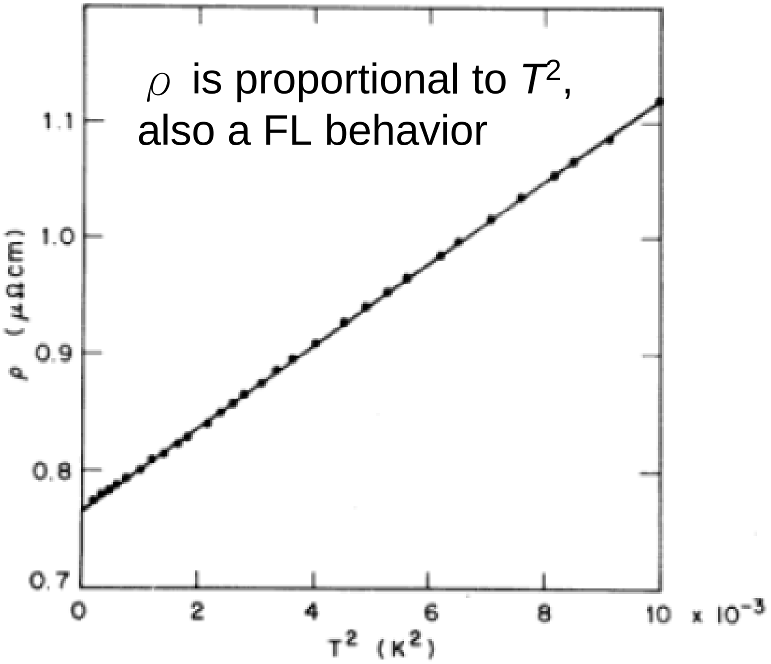


(30 bar)

Specific is linear in T below 20 mK



C is proportional to *T*, but the slope gives an effective mass 10^3 times larger!

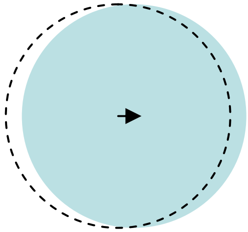


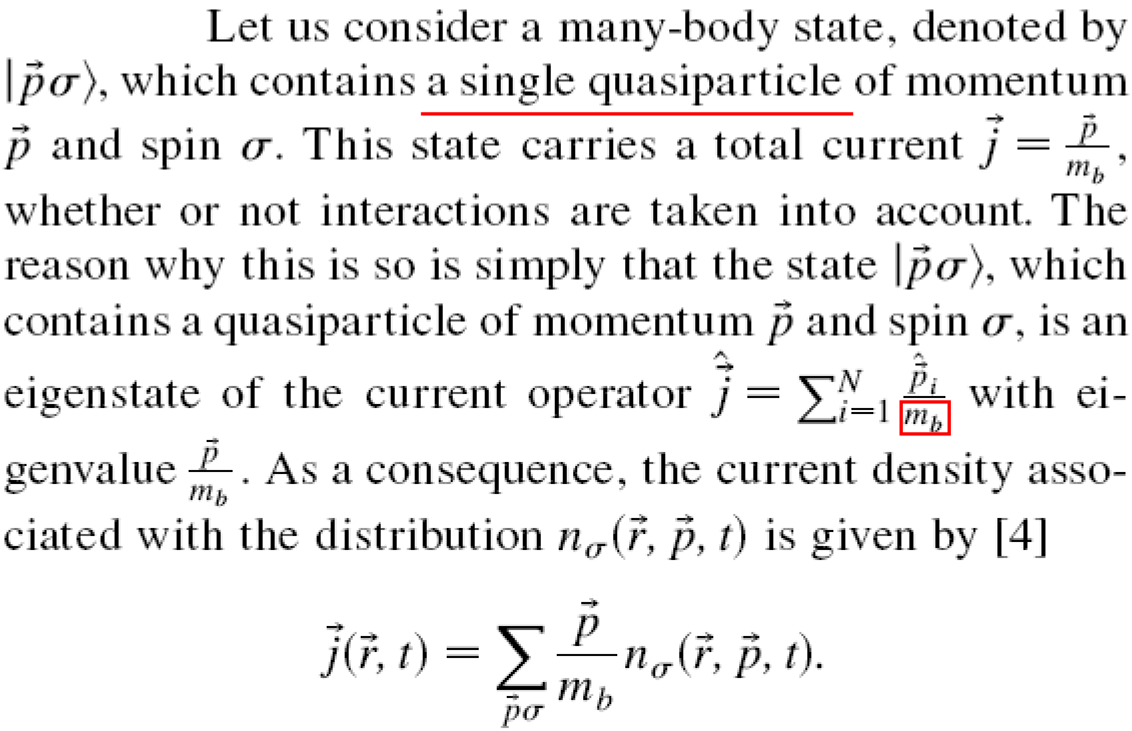
ρ is proportional to T^2, also a FL behavior

Giamarchi's note, p.88

Effective mass of a QP (I)

(total) "Particle" current $\vec{J}_N = \sum_{k\sigma} \frac{\hbar \vec{k}}{m} f_{k\sigma} = \sum_{k\sigma} \frac{\hbar \vec{k}}{m} \delta f_{k\sigma}$ (1)



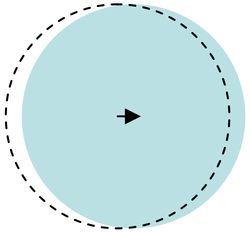


Let us consider a many-body state, denoted by $|\vec{p}\sigma\rangle$, which contains a single quasiparticle of momentum $\vec{p}$ and spin σ. This state carries a total current $\vec{j} = \frac{\vec{p}}{m_b}$, whether or not interactions are taken into account. The reason why this is so is simply that the state $|\vec{p}\sigma\rangle$, which contains a quasiparticle of momentum $\vec{p}$ and spin σ, is an eigenstate of the current operator $\hat{\vec{j}} = \sum_{i=1}^{N} \frac{\hat{\vec{p}}_i}{\boxed{m_b}}$ with eigenvalue $\frac{\vec{p}}{m_b}$. As a consequence, the current density associated with the distribution $n_\sigma(\vec{r}, \vec{p}, t)$ is given by [4]

$$\vec{j}(\vec{r}, t) = \sum_{\vec{p}\sigma} \frac{\vec{p}}{m_b} n_\sigma(\vec{r}, \vec{p}, t).$$

Z.Qian et al, PRL 93, 106601 (2004)

Effective mass of a QP (II)



On the other hand, $\vec{J}_N = \frac{dE}{\hbar d\vec{k}}$ (see next page)

give particles an active boost (with p-h excitations) $\vec{k} \to \vec{k} + d\vec{k}$

$$f_{k\sigma} \to f_{k-dk,\sigma} = f_{k\sigma} - d\vec{k} \cdot \frac{\partial f_{k\sigma}}{\partial \vec{k}}$$

$$dE = -\sum_{k\sigma} \varepsilon_k \left(d\vec{k} \cdot \frac{\partial f_{k\sigma}}{\partial \vec{k}} \right)$$

$$-\frac{\partial f^0_{k'}}{\partial \varepsilon_{k'}}$$

$$\to \quad \vec{J}_N = \sum_{k\sigma} \frac{\partial \varepsilon_{k\sigma}}{\hbar \partial \vec{k}} f_{k\sigma} = \ldots = \sum_{k\sigma} \left[\vec{v}_{k\sigma} + \sum_{k'\sigma'} u^{\sigma\sigma'}_{kk'} \vec{v}_{k'\sigma'} \delta(\varepsilon^0_{k'} - \varepsilon_F) \right] \delta f_{k\sigma} \qquad (2)$$

(1)=(2) $\longrightarrow$ (δ *f* is arbitrary)

$$\frac{\hbar\vec{k}}{m} = \frac{\hbar\vec{k}}{m^*_\sigma} + \sum_{k'\sigma'} u^{\sigma\sigma'}_{kk'} \frac{\hbar\vec{k}\,'}{m^*_{\sigma'}} \delta(\varepsilon^0_{k'} - \varepsilon_F)$$

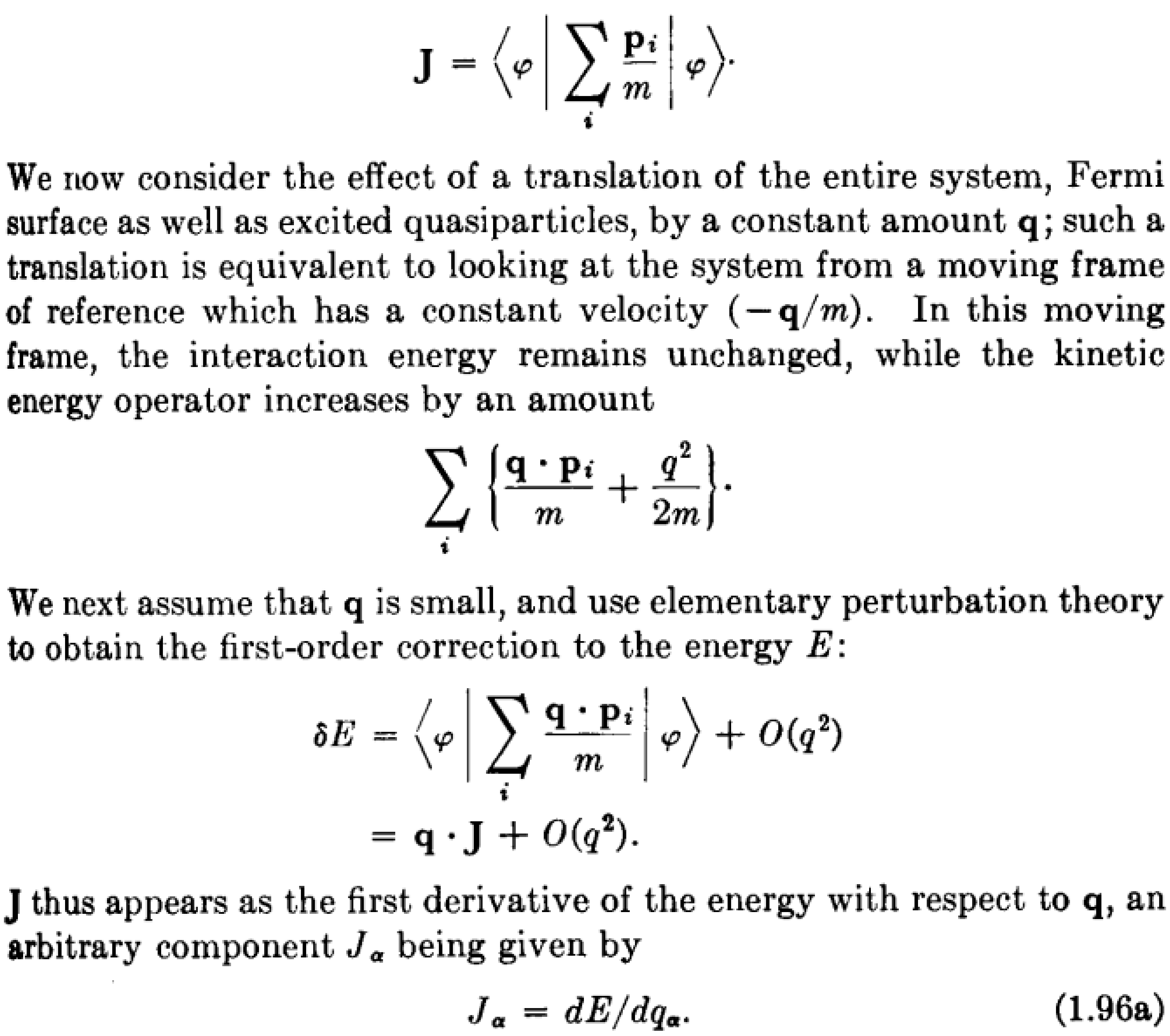
If m* is spin-indep, (nonmagnetic FL) then $\to$

$$\frac{m^*}{m} = 1 + \sum_{k'\sigma'} u^{\sigma\sigma'}_{kk'} \frac{\vec{k}_F \cdot \vec{k}_F\,'}{k_F^2} \delta(\varepsilon^0_{k'} - \varepsilon_F)$$

(an integral over the FS)

(Only for QPs near the Fermi surface)

see Fradkin's note,
Pathria p.296



$$\mathbf{J} = \left\langle \varphi \left| \sum_i \frac{\mathbf{p}_i}{m} \right| \varphi \right\rangle.$$

We now consider the effect of a translation of the entire system, Fermi surface as well as excited quasiparticles, by a constant amount $\mathbf{q}$; such a translation is equivalent to looking at the system from a moving frame of reference which has a constant velocity $(-\mathbf{q}/m)$. In this moving frame, the interaction energy remains unchanged, while the kinetic energy operator increases by an amount

$$\sum_i \left\{ \frac{\mathbf{q} \cdot \mathbf{p}_i}{m} + \frac{q^2}{2m} \right\}.$$

We next assume that $\mathbf{q}$ is small, and use elementary perturbation theory to obtain the first-order correction to the energy E:

$$\begin{aligned} \delta E &= \left\langle \varphi \left| \sum_i \frac{\mathbf{q} \cdot \mathbf{p}_i}{m} \right| \varphi \right\rangle + O(q^2) \\ &= \mathbf{q} \cdot \mathbf{J} + O(q^2). \end{aligned}$$

$\mathbf{J}$ thus appears as the first derivative of the energy with respect to $\mathbf{q}$, an arbitrary component J_α being given by

$$J_\alpha = dE/dq_\alpha. \qquad (1.96a)$$

(a passive “boost”)

Nozieres and Pines, p.37

Introducing Fermi liquid parameters

- Moments of $u_{kk'}^{\sigma\sigma'}$ over the FS provide the most important information about interactions (e.g., see the previous m* formula)

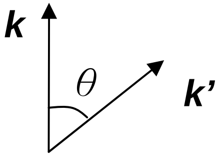
let

$$u_{kk'}^{\uparrow\uparrow} = u_{kk'}^{\downarrow\downarrow} = u_{kk'}^{s} + u_{kk'}^{a}$$

$$u_{kk'}^{\uparrow\downarrow} = u_{kk'}^{\downarrow\uparrow} = u_{kk'}^{s} - u_{kk'}^{a}$$

For spherical FS, $u_{kk'}$ depends only on θ

and decompose



$$u_{kk'}^{s} = \sum_{\ell=0}^{\infty} u_\ell^s P_\ell(\cos\theta)$$

$$u_{kk'}^{a} = \sum_{\ell=0}^{\infty} u_\ell^a P_\ell(\cos\theta)$$

$$\rightarrow \begin{cases} u_\ell^s = \dfrac{2\ell+1}{2}\displaystyle\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\dfrac{u_{kk'}^{\uparrow\uparrow}+u_{kk'}^{\uparrow\downarrow}}{2} \\ u_\ell^a = \dfrac{2\ell+1}{2}\displaystyle\int_{-1}^{1} d\cos\theta\, P_\ell(\cos\theta)\dfrac{u_{kk'}^{\uparrow\uparrow}-u_{kk'}^{\uparrow\downarrow}}{2} \end{cases}$$

- Dimensionless parameters

$$F_\ell^s \equiv VD^*(\varepsilon_F)u_\ell^s, \quad F_\ell^a \equiv VD^*(\varepsilon_F)u_\ell^a$$

A small set of parameters for various phenomena

For example,

$$\frac{m^*}{m} = 1 + V\int \frac{k'^2\,dk'}{(2\pi)^3} d\Omega'\left(u_{kk'}^{\uparrow\uparrow} + u_{kk'}^{\uparrow\downarrow}\right)\cos\theta'\,\delta(\varepsilon_{k'}^0 - \varepsilon_F)$$

$$\cdots = 1 + \frac{1}{3}F_1^s$$

H.W.

determined from specific heat.

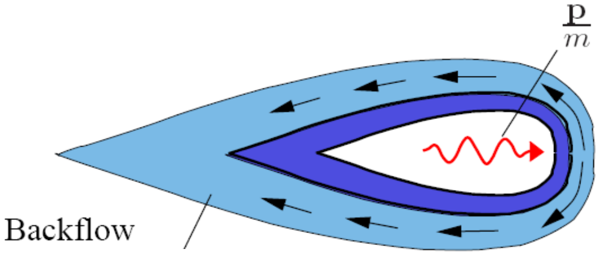
m*/m ∼ 3 for He-3

More on the effective mass

- recall

$$\frac{m^*}{m} = 1 + \frac{1}{3} F_1^s$$

$$\rightarrow \frac{\vec{p}_F}{m^*} = \frac{\vec{p}_F}{m} - \left(\frac{F_1^s}{3 + F_1^s} \right) \frac{\vec{p}_F}{m}$$



Backflow correction
(to ensure current conservation)

- $$\frac{m^*}{m} = 1 + \frac{1}{3} V D^*(\varepsilon_F) u_1^s, \quad D^*(\varepsilon_F) = \frac{m^* k_F}{\pi^2 \hbar^2}$$

$$\rightarrow m^* = \frac{m}{1 - \frac{V}{3} D(\varepsilon_F) u_1^s}$$

diverges when $\frac{V}{3} D(\varepsilon_F) u_1^s = 1$ ($\sim$ Mott transition)

Compressibility of Fermi liquid

$$\kappa \equiv -\frac{1}{V}\frac{\partial V}{\partial P}$$

At fixed S or T
(little difference near T=0)

$$\rightarrow \kappa_T = \frac{1}{n^2}\left.\frac{\partial n}{\partial \mu}\right|_T$$

$$dP = SdT + nd\mu$$

$$\rightarrow \left.\frac{\partial \mu}{\partial P}\right|_T = \frac{1}{n}$$

$$\kappa_T = -n\left.\frac{\partial(1/n)}{\partial P}\right|_{N,T}$$

$$= \frac{1}{n}\left.\frac{\partial n}{\partial \mu}\right|_T \left.\frac{\partial \mu}{\partial P}\right|_T = \frac{1}{n^2}\left.\frac{\partial n}{\partial \mu}\right|_T$$

Note: $\delta\rho \leftrightarrow \delta\mu$

Before compression

At T=0, $\delta f_{k\sigma} = \theta(\mu - \varepsilon_{k\sigma}) - \theta(\varepsilon_F - \varepsilon^0_{k\sigma})$

$$= \left.\frac{\partial \theta}{\partial \mu}\right|_{\varepsilon_F}(\delta\mu - \delta\varepsilon_{k\sigma})$$

$$\frac{\partial \delta f_{k\sigma}}{\partial \mu} = \delta(\varepsilon_F - \varepsilon^0_{k\sigma})\Bigg(1 - \underbrace{\sum_{k'\sigma'} u^{\sigma\sigma'}_{kk'}\frac{\partial \delta f_{k'\sigma'}}{\partial \mu}}_{\equiv A_k}\Bigg)$$

$\equiv A_k$ (indep. of σ if not magnetized)

Note: Slightly different from Marder's (see Baym and Pethick, p.11)

Both ***k*** and ***k***' lie on FS, and $U_{kk'}$ depends only on cos θ, ∴ A_k is indep. of ***k***.

$$\rightarrow A = \sum_{k'\sigma'} u^{\sigma\sigma'}_{kk'}\frac{\partial \delta f_{k'\sigma'}}{\partial \mu} = \underbrace{\sum_{k'\sigma'} u^{\sigma\sigma'}_{kk'}\delta(\varepsilon_F - \varepsilon^0_{k'})}_{= F_0^s}(1 - A)$$

$$\rightarrow A = \frac{F_0^s}{1 + F_0^s}$$

Dependence of various quantities on $\delta\mu$

$$\delta f_{k\sigma} = \delta\left(\varepsilon_F - \varepsilon^0_{k\sigma}\right)(1-A)\delta\mu$$

$$=\delta\left(\varepsilon_F - \varepsilon^0_{k\sigma}\right)\frac{1}{1+F^s_0}\delta\mu$$

$$\delta\varepsilon_{k\sigma} = \sum_{k'\sigma'} u^{\sigma\sigma'}_{kk'}\delta f_{k'\sigma'}$$

$$= \frac{F^s_0}{1+F^s_0}\delta\mu$$

also, $$\frac{\delta N}{V} = \frac{1}{V}\sum_{k\sigma}\delta f_{k\sigma} = D^*(\varepsilon_F)\frac{1}{1+F^s_0}\delta\mu$$

$$\left.\frac{\partial n}{\partial\mu}\right|_V = D^*(\varepsilon_F)\frac{1}{1+F^s_0}$$

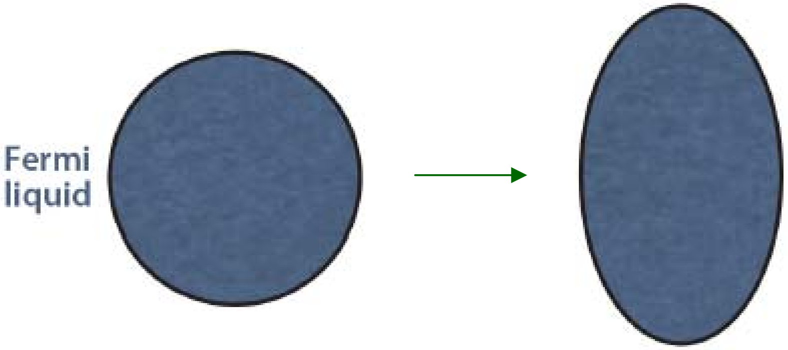
$$\to \kappa_T = \frac{1}{n^2}\left.\frac{\partial n}{\partial\mu}\right|_{V,T} = \frac{D^*(\varepsilon_F)}{n^2}\frac{1}{1+F^s_0}$$

$$\therefore \frac{\kappa_T}{\kappa^0_T} = \frac{m^*/m}{1+F^s_0}$$

Note:

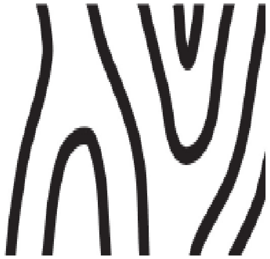
- For attractive interaction, $F^s_\ell < 0$

If $F^s_0 = -1$, then κ_T diverges, and FS will become unstable to deformation (spontaneous breaking of rotational symmetry).



- This is called Pomeranchuk instability (1958). For example, nematic FL.

向列型



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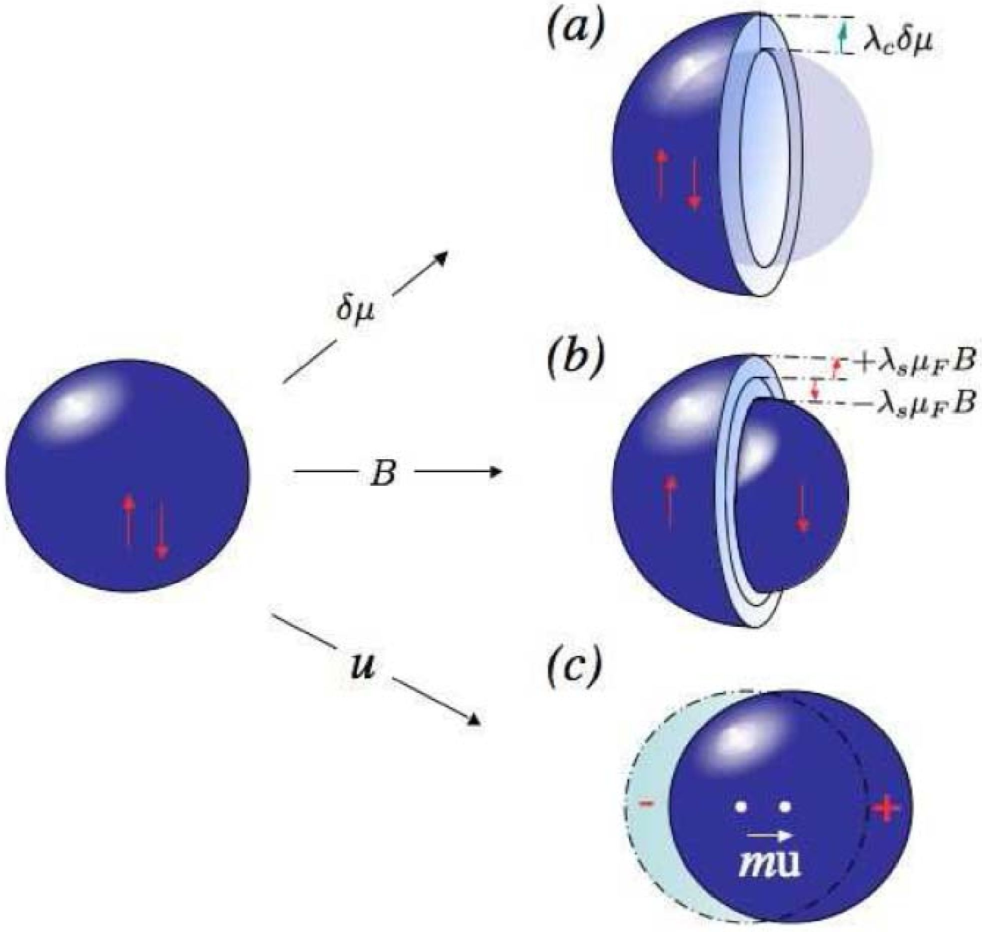
Deformation of Fermi sphere and the FL parameters



$$F_0^s = \sum_{k'\sigma'} u_{kk'}^{\sigma\sigma'}\delta\left(\varepsilon_{k'}^0 - \varepsilon_F\right)$$

$$F_0^a$$

$$F_1^s = \frac{1}{3}\sum_{k'\sigma'} u_{kk'}^{\sigma\sigma'}\frac{\vec{k}_F\cdot\vec{k}_F\,'}{k_F^2}\delta(\varepsilon_{k'}^0 - \varepsilon_F)$$

From Coleman’s note

summary

PROPERTY	NON-INTERACTING	LANDAU FERMI LIQUID
Fermi momentum	p_F	unchanged
Density of particles	$2\frac{V_{FS}}{(2\pi)^3}$	unchanged
Density of states	$\mathcal{N}(0) = \frac{mp_F}{\pi^2\hbar^3}$	$\mathcal{N}^*(0) = \frac{m^* p_F}{\pi^2\hbar^3}$
Effective mass	m	$\frac{m^*}{m} = 1 + \frac{F_1^s}{3}$
Specific heat Coefficient $C_V = \gamma T$	$\gamma = \frac{\pi^2}{3}k_B^2\mathcal{N}(0)$	$c_V = \frac{m^*}{m}c_V^0$
Spin susceptibility	$\chi_s = \mu_F^2\mathcal{N}(0)$	$\chi_s = \frac{m^*/m}{1+F_0^a}\chi_s^0$
compressibility	$\kappa = \mathcal{N}(0)$	$\kappa = \frac{m^*/m}{1+F_0^s}\kappa^0$

For He-3,

$F_1^s \cong 6$

(larger effective mass)

$F_0^a \cong -0.5$

(more spin polarizable)

$F_0^s \cong 10$

(less compressible)

From Coleman's note

Travelling wave: firstly, 1st sound (i.e., the usual pressure wave)

Velocity of the 1st sound

$$c_1 = \sqrt{\left.\frac{\partial P}{\partial \rho}\right|_s}, \qquad \rho = mn$$

$$\text{so } \left.\frac{\partial P}{\partial \rho}\right|_{s,N} = -\frac{V}{n}\left.\frac{\partial P}{\partial V}\right|_s = \frac{1}{\rho \kappa_s}$$

$$= \left[\frac{n}{mD^*(\varepsilon_F)}\left(1+F_0^s\right)\right]^{1/2}$$

$$D^*(\varepsilon_F) = \frac{m^* k_F}{\pi^2 \hbar^2}, \qquad n = \frac{k_F^3}{3\pi^2}$$

$$\Rightarrow c_1 = v_F\left[\frac{m^*}{3m}\left(1+F_0^s\right)\right]^{1/2} \rightarrow \frac{v_F}{\sqrt{3}} \quad \text{(w/o interaction)}$$

F_0^S=10.8 for He-3,
determined by measuring C_1

Zero sound (predicted by Landau, verified by Wheatley et al 1966)

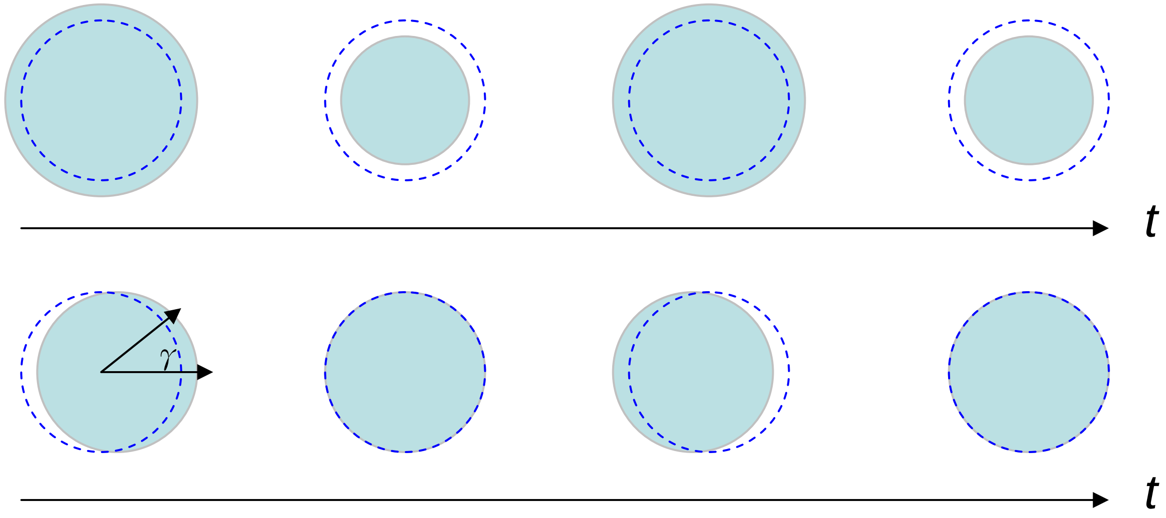
- usual sound requires $\omega\tau$ <<1 (mean free path $\ell << \lambda$)

  when $\omega\tau \rightarrow 1$, sound is strongly absorbed
- when $\omega\tau$ >>1, sound propagation is again possible
- zero sound is a collisionless sound $\sim$ plasma wave in charged FL

  no thermal equilibrium in each volume element
- to get the zero sound, one can increase ω or decrease T (to increase τ)

Can exist at 0 K

Oscillation of Fermi sphere



1st sound

zero sound

$$\delta f_k \propto \frac{\cos\gamma}{s-\cos\gamma}$$

(egg-like shape)

Giamarchi’s note, p.102

Boltzmann-like approach (requires $\hbar\omega << \varepsilon_F, q << k_F$)

$$\frac{\partial f}{\partial t}+\dot{\vec{r}}\cdot\frac{\partial f}{\partial \vec{r}}+\dot{\vec{k}}\cdot\frac{\partial f}{\partial \vec{k}}=0 \quad \text{(collisionless)}$$

Instead of the semiclassical equations, one uses

$$\dot{\vec{r}}_k=\frac{\partial \varepsilon_k}{\hbar\partial \vec{k}}; \quad \hbar\dot{\vec{k}}=-\frac{\partial \varepsilon_k}{\partial \vec{r}}$$

consider

No r-dependence hidden in ε .

$$f_{k\sigma}=f_k^0+\delta f_{k\sigma}(\vec{r},t), \qquad f_k^0=\theta\left(\varepsilon_k^0-\varepsilon_F\right)$$

$$\delta f_{k\sigma}=\alpha_k e^{i(\vec{q}\cdot\vec{r}-\omega t)} \quad \text{(indep of spin)}$$

$$\rightarrow \frac{\partial f_{k\sigma}}{\partial t}+\dot{\vec{r}}\cdot\frac{\partial f_{k\sigma}}{\partial \vec{r}}=-i(\omega-\vec{v}_k\cdot\vec{q})\delta f_{k\sigma}$$

To order δ f

$$\dot{\vec{k}}\cdot\frac{\partial f_{k\sigma}}{\partial \vec{k}}\cong-\frac{\partial \varepsilon_k}{\partial \vec{r}}\cdot\vec{v}_F\frac{\partial f_k^0}{\partial \varepsilon_k}=\frac{\partial \varepsilon_k}{\partial \vec{r}}\cdot\vec{v}_F\delta\left(\varepsilon_k^0-\varepsilon_F\right)$$

$$\frac{\partial \varepsilon_k}{\partial \vec{r}}=\sum_{k'\sigma'}u_{kk'}^{\sigma\sigma'}\frac{\partial\delta f_{k'\sigma'}}{\partial \vec{r}}=i\vec{q}\sum_{k'\sigma'}u_{kk'}^{\sigma\sigma'}\delta f_{k'\sigma'}$$

$$\rightarrow(\omega-\vec{v}\cdot\vec{q})\alpha_k-\vec{v}_k\cdot\vec{q}\delta\left(\varepsilon_k^0-\varepsilon_F\right)\sum_{k'\sigma'}u_{kk'}^{\sigma\sigma'}\alpha_{k'}=0$$

$$\rightarrow\alpha_k=\delta\left(\varepsilon_k^0-\varepsilon_F\right)\frac{\vec{v}_k\cdot\vec{q}}{\omega-\vec{v}_k\cdot\vec{q}}\sum_{k'\sigma'}u_{kk'}^{\sigma\sigma'}\alpha_{k'}$$

$$\text{let}\quad \alpha_k = \delta\left(\varepsilon_k^0 - \varepsilon_F\right)\phi_k$$

$$\text{then } \phi_k = \frac{\vec{v}_k \cdot \vec{q}}{\omega - \vec{v}_k \cdot \vec{q}} \sum_{k'\sigma'} u_{kk'}^{\sigma\sigma'} \alpha_{k'} \quad (1)$$

$$\sum_{k'\sigma'} u_{kk'}^{\sigma\sigma'} \alpha_{k'} = \sum_{k'\sigma'} u_{kk'}^{\sigma\sigma'} \delta\left(\varepsilon_{k'}^0 - \varepsilon_F\right)\phi_{k'}$$

$$= \int \frac{d\Omega'}{4\pi} a(\theta')\phi_{k'}$$

$$a(\theta) \equiv \frac{V}{2} D^*\left(u_{kk'}^{\uparrow\uparrow} + u_{kk'}^{\uparrow\downarrow}\right)$$

Assume a(θ) $\sim$const.

$$\cong \int \frac{d\Omega}{4\pi} a(\theta) = F_0^s$$

decompose

$$\phi_k(\theta) = \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\phi_\ell$$

$$\to \quad \int \frac{d\Omega'}{4\pi} \phi_k(\theta') = \phi_0$$

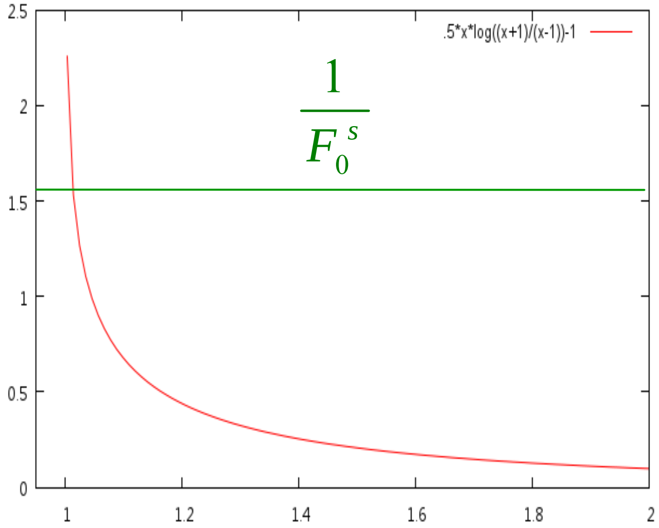
(1) $\to \phi_0 = \left(F_0^s \int \frac{d\Omega}{4\pi} \frac{\vec{v}_k \cdot \vec{q}}{\omega - \vec{v}_k \cdot \vec{q}} \right) \phi_0$

$$\text{let } \gamma = \angle(\vec{q}, \vec{k})$$

$$\to 1 = \frac{F_0^s}{2} \int d\cos\gamma \frac{\cos\gamma}{s - \cos\gamma}, \qquad s = \frac{\omega}{vq}$$

$$= \frac{F_0^s}{2}\left(-1 - \frac{s}{2}\ln\frac{s-1}{s+1}\right)$$

$$\text{or } \frac{s}{2}\ln\frac{s+1}{s-1} - 1 = \frac{1}{F_0^s}$$

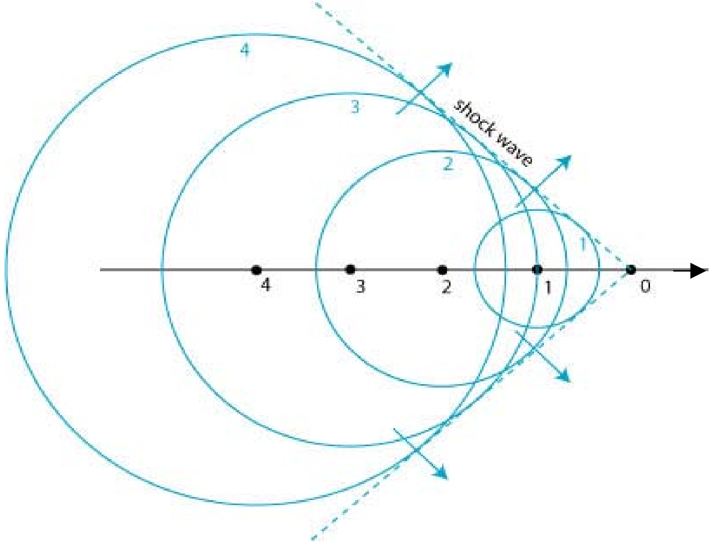


- only F$_0^s$ > 0 (repulsion) can have a solution
- for $\quad F_0^s \to 0, \quad S \to 1^+$

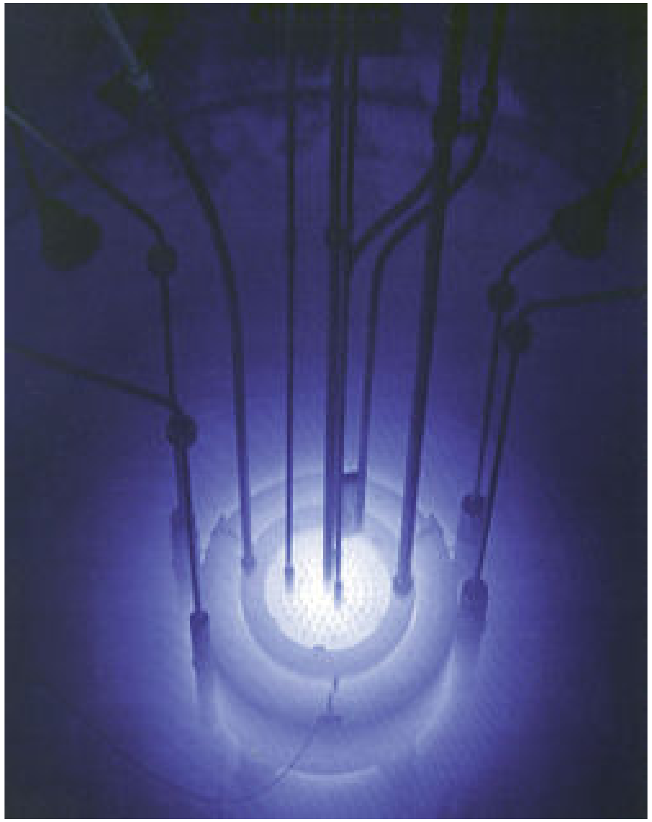
$$\omega = s v_F q \quad \therefore c_0 = s v_F \cong \sqrt{3} c_1$$

- when QP velocity > C_0 (s<1), the integral has a pole at $\cos\gamma_0 = s$, a QP would emit “supersonic” zero sound

Analogies:

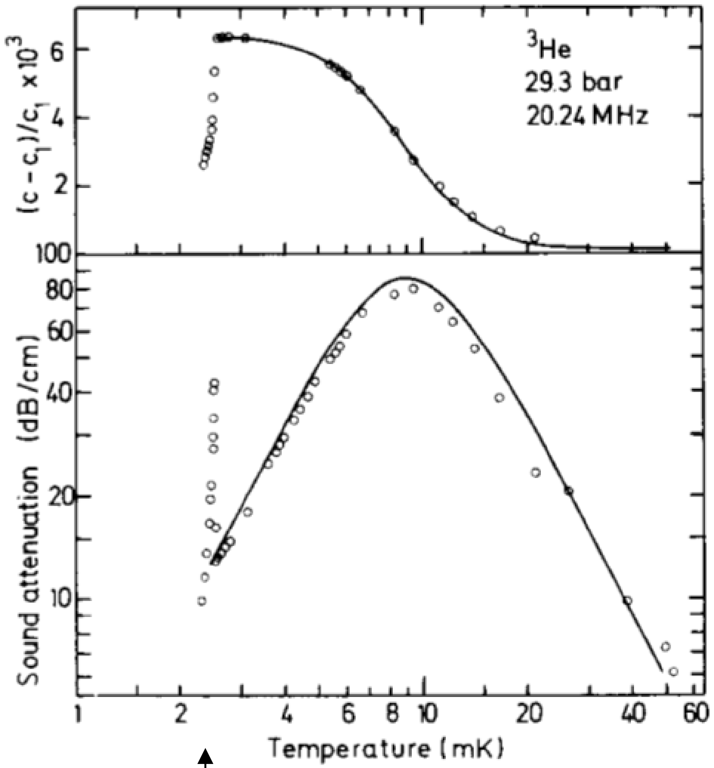


- Supersonic shock wave



- Cherenkov EM radiation from “superluminal” charged particles

Transition from the 1st sound to the zero sound



Superfluid transition

Dispersion of zero sound in He-3

(from neutron scattering exp't)

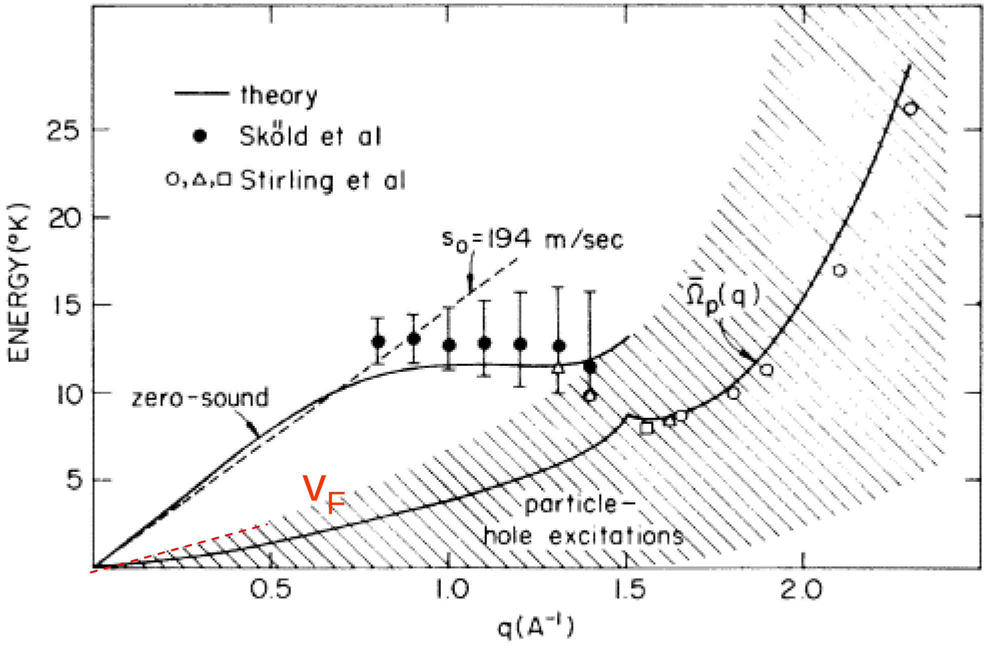


FIG. 2. Comparison between theory and experiment for the zero sound energy, and $\overline{\Omega}_p$, the mean excitation energy of a single quasiparticle–quasipole pair. The dashed line is a linear extrapolation of the line is a linear extrapolation of the long wavelength zero sound spectrum, using the experimentally determined velocity of 194 msec^{-1}.

Vollhardt and Woelfle, p.45

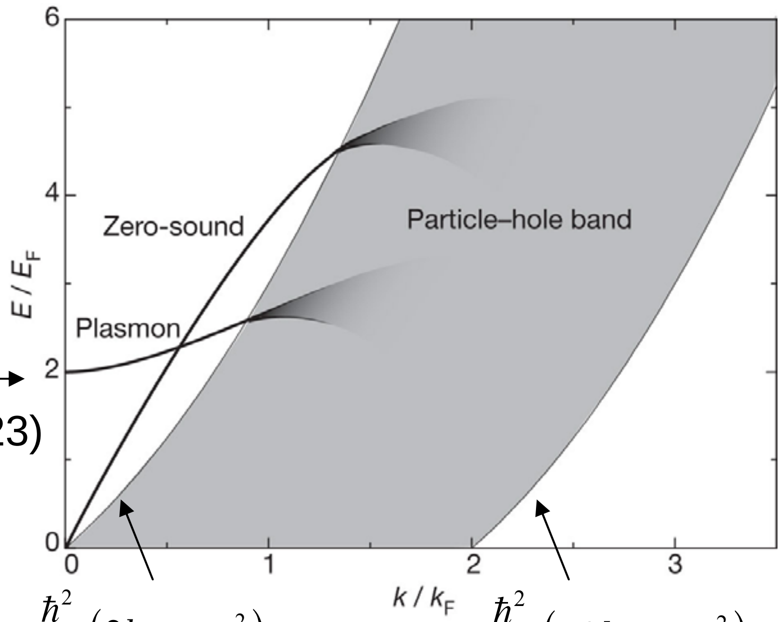
Aldrich et al, PRL 1976

coherent

In addition to collective excitations (zero sound, plasma),
there are also particle-hole excitations

incoherent

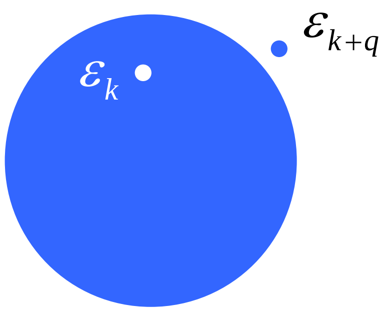
For charged
FL only →
(more in ch 23)



$$\frac{\hbar^2}{2m}\left(2k_F q+q^2\right)$$

$$\frac{\hbar^2}{2m}\left(-2k_F q+q^2\right)$$

particle-hole excitation:



$$\varepsilon_{k+q}-\varepsilon_k=\frac{\hbar^2}{2m}\left(2\vec{k}\cdot\vec{q}+q^2\right)$$

Q: what is the particle-hole band for 1-dim electron liquid?

H. Godfrin et al, Nature 2012

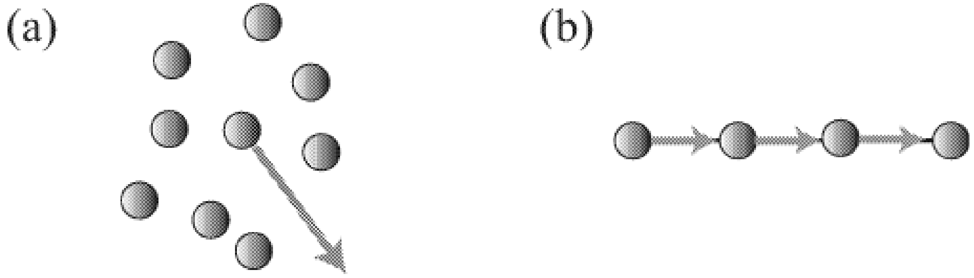
From Altshuler’s slide

Beyond the Fermi liquid

Quasiparticle decay rate at T = 0 in a Fermi Liquid:

$$\boxed{\frac{\hbar}{\tau_{e-e}(\varepsilon)} \propto \begin{cases} \varepsilon^2/\varepsilon_F & d=3 \\ \left(\varepsilon^2/\varepsilon_F\right)\log\left(\varepsilon_F/\varepsilon\right) & d=2 \\ \varepsilon & d=1 \end{cases}}$$

- For *d* =3,2, from $\varepsilon << \varepsilon_F$ it follows that $\tau_{ee} >> \hbar/\varepsilon$, i.e., the QPs are well determined and the Fermi-liquid approach is applicable.
- For *d* =1, τ_{ee} is of the order of $\hbar/\varepsilon$, i.e., the QP is not well defined and the Fermi-liquid approach is not valid.



(a) In high dimensions, nearly free quasiparticle excitations, that look nearly as individual particles are possible. (b) In a one-dimensional interacting system, an individual electron cannot move without pushing all the electrons. Thus, only *collective* excitations can exist.

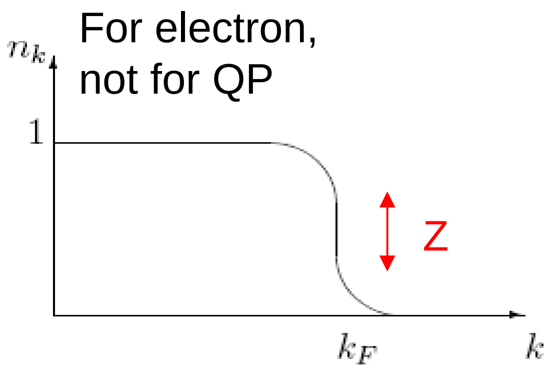
$\longrightarrow$ Tomonaga-Luttinger liquid in 1D (1950,1963)

For more, see Marder, Sec 18.6

Giamarchi, p.6

Features of a Luttinger liquid (cited from wiki)

- Even at T=0, the particles' momentum distribution function has no sharp jump (Z=0). (in contrast to QP dist)
- Charge and spin waves are the elementary excitations of the Luttinger liquid, unlike the QPs of the FL (which carry both spin and charge).
- spin density waves propagate independently from the charge density waves (spin-charge separation).



For electron, not for QP

Luttinger liquids reported in literatures

- electrons moving along edge states in the fractional Quantum Hall Effect (Th: Kane and Fisher 1994; Ep: 1996)
- electrons in carbon nanotubes (McEuen group, 1998)
- 'quantum wires' defined by applying gate voltages to a two-dimensional electron gas. (Auslaender et al, 2000)

Non-Fermi-liquid in 2D?